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Research Article

A Coincidence Best Proximity Point Problem in G-Metric Spaces

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The aim of this paper is to initiate the study of coincidence best proximity point problem in the setup of generalized metric spaces. Some results dealing with existence and uniqueness of a coincidence best proximity point of mappings satisfying certain contractive conditions in such spaces are obtained. An example is provided to support the result proved herein. Our results generalize, extend, and unify various results in the existing literature.

1. Introduction and Preliminaries

Let Y be any nonempty subset of a metric space X and $T : Y \to X$. A fixed point problem $\mathrm{Fix}(X, Y, T)$ defined by X, Y and T is to find a point x^* in Y such that $d(x^*, Tx^*) = 0$. A point x^* in Y, where $\inf\{d(y, Tx^*) : y \in Y\}$, is attained; that is, $d(x^*, Tx^*) = \inf\{d(y, Tx^*) : y \in Y\}$ holds and is called an approximate fixed point of T. In case it is not possible to solve $\mathrm{Fix}(X, Y, T)$, it could be interesting to study the conditions that assure existence and uniqueness of approximate fixed point of a mapping T.

Let A and B be two nonempty subsets of X and $T : A \to B$. Suppose that $\Delta_{AB} = d(A, B) = \inf(\{d(a, b) : a \in A, b \in B\})$ is the measure of a distance between two sets A and B. A point x^* is called the best proximity point of T if $d(x^*, Tx^*) = \Delta_{AB}$. Thus the best proximity point problem defined by a mapping T and a pair of sets (A, B) is to find a point x^* in A such that $d(x^*, Tx^*) = \Delta_{AB}$. If $A \cap B = \phi$, the fixed point problem defined by A, B and T has no solution. If $A = B$, the best proximity point problem reduces to a fixed point problem. In this way, the best proximity point problem can be viewed as a natural generalization of a fixed point problem. Furthermore, results dealing with existence and uniqueness of the best proximity point of certain mappings are more general than the ones dealing with fixed point problem of those mappings. A coincidence best proximity point problem is defined as follows: find a point x^* in A such that $d(gx^*, Tx^*) = \Delta_{AB}$, where g is a self-mapping on A. This is an extension of the best proximity point problem. There are several results dealing with proximity point problem in the setup of metric spaces (see, e.g., [1–11] and references mentioned therein).

Mustafa and Sims [12] introduced the concept of a G-metric space as a substantial generalization of metric space. They [13] obtained some fixed point theorems for mappings satisfying different contractive conditions in such spaces. Based on the notion of generalized metric spaces, Mustafa et al. [14–16] obtained several fixed point theorems for mappings satisfying different contractive conditions. Mustafa et al. [17–19] obtained some fixed point theorems for mappings satisfying different contractive conditions. Chugh et al. [20] obtained some fixed point results for maps satisfying property P in G-metric spaces. Saadati et al. [21] studied fixed point of contractive mappings in partially ordered G-metric spaces. Shatanawi [22] obtained fixed points of Φ-maps in G-metric spaces. For more details, we refer to, for example, [22–39] and references therein.

A study of the best proximity point problem in the setup of G-metric space is a recent development by Hussain et al. [40]. This motivates us to extend the scope of this investigation and extend this study to coincidence proximity point problem of certain mappings in the framework of generalized metric spaces.

Consistent with Mustafa and Sims [12], the following definitions and results will be needed in the sequel.

Definition 1. Let X be a nonempty set. Suppose that a mapping $G : X \times X \times X \to R^+$ satisfies

(G1) $0 \le G(x, y, z)$ for all $x, y, z \in X$ and $G(x, y, z) = 0$ if and only if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \ne y$,

(G3) $G(x, x, y) \le G(x, y, z)$ for all $x, y, z \in X$, with $y \ne z$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetric in all three variables),

(G5) $G(x, y, z) \le G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a generalized metric on X or G-metric on X and (X, G) is called a *G-metric space*.

Definition 2. Let (X, G) be a G-metric space, $\{x_n\}$ a sequence in X, and $x \in X$. One says that $\{x_n\}$ is

(i) a *G-Cauchy* sequence if, for any $\varepsilon > 0$, there exists a natural number N such that, for all $n, m, l \ge N$, $G(x_n, x_m, x_l) < \varepsilon$;

(ii) a *G-convergent* sequence if, for any $\varepsilon > 0$, there exists a natural number N such that, for all $n, m \ge N$, $G(x_n, x_m, x) < \varepsilon$ for some x in X.

A G-metric space X is said to be complete if every G-Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in (X, G)$ if and only if $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

Proposition 3. *Let (X, G) be a G-metric space; then the following are equivalent.*

(1) *$\{x_n\}$ converges to $x \in X$.*

(2) *$G(x_n, x_m, x) \to 0$, as $m, n \to \infty$.*

(3) *$G(x_n, x_n, x) \to 0$, as $n \to \infty$.*

(4) *$G(x_n, x, x) \to 0$, as $n \to \infty$.*

Definition 4. A G-metric on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 5. *Every G-metric on X will define a metric d_G on X by*

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \quad \forall x, y \in X. \tag{1}$$

Remark 6. Let $\{x_n\}$ be a sequence in G-metric space X. If $\{G(x_n, x_{n+1}, x_{n+1})\} \to 0$ and $\{x_n\}$ is not a Cauchy sequence, then there exist $\epsilon_0 > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ such that, for all $k \in \mathbb{N}$, $k \le m(k) < n(k)$, $G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \ge \epsilon_0$, and $G(x_{m(k)}, x_l, x_l) < \epsilon_0$ for all $l \in \{m(k) + 1, m(k) + 2, \ldots, n(k) - 2, n(k) - 1\}$. If $\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon_0$, then

$$\begin{aligned} &\lim_{k\to\infty} G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}) = \epsilon_0, \\ &\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)+l}, x_{n(k)+l}) = \epsilon_0, \end{aligned} \tag{2}$$

for all $l \ge 0$. Indeed, if $\{G(x_n, x_{n+1}, x_{n+1})\} \to 0$, then, for all $k \in \mathbb{N}$, we have

$$\begin{aligned} &G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}) \\ &\quad \le G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) \\ &\quad \le G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\qquad + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}), \\ &G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\quad \le G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}) \\ &\qquad + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}). \end{aligned} \tag{3}$$

From (3) we have

$$\begin{aligned} &G(x_{m(k)}, x_{n(k)}, x_{n(k)}) - G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) \\ &\qquad - G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) \\ &\quad \le G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}) \\ &\quad \le G(x_{m(k)-1}, x_{m(k)}, x_{m(k)}) + G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\qquad + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}). \end{aligned} \tag{4}$$

Taking limit as $k \to \infty$, we obtain that $\lim_{k\to\infty} G(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)-1}) = \epsilon_0$. To prove

$$\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)+l+1}, x_{n(k)+l+1}) = \epsilon_0 \tag{5}$$

for all $l \ge 0$, we use induction on l. Equation (5) for $l = 0$ holds obviously. Suppose that (5) holds for some $l > 0$. Consider

$$\begin{aligned} G(x_{m(k)}, x_{n(k)+l+1}, x_{n(k)+l+1}) &\le G(x_{m(k)}, x_{n(k)+l}, x_{n(k)+l}) \\ &\quad + G(x_{n(k)+l}, x_{n(k)+l+1}, x_{n(k)+l+1}). \end{aligned} \tag{6}$$

Also,

$$\begin{aligned} G(x_{m(k)}, x_{n(k)+l}, x_{n(k)+l}) &\le G(x_{m(k)}, x_{n(k)+l+1}, x_{n(k)+l+1}) \\ &\quad + G(x_{n(k)+l+1}, x_{n(k)+1}, x_{n(k)+1}). \end{aligned} \tag{7}$$

From (6) and (7), we obtain that

$$\begin{aligned} &G(x_{m(k)}, x_{n(k)+l}, x_{n(k)+l}) - G(x_{n(k)+l+1}, x_{n(k)+1}, x_{n(k)+1}) \\ &\quad \le G(x_{m(k)}, x_{n(k)+l+1}, x_{n(k)+l+1}) \\ &\quad \le G(x_{m(k)}, x_{n(k)+l}, x_{n(k)+l}) + G(x_{n(k)+l+1}, x_{n(k)+1}, x_{n(k)+1}). \end{aligned} \tag{8}$$

Taking limit as $k \to \infty$, we have $\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)+l+1}, x_{n(k)+l+1}) = \epsilon_0$.

Definition 7. Let X be a G-metric space and A and B two nonempty subsets of X. Define

$$\begin{aligned}
\Delta_{AB}^G &= G(A,B,B) = \inf\{G(a,b,b) : a \in A,\ b \in B\},\\
A_0 &= \{a \in A : \text{there exists some } b \in B \text{ such that } G(a,b,b),\\
&\qquad = \Delta_{AB}^G\},\\
B_0 &= \{b \in B : \text{there exists some } a \in A \text{ such that } G(a,b,b)\\
&\qquad = \Delta_{AB}^G\}.
\end{aligned} \tag{9}$$

Now we define the concept of g-best proximity point of a mapping in the setup of G-metric spaces.

Definition 8. Let X be a G-metric space and A and B two nonempty subsets of X. Suppose that $T : A \to B$, and $g : A \to A$. A point $x \in A$ is called g-best proximity point of T if $G(gx, Tx, Tx) = \Delta_{AB}^G$.

Note that if g is an identity mapping on A, then x in above definition becomes the best proximity point of T.

Consistent with [41], we consider the following classes of mappings.

$\Psi = \{\varphi : [0,\infty) \to [0,\infty)$ such that, for all $t > 0$, the series $\sum_{n\geq 1} \varphi^n(t)$ converges$\}$. Elements in Ψ are called (c)-comparison functions.

$\Phi = \{\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) < t$ and $\lim_{r\to t^+} \phi(r) < t$ for all $t > 0\}$.

$\Theta = \{\theta : [0,\infty)^4 \to [0,\infty)$ such that $\theta(a,b,c,d) = 0$ if one or more arguments take the value zero and θ is continuous$\}$.

$\Omega_1 = \{\theta : [0,\infty)^4 \to [0,\infty)$ such that $\theta(a,b,c,d) = 0$ if one or more arguments take the value zero$\}$.

$\Omega_1 = \{\theta : [0,\infty)^4 \to [0,\infty)$ such that $\theta(0,b,c,d) = 0\}$.

$\Omega_2 = \{\theta : [0,\infty)^4 \to [0,\infty)$ such that $\lim_{n\to\infty} \theta(t_n^1, t_n^2, t_n^3, t_n^4) = 0$, whenever the sequences $\{t_n^1\}$, $\{t_n^1\}$, $\{t_n^1\}$, $\{t_n^1\} \subset [0,\infty)$ are such that at least one of them is convergent to zero$\}$.

Definition 9. Let X be a G-metric space and A and B two nonempty subsets of X, $g : A \to A$, and $\alpha : X \times X \to [0,\infty)$. A mapping $T : A \to B$ is said to be $(\varphi,\theta,\alpha,g)$-contraction if, for all $x, y \in A_0$ with $G(gy, Tx, Tx) = \Delta_{AB}^G$ and $\alpha(gx, gy) \geq 1$, one has

$$\begin{aligned}
&\alpha(gx,gy)\, G(Tx,Ty,Ty)\\
&\leq \varphi(M^g(x,y,y))\\
&\quad + \theta\big(G(gy,Tx,Tx) - \Delta_{AB}^G, G(gx,Ty,Ty) - \Delta_{AB}^G,\\
&\qquad G(gx,Tx,Tx) - \Delta_{AB}^G, G(gy,Ty,Ty) - \Delta_{AB}^G\big),
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
M^g(x,y,y) = \max\Big(&G(gx,gy,gy), G(gx,Tx,Tx) - \Delta_{AB},\\
&G(gy,Ty,Ty) - \Delta_{AB}^G,\\
&\frac{G(gx,Ty,Ty) + G(gy,Ty,Ty)}{2} - \Delta_{AB}^G\Big),
\end{aligned} \tag{11}$$

$\psi \in \Psi$ and $\theta \in \Theta$.

Definition 10. Let X be a G-metric space and A and B two nonempty subsets of X, $g : A \to A$, and $\alpha : X \times X \to [0,\infty)$. A mapping $T : A \to B$ is said to be (α, g)-proximinal and admissible if $a_1, a_2, b_1, b_2 \in A_0$, $\alpha(gb_1, gb_2) \geq 1$, $G(ga_1, Tb_1, Tb_1) = \Delta_{AB}^G$, $G(ga_2, Tb_2, Tb_2) = \Delta_{AB}^G$, and

$$\Longrightarrow \alpha(ga_1, ga_2) \geq 1. \tag{12}$$

Definition 11. Let X be a G-metric space and A and B two subsets of X such that A_0 is nonempty, $T : A \to B$, and $g : A \to A$. For $a_1, a_2, a_3, a_4 \in A_0$, the quadruple (A, B, T, g) has

(1) weak P-property of the first kind if

$$\begin{aligned}
&G(ga_1, Ta_3, Ta_3) = \Delta_{AB}^G, \qquad G(ga_2, Ta_4, Ta_4) = \Delta_{AB}^G\\
&\text{implies that } G(ga_1, ga_2, ga_2) \leq G(Ta_3, Ta_4, Ta_4);
\end{aligned} \tag{13}$$

(2) weak P-property of the second kind if

$$\begin{aligned}
&G(ga_1, Ta_3, Ta_3) = \Delta_{AB}^G, \qquad G(ga_2, Ta_4, Ta_4) = \Delta_{AB}^G\\
&\text{implies that } G(ga_1, ga_2, ga_2) = G(Ta_3, Ta_4, Ta_4);
\end{aligned} \tag{14}$$

(3) weak P-property of the third kind if

$$\begin{aligned}
&G(ga_1, Ta_3, Ta_3) = \Delta_{AB}^G, \qquad G(ga_2, Ta_4, Ta_4) = \Delta_{AB}^G\\
&\text{implies that } G(ga_1, ga_2, ga_2) \leq G(Tb_1, Tb_2, Tb_2).
\end{aligned} \tag{15}$$

Definition 12 (see [41]). Let $g : A \to A$ and $\alpha : X \times X \to [0,1)$ be two mappings and let $N \in \mathbb{N}$, $N \geq 2$. One will say that α is (N, g)-transitive on A_0 if $x_1, x_2, \ldots, x_{N+1} \in A_0$, $\alpha(gx_i, gx_{i+1}) \geq 1$ $\forall i \in \{1, 2, \ldots, N\} \Rightarrow \alpha(gx_1, gx_{N+1}) \geq 1$.

Indeed, we will only use the notion of $(2, g)$-transitive mapping on A_0; that is, $x_1, x_2, x_3 \in A_0$, $\alpha(gx_1, gx_2) \geq 1$, $\alpha(gx_2, gx_3) \geq 1$, and

$$\Longrightarrow \alpha(gx_1, gx_3) \geq 1. \tag{16}$$

2. Coincidence Best Proximity Point Results

In this section, we obtain several coincidence best proximity results in the setup of generalized metric spaces.

Theorem 13. *Let X be a complete G-metric space, A and B two closed subsets of X, and g a continuous self-mapping on A such that $\phi \neq A_0 \subseteq gA_0$. Suppose that $T : A \to B$ is continuous (α, g)-proximal and admissible and $(\varphi, \theta, \alpha, g)$-contraction, where $\varphi \in \Psi$, $\theta \in \Omega_1$, and $T(A_0) \subseteq B_0$. If the following conditions hold:*

(a) *quadruple (A, B, T, g) satisfies weak P-property of the first kind;*

(b) *if a sequence $\{z_n\}$ in A_0 such that $\{gz_n\} \subseteq A_0$ is Cauchy, then $\{z_n\}$ is also a Cauchy;*

(c) *there exists $(x_0, x_1) \in A_0 \times A_0$ such that $G(gx_1, Tx_0, Tx_0) = \Delta^G_{AB}$ and $\alpha(gx_0, gx_1) \geq 1$.*

Then there exists a convergent sequence $\{x_n\} \subseteq A_0$ which satisfies

$$G(gx_{n+1}, Tx_n, Tx_n) = \Delta^G_{AB} \quad \forall n \geq 0, \tag{17}$$

and the limit of $\{x_n\}$ is a g-best proximity point of T.

Proof. Let $x_1 \in A_0$. Then $Tx_1 \in T(A_0) \subseteq B_0$. Hence there is $z_2 \in A$ such that $G(z_2, Tx_1, Tx_1) = \Delta^G_{AB}$ which implies that $z_2 \in A_0$. As $A_0 \subseteq gA_0$, there is $x_2 \in A_0$ such that $g(x_2) = z_2$, so $G(gx_2, Tx_1, Tx_1) = G(z_2, Tx_1, Tx_1) = \Delta^G_{AB}$. In a similar way, there is $x_3 \in A_0$ such that $G(gx_3, Tx_2, Tx_2) = \Delta^G_{AB}$. Inductively we construct a sequence $\{x_n\} \subseteq A_0$ such that

$$G(gx_{n+1}, Tx_n, Tx_n) = \Delta^G_{AB} \quad \forall n \geq 0. \tag{18}$$

If there exists some $n_0 \in N$, such that $gx_{n_0} = gx_{n_0+1}$, then $G(gx_{n_0}, Tx_{n_0}, Tx_{n_0}) = G(gx_{n_0+1}, Tx_{n_0}, Tx_{n_0}) = \Delta^G_{AB}$ implies that x_{n_0} is a g-best proximity point of T. If we define $x_m = x_{n_0}$ for all $m \geq n_0$, then $\{x_n\}$ converges to a g-best proximity point of T. The proof is complete. Assume that

$$G(gx_n, gx_{n+1}, gx_{n+1}) > 0 \quad \forall n \geq 0. \tag{19}$$

Note that $x_n, gx_{n+1} \in A_0$ and $Tx_n \in B_0$ for all $n \geq 0$. We claim that

$$\alpha(gx_n, gx_{n+1}) \geq 1 \quad \forall n \geq 0. \tag{20}$$

If $n = 0$, then $\alpha(gx_n, gx_{n+1}) \geq 1$ holds by given hypothesis. Suppose that $\alpha(gx_n, gx_{n+1}) \geq 1$ for some $n > 0$. As T is (α, g)-proximal and admissible, for $x_n, x_{n+1}, x_{n+2} \in A_0$, $\alpha(gx_n, gx_{n+1}) \geq 1$, $G(gx_{n+1}, Tx_n, Tx_n) = \Delta^G_{AB}$, and $G(gx_{n+2}, Tx_{n+1}, Tx_{n+1}) = \Delta^G_{AB}$, we have $\alpha(gx_{n+1}, gx_{n+2}) \geq 1$. Thus (20) holds.

Use weak P-property of the first kind, for all $n > 0$, $x_n, x_{n+1}, x_{n+2} \in A_0$, $G(gx_{n+1}, Tx_n, Tx_n) = \Delta^G_{AB}$, $G(gx_{n+2}, Tx_{n+1}, Tx_{n+1}) = \Delta^G_{AB}$ imply the following inequality:

$$\Longrightarrow G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \leq G(Tx_n, Tx_{n+1}, Tx_{n+1}). \tag{21}$$

Now by (20), (21), and $(\varphi, \theta, \alpha, g)$-contractive property of T, we have

$$\begin{aligned}
&G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \\
&\quad \leq G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\
&\quad \leq \alpha(gx_n, gx_{n+1}) G(Tx_n, Tx_{n+1}, Tx_{n+1}) \\
&\quad \leq \varphi(M^g(x_n, x_{n+1}, x_{n+1})) \\
&\qquad + \theta\big(G(gx_{n+1}, Tx_n, Tx_n) \\
&\qquad\quad - \Delta^G_{AB}, G(gx_n, Tx_{n+1}, Tx_{n+1}) \\
&\qquad\quad - \Delta^G_{AB}, G(gx_n, Tx_n, Tx_n) - \Delta^G_{AB}, \\
&\qquad\quad G(gx_{n+1}, Tx_{n+1}, Tx_{n+1}) - \Delta^G_{AB}\big) \\
&\quad = \varphi(M^g(x_n, x_{n+1}, x_{n+1}))
\end{aligned} \tag{22}$$

for all $n > 0$, where

$$\begin{aligned}
&\varphi(M^g(x_n, x_{n+1}, x_{n+1})) \\
&= \max\Big(G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_n, Tx_n, Tx_n) - \Delta^G_{AB}, \\
&\qquad G(gx_{n+1}, Tx_{n+1}, Tx_{n+1}) - \Delta^G_{AB}, \\
&\qquad \frac{G(gx_n, Tx_{n+1}, Tx_{n+1}) + G(x_{n+1}, Tx_n, Tx_n)}{2} \\
&\qquad - \Delta^G_{AB}\Big) \\
&\leq \max\Big(G(gx_n, gx_{n+1}, gx_{n+1}), G(x_n, x_{n+1}, x_{n+1}) \\
&\qquad + G(gx_{n+1}, Tx_n, Tx_n) - \Delta^G_{AB}, \\
&\qquad G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \\
&\qquad + G(gx_{n+2}, Tx_{n+1}, Tx_{n+1}) - \Delta^G_{AB}, \\
&\qquad \frac{1}{2}[G(x_n, gx_{n+1}, gx_{n+1}) \\
&\qquad\quad + G(gx_{n+1}, Tx_{n+1}, Tx_{n+1}) \\
&\qquad\quad + G(x_{n+1}, gx_{n+2}, gx_{n+2}) \\
&\qquad\quad + G(gx_{n+2}, Tx_n, Tx_n)] - \Delta^G_{AB}\Big) \\
&= \max\Big(G(gx_n, gx_{n+1}, gx_{n+1}), G(x_n, x_{n+1}, x_{n+1}) \\
&\qquad + \Delta^G_{AB} - \Delta^G_{AB}, \\
&\qquad G(gx_{n+1}, gx_{n+2}, gx_{n+2}) + \Delta^G_{AB} - \Delta^G_{AB}, \\
&\qquad \big(\big(G(x_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \\
&\qquad\quad + 2\Delta^G_{AB}\big) \times (2)^{-1}\big) - \Delta^G_{AB}\Big)
\end{aligned}$$

$$= \max\Big(G(gx_n, gx_{n+1}, gx_{n+1}), G(x_n, x_{n+1}, x_{n+1}), G(gx_{n+1}, gx_{n+2}, gx_{n+2}), \frac{G(x_n, gx_{n+1}, gx_{n+1}) + G(gx_{n+1}, gx_{n+2}, gx_{n+2})}{2}\Big)$$
$$= \max(G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n+1}, gx_{n+2}, gx_{n+2})). \tag{23}$$

That is,

$$\varphi(M^g(x_n, x_{n+1}, x_{n+1})) \le \max(G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n+1}, gx_{n+2}, gx_{n+2})). \tag{24}$$

From (22) and (24), we have

$$G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \le \varphi(\max(G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n+1}, gx_{n+2}, gx_{n+2}))) \tag{25}$$

for all $n > 0$.

If there exists some $n_0 \in N$ such that

$$\max\left(G\left(gx_{n_0}, gx_{n_0+1}, gx_{n_0+1}\right), G\left(gx_{n_0+1}, gx_{n_0+2}, gx_{n_0+2}\right)\right) = G\left(gx_{n_0+1}, gx_{n_0+2}, gx_{n_0+2}\right), \tag{26}$$

then, using (19) and the fact that $\varphi(t) < t$ for all $t > 0$, we have

$$G\left(gx_{n_0+1}, gx_{n_0+2}, gx_{n_0+2}\right) \le \varphi\left(G\left(gx_{n_0+1}, gx_{n_0+2}, gx_{n_0+2}\right)\right) < G\left(gx_{n_0+1}, gx_{n_0+2}, gx_{n_0+2}\right), \tag{27}$$

which is a contradiction. Hence

$$\max(G(gx_n, gx_{n+1}, gx_{n+1}), G(gx_{n+1}, gx_{n+2}, gx_{n+2})) = G(gx_n, gx_{n+1}, gx_{n+1}) \tag{28}$$

for all $n > 0$. Now (25) implies that

$$G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \le \varphi(G(gx_n, gx_{n+1}, gx_{n+1})) \tag{29}$$

for all $n > 0$.

In particular, for all $n \ge 1$, we have

$$\begin{aligned} G(gx_n, gx_{n+1}, gx_{n+1}) &\le \varphi(G(gx_{n-1}, gx_n, gx_n)) \\ &\le \varphi^2(G(gx_{n-2}, gx_{n-1}, gx_{n-1})) \\ &\le \cdots \le \varphi^n(G(gx_0, gx_1, gx_1)). \end{aligned} \tag{30}$$

Fix $\epsilon > 0$ and $t_0 = G(gx_0, gx_1, gx_1) > 0$. Since $\varphi \in \Psi$, $\sum_{n\ge1} \varphi^n(t_0)$ converges. In particular, there exists some $m_0 \in N$ such that $\sum_{k=m_0}^{\infty} \varphi^n(t_0) < \epsilon$. Hence, for $m > n \ge m_0$, we have

$$\begin{aligned} G(gx_n, gx_m, gx_m) &\le \sum_{k=n}^{m-1} G(gx_k, gx_{k+1}, gx_{k+1}) \\ &\le \sum_{k=n}^{m-1} \varphi^k G(gx_0, gx_1, gx_1) \\ &\le \sum_{k=m_0}^{\infty} \varphi^n(t_0) < \epsilon. \end{aligned} \tag{31}$$

This implies that $\{gx_n\}$ is a Cauchy sequence. By given hypothesis, $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $z \in X$ such that $\{x_n\} \to z$. As $x_n \in A_0 \subseteq A$ for all n, so $z \in A$. Since T and g are continuous mappings, $\{Tx_n\} \to Tz$ and $\{gx_n\} \to gz$. Taking limit in (18) as $n \to \infty$, we conclude that z is a g-best proximity point of T. □

Remark 14. If g is an identity map in Theorem 13, then we obtain the best proximity point of mapping T.

Corollary 15. *Let X be a complete G-metric space, A and B two closed subsets of X, and g a continuous self-mapping on A such that $\phi \ne A_0 \subseteq gA_0$. Suppose that $T : A \to B$ is continuous (α, g)-proximal and admissible and $(\varphi, \theta, \alpha, g)$-contraction, where $\varphi \in \Psi$, $\theta \in \Omega_1$, and $T(A_0) \subseteq B_0$. If following conditions hold:*

(a) *quadruple (A, B, T, g) satisfies weak P-property of the first kind,*

(b) *for $x, y, z \in A_0$ with $G(gy, Tx, Tx) = \Delta_{AB}^G$ and $\alpha(gx, gy) \ge 1$, the following holds:*

$$\begin{aligned} \alpha(gx, gy) G(Tx, Ty, Tz) &\le kM^g(x, y, z) \\ &\quad + \theta\Big(G(gy, Tx, Tx) - \Delta_{AB}^G, G(gx, Ty, Ty) - \Delta_{AB}^G, G(gx, Tx, Tx) - \Delta_{AB}^G, G(gy, Ty, Ty) - \Delta_{AB}^G\Big), \end{aligned} \tag{32}$$

(c) *if a sequence $\{z_n\}$ in A_0 with $\{gz_n\} \subseteq A_0$ is Cauchy, then $\{z_n\}$ is Cauchy,*

(d) *there is $(x_0, x_1) \in A_0 \times A_0$ such that $G(gx_1, Tx_0, Tx_0) = \Delta_{AB}^G$ and $\alpha(gx_0, gx_1) \ge 1$.*

Then there exists a convergent sequence $\{x_n\} \subseteq A_0$ which satisfies

$$G(gx_{n+1}, Tx_n, Tx_n) = \Delta_{AB}^G \quad \forall n \ge 0, \tag{33}$$

and $\{x_n\}$ converges to g-best proximity point of T.

Example 16. Let $X = \{0, 1, 2, 3, \ldots\}$ and $G : X \times X \times X \to \mathbb{R}^+$ defined by

$$G(x, y, z) = \begin{cases} x + y + z & \text{if } x \neq y \neq z \neq 0, \\ x + y & \text{if } x = y \neq z,\ x, y, z \neq 0, \\ y + z + 1 & \text{if } x = 0, y \neq z,\ y, z \neq 0, \\ y + 2 & \text{if } x = 0, y = z \neq 0, \\ z + 1 & \text{if } x = y = 0, z \neq 0, \\ 0 & \text{if } x = y = z. \end{cases} \tag{34}$$

It is known that X is a complete G-metric space. Let $A = \{0, 2, 4\}$ and $B = \{1, 3, 5, \ldots\}$. Obviously A and B are closed subsets of X and $\Delta_{AB}^G = G(A, B, B) = G(0, 1, 1) = 3$. Take $A_0 = \{0, 2\}$. Define the mapping $g : A \to A$ by

$$g(x) = \begin{cases} x & \text{if } x = 0, 2 \\ \dfrac{x}{2} & \text{if } x = 4. \end{cases} \tag{35}$$

Obviously g is continuous and $A_0 \subseteq g(A_0)$. A mapping $T : A \to B$ defined by $T(x) = 1$ is continuous. Define $\alpha : X \times X \to [0, \infty)$ by $\alpha(x, y) = x + y$. Clearly

$$\begin{aligned} \alpha(g(0), g(2)) &= \alpha(0, 2) = 2 > 1, \\ G(g(0), T(2), T(2)) &= G(0, 1, 1) = 3, \\ G(g(2), T(0), T(0)) &= G(2, 1, 1) = 3. \end{aligned} \tag{36}$$

As $\alpha(0, 2) = 2 > 1$, so T is (α, g)-proximal and admissible. Since

$$\alpha(g(0), g(2)) G(T(0), T(2), T(2)) = 2G(1, 1, 1) = 0 \tag{37}$$

and $M^g(0, 2, 2) = 4$, therefore T is $(\varphi, \theta, \alpha, g)$-contraction. Now

$$\begin{aligned} G(g(0), T(2), T(2)) &= G(0, 1, 1) = 3 = \Delta_{AB}^G, \\ G(g(2), T(0), T(0)) &= G(2, 1, 1) = 3 = \Delta_{AB}^G \end{aligned} \tag{38}$$

imply that $G(g(0), g(0), g(0)) = G(0, 0, 0) = 0 = G(T(2), T(0), T(0)) = G(1, 1, 1)$. Hence quadruple (A, B, T, g) has weak P-property of the first kind. Note that $(0, 2) \in A_0 \times A_0$ with $G(g(2), T(0), T(0)) = 3 = \Delta_{AB}^G$ and $\alpha(g(0), g(2)) > 1$. Thus T has g-best proximity point (0 and 2 are g-best proximity point of T).

Lemma 17. *Let $\phi \in \Phi$ be a mapping and let $\{a_m\} \subset \mathbb{R}^+$ be a sequence. If $a_{m+1} \leq \phi(a_m)$ and $a_m \neq 0$ for all m, then $\{a_m\} \to 0$.*

Theorem 18. *If condition (h) in Theorem 13 is replaced by the following:*

(h') *$\varphi \in \Phi$, $\theta \in \Omega_2$ and α is $(2, g)$-transitive, then there exists a sequence $\{x_n\} \subseteq A_0$ which satisfies*

$$G(gx_{n+1}, Tx_n, Tx_n) = \Delta_{AB}^G \quad \forall n \geq 0 \tag{39}$$

and converges to a g-best proximity point of T.

Proof. Following arguments similar to those in the proof of Theorem 13, we have

$$\begin{aligned} &G(gx_{n+1}, Tx_n, Tx_n) = \Delta_{AB}, \\ &G(gx_n, gx_{n+1}, gx_{n+1}) > 0, \\ &\alpha(gx_n, gx_{n+1}) \geq 1, \\ &\qquad x_n \in A_0, \\ &G(gx_{n+1}, gx_{n+2}, gx_{n+2}) \leq \varphi(G(gx_n, gx_{n+1}, gx_{n+2})) \\ &\qquad\qquad \forall n \geq 0. \end{aligned} \tag{40}$$

By Lemma 17, we have

$$\{G(gx_n, gx_{n+1}, gx_{n+1})\} \longrightarrow 0. \tag{41}$$

Next, we show that $\{gx_n\}$ is a Cauchy sequence. Assume on the contrary that $\{gx_n\}$ is not a Cauchy sequence. Then, by Remark 6, there exist $\epsilon_0 > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ such that the following hold:

$$\begin{aligned} &k \leq m(k) < n(k), \qquad G(gx_{m(k)}, gx_{n(k)}, gx_{n(k)}) > \epsilon_0, \\ &\qquad\qquad \forall k \in \mathbb{N}, \\ &G(gx_{m(k)}, gx_p, gx_p) \leq \epsilon_0 \\ &\forall p \in \{m(k) + 1, m(k) + 2, \ldots, n(k) - 2, n(k) - 1\}, \end{aligned} \tag{42}$$

$$\begin{aligned} \lim_{k \to \infty} G(gx_{m(k)-1}, gx_{n(k)-1}, gx_{n(k)-1}) &= \epsilon_0, \\ \lim_{k \to \infty} G(gx_{m(k)-1}, gx_{n(k)+p}, gx_{n(k)+p}) &= \epsilon_0, \end{aligned} \tag{43}$$

$\forall p \geq 0$. Note that

$$\begin{aligned} 0 &\leq G(gx_{n(k)}, Tx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}^G \\ &\leq G(gx_{n(k)}, gx_{n(k)+1}, gx_{n(k)+1}) \\ &\quad + G(gx_{n(k)+1}, Tx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}^G \\ &= G(gx_{n(k)}, gx_{n(k)+1}, gx_{n(k)+1}). \end{aligned} \tag{44}$$

Therefore

$$\lim_{k \to \infty} \left[G(gx_{n(k)}, Tx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}^G\right] = 0. \tag{45}$$

Similarly,

$$\lim_{k \to \infty} \left[G(gx_{m(k)}, Tx_{m(k)}, Tx_{m(k)}) - \Delta_{AB}^G\right] = 0. \tag{46}$$

Furthermore,

$$\epsilon_0 < G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq M^g(x_{m(k)}, x_{n(k)}, x_{n(k)}) \quad \forall k \geq 0, \tag{47}$$

where

$$\begin{aligned}
&M^g\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right)\\
&=\max\Bigg(G\left(gx_{m(k)}, gx_{n(k)}, gx_{n(k)}\right), G\left(gx_{m(k)}, Tx_{m(k)}, Tx_{m(k)}\right)\\
&\qquad - \Delta_{AB}^G, G\left(gx_{n(k)}, Tx_{n(k)}, Tx_{n(k)}\right) - \Delta_{AB}^G,\\
&\qquad \frac{G\left(gx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}\right)+G\left(gx_{n(k)}, Tx_{n(k)}, Tx_{m(k)}\right)}{2}\\
&\qquad - \Delta_{AB}^G\Bigg).
\end{aligned} \tag{48}$$

$$\begin{aligned}
&\frac{G\left(gx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}\right) + G\left(gx_{n(k)}, Tx_{n(k)}, Tx_{m(k)}\right)}{2}\\
&\quad - \Delta_{AB}^G\\
&\le \frac{1}{2}\left[G\left(gx_{m(k)}, gx_{n(k)+1}, gx_{n(k)+1}\right)\right.\\
&\qquad + G\left(gx_{n(k)+1}, Tx_{n(k)}, Tx_{n(k)}\right)\\
&\qquad + G\left(gx_{n(k)}, gx_{m(k)+1}, gx_{m(k)+1}\right)\\
&\qquad \left. + G\left(gx_{m(k)+1}, Tx_{n(k)}, Tx_{m(k)}\right)\right]\\
&\quad - \Delta_{AB}^G\\
&= \left(G\left(gx_{m(k)}, gx_{n(k)+1}, gx_{n(k)+1}\right) + \Delta_{AB}\right.\\
&\qquad \left. + G\left(gx_{n(k)}, gx_{m(k)+1}, gx_{m(k)+1}\right) + \Delta_{AB}\right) \times (2)^{-1}\\
&\quad - \Delta_{AB}\\
&= \left(G\left(gx_{m(k)}, gx_{n(k)+1}, gx_{n(k)+1}\right)\right.\\
&\qquad \left. + G\left(gx_{n(k)}, gx_{m(k)+1}, gx_{m(k)+1}\right)\right) \times (2)^{-1}.
\end{aligned} \tag{49}$$

Taking limit as $k \to \infty$ in (49) and using (45), we obtain that

$$\begin{aligned}
&\lim_{k\to\infty}\left(\frac{G\left(gx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}\right) + G\left(gx_{n(k)}, Tx_{n(k)}, Tx_{m(k)}\right)}{2}\right.\\
&\qquad \left. - \Delta_{AB}^G\right) \le \frac{\epsilon+\epsilon}{2} = \epsilon_0.
\end{aligned} \tag{50}$$

Taking limit as $k \to \infty$ in (48) and using (43), (45), (46), and (50), we have

$$\lim_{k\to\infty} M^g\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right) = \max\left(\epsilon_0, 0, 0, \epsilon_0\right) = \epsilon_0. \tag{51}$$

Thus a sequence $\{M^g(x_{m(k)}, x_{n(k)}, x_{n(k)})\}$ converges to ϵ_0 and terms of this sequence are strictly greater than ϵ_0. In particular, since $\varphi \in \Phi$,

$$\lim_{k\to\infty} \varphi\left(\left\{M^g\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right)\right\}\right) = \lim_{t\to\epsilon_0^+} \varphi(t) < \epsilon_0. \tag{52}$$

From the fact that $\alpha(gx_n, gx_{n+1}) \ge 1$ for all $n \ge 0$ and α is $(2, g)$-transitive, we deduce that

$$\alpha\left(gx_{m(k)}, gx_{n(k)}\right) \ge 1 \quad \forall k \ge 0. \tag{53}$$

As (A, B, T, g) has the weak P-property of the first kind, so, for all $k \ge 0$,

$$\begin{aligned}
&x_{m(k)}, x_{m(k)+1}, x_{n(k)}, x_{n(k)+1} \in A_0,\\
&G\left(gx_{m(k)+1}, Tx_{m(k)}, Tx_{m(k)}\right) = \Delta_{AB}^G,\\
&G\left(gx_{n(k)+1}, Tx_{n(k)}, Tx_{n(k)}\right) = \Delta_{AB}^G.
\end{aligned} \tag{54}$$

This implies that

$$G\left(gx_{m(k)+1}, gx_{n(k)+1}, gx_{n(k)+1}\right) \le G\left(Tx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}\right). \tag{55}$$

As T is $(\varphi, \theta, \alpha, g)$-contraction, so we have

$$\begin{aligned}
&G\left(gx_{m(k)+1}, gx_{n(k)+1}, gx_{n(k)+1}\right)\\
&\le G\left(gx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}\right)\\
&\le \alpha\left(Tx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}\right) G\left(gx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}\right)\\
&\le \varphi\left(M^g\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right)\right.\\
&\qquad + \theta\left(G\left(gx_{n(k)}, Tx_{m(k)}, Tx_{m(k)}\right)\right.\\
&\qquad\qquad - \Delta_{AB}, G\left(gx_{m(k)}, Tx_{n(k)}, Tx_{n(k)}\right) - \Delta_{AB},\\
&\qquad\qquad G\left(gx_{m(k)}, Tx_{m(k)}, Tx_{m(k)}\right) - \Delta_{AB},\\
&\qquad\qquad \left.\left. G\left(gx_{n(k)}, Tx_{n(k)}, Tx_{n(k)}\right) - \Delta_{AB}\right)\right).
\end{aligned} \tag{56}$$

Using (45), the third and the fourth arguments of θ converge to zero as $k \to \infty$. Since $\theta \in \Omega_2$, all the terms tend to zero as $k \to \infty$. Taking limit as $k \to \infty$ in (56), using (45) and (52), we have

$$\begin{aligned}
\epsilon_0 &= \lim_{k\to\infty} G\left(gx_{m(k)+1}, gx_{n(k)+1}, gx_{n(k)+1}\right)\\
&\le \lim_{k\to\infty} \varphi\left(M^g\left(x_{m(k)}, x_{n(k)}, x_{n(k)}\right)\right) < \epsilon_0,
\end{aligned} \tag{57}$$

which is an absurd statement. Hence $\{gx_n\}$ is a Cauchy sequence. The rest follows from Theorem 13. □

Theorem 19. *Theorem 13 also holds if contractive condition (10) is valid for all $x \in A_0$ and $y \in A$; conditions (b) and (g) are replaced by the following:*

(b') *quadruple (A, B, T, g) has the weak P-property of the second kind;*

(g') *for a sequence $\{x_n\} \subseteq A_0$ converging to $x \in A$ and $\alpha(gx_n, gx_{n+1}) \ge 1$ for all $n \ge 0$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(gx_{n(k)}, gx) \ge 1$ for all $k \ge 0$.*

Proof. Following similar arguments to those given in proof of Theorem 13, we deduce that $\{gx_n\}$ and $\{x_n\}$ are Cauchy sequences in closed subset A of X. So we obtain an x in A such that $\{x_n\} \to x$ and $\{gx_n\} \to gx$. We show that x is a g-best proximity point of T.

Given that (A, B, T, g) has the weak P-property of the second kind, for all $n, m \in \mathbb{N}$,

$$\begin{gathered} x_m, x_{m+1}, x_n, x_{n+1} \in A_0, \\ G(gx_{m+1}, Tx_m, Tx_m) = \Delta_{AB}^G, \\ G(gx_{n+1}, Tx_n, Tx_n) = \Delta_{AB}^G \end{gathered} \tag{58}$$

imply that

$$G(gx_{m+1}, gx_{n+1}, gx_{n+1}) \leq G(Tx_m, Tx_n, Tx_n). \tag{59}$$

It follows that $\{Tx_n\}$ is also a Cauchy sequence in B. Hence, there is $z \in B$ such that $\{Tx_n\} \to z$. Thus

$$G(gx_n, gx, gx) \longrightarrow 0, \qquad G(Tx_n, z, z) \longrightarrow 0. \tag{60}$$

Since $G(gx_{n+1}, Tx_n, Tx_n) = \Delta_{AB}^G$ for all $n \geq 0$, we deduce that

$$G(gx, z, z) = \Delta_{AB}^G; \tag{61}$$

that is, $gx \in A_0$ and $z \in B_0$. Using condition (g'), we conclude that there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(gx_{n(k)}, gx) \geq 1 \quad \forall k \geq 0. \tag{62}$$

Note that

$$\begin{aligned} 0 &\leq G(gx_{n(k)}, Tx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}^G \\ &\leq G(gx_{n(k)}, gx_{n(k)+1}, gx_{n(k)+1}) \\ &\quad + G(gx_{n(k)+1}, Tx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}^G \\ &= G(gx_{n(k)}, gx_{n(k)+1}, gx_{n(k)+1}). \end{aligned} \tag{63}$$

Therefore

$$\lim_{k\to\infty} \left[G(gx_{n(k)}, Tx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}^G\right] = 0. \tag{64}$$

The first and the second arguments of

$$\begin{aligned} &M^g(x_{n(k)}, x, x) \\ &= \max\Bigg(G(gx_{n(k)}, gx, gx), G(gx_{n(k)}, Tx_{n(k)}, Tx_{n(k)}) \\ &\qquad - \Delta_{AB}^G, G(gx, Tx, Tx) - \Delta_{AB}^G, \\ &\qquad \frac{G(gx_{n(k)}, Tx, Tx) + G(gx, Tx_{n(k)}, Tx_{n(k)})}{2} - \Delta_{AB}^G\Bigg) \end{aligned} \tag{65}$$

tend to zero, while the last argument gives

$$\begin{aligned} &\lim_{k\to\infty} \frac{G(gx_{n(k)}, Tx, Tx) + G(gx, Tx_{n(k)}, Tx_{n(k)})}{2} - \Delta_{AB}^G \\ &\leq \lim_{k\to\infty} \big(G(gx_{n(k)}, Tx, Tx) + G(gx, gx_{n(k)+1}, gx_{n(k)+1}) \\ &\qquad + G(gx_{n(k)+1}, Tx_{n(k)}, Tx_{n(k)})\big) \times (2)^{-1} \\ &\quad - \Delta_{AB}^G \\ &= \lim_{k\to\infty} \big(G(gx_{n(k)}, Tx, Tx) + G(gx, gx_{n(k)+1}, gx_{n(k)+1}) \\ &\qquad + \Delta_{AB}\big) \times (2)^{-1} \\ &\quad - \Delta_{AB}^G \\ &= \frac{G(gx, Tx, Tx) + 0 + \Delta_{AB}}{2} - \Delta_{AB}^G \\ &= \frac{G(gx, Tx, Tx) + 0 - \Delta_{AB}^G}{2}. \end{aligned} \tag{66}$$

Therefore,

$$\lim_{k\to\infty} M^g(x_{n(k)}, x, x) = G(gx, Tx, Tx) - \Delta_{AB}^G. \tag{67}$$

Suppose that $G(gx, Tx, Tx) \neq \Delta_{AB}^G$; that is,

$$t_0 = G(gx, Tx, Tx) - \Delta_{AB}^G > 0. \tag{68}$$

Since the first and the second terms in (65) tend to zero, and the fourth term tends to $t_0/2$, there exists $k_0 \in \mathbb{N}$ such that

$$M^g(x_{n(k)}, x, x) = G(gx, Tx, Tx) - \Delta_{AB}^G = t_0 > 0 \quad \forall k \geq k_0. \tag{69}$$

Using the contractivity condition, we have

$$\begin{aligned} &G(Tx_{n(k)}, Tx, Tx) \\ &\leq \alpha(gx_{n(k)}, gx, gx)\, G(Tx_{n(k)}, Tx, Tx) \\ &\leq \varphi(M^g(x_{n(k)}, x, x)) \\ &\quad + \theta\big(G(gx, Tx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}^G, \\ &\qquad G(gx_{n(k)}, Tx, Tx) - \Delta_{AB}^G, \\ &\qquad G(gx_{n(k)}, Tx_{n(k)}, Tx_{n(k)}) - \Delta_{AB}^G\big), \end{aligned}$$

$$\begin{aligned}
&G(gx,Tx,Tx)-\Delta_{AB}^{G}\\
&\quad=\varphi\left(G(gx,Tx,Tx)-\Delta_{AB}^{G}\right)\\
&\qquad+\theta\Big(G(gx,Tx_{n(k)},Tx_{n(k)})-\Delta_{AB}^{G},\\
&\qquad\quad G\left(gx_{n(k)},Tx,Tx-\Delta_{AB}^{G},G(gx_{n(k)},Tx_{n(k)},Tx_{n(k)})\right.\\
&\qquad\quad\left.-\Delta_{AB}^{G},\ G(gx,Tx,Tx)-\Delta_{AB}^{G}\right)\Big).
\end{aligned}\tag{70}$$

Since the third argument of θ in (70) tends to zero and $\theta\in\Omega_2$, its limit as $k\to\infty$ is zero. Therefore, we have

$$\begin{aligned}
G(z,Tx,Tx)&=\lim_{k\to\infty}G(Tx_{n(k)},Tx,Tx)\\
&\le\varphi\left(G(gx,Tx,Tx)-\Delta_{AB}^{G}\right).
\end{aligned}\tag{71}$$

As $G(gx,Tx,Tx)-\Delta_{AB}^{G}>0$, then $\varphi(G(gx,Tx,Tx)-\Delta_{AB})<G(gx,Tx,Tx)-\Delta_{AB}^{G}$. Thus,

$$\begin{aligned}
&G(z,Tx,Tx)\\
&\quad\le\varphi\left(G(gx,Tx,Tx)-\Delta_{AB}^{G}\right)<G(gx,Tx,Tx)-\Delta_{AB}\\
&\quad\le G(gx,z,z)+G(z,Tx,Tx)-\Delta_{AB}^{G}\\
&\quad\le\Delta_{AB}+G(gx,z,z)-\Delta_{AB}^{G}\\
&\quad=G(z,Tx,Tx),
\end{aligned}\tag{72}$$

which is a contradiction. Hence $G(gx,Tx,Tx)=\Delta_{AB}$ and the result follows. □

2.1. Uniqueness of g-Best Proximity Points. In this section, we study sufficient conditions in order to prove the uniqueness of g-best proximity point.

Definition 20. Let $T:A\to B$, $g:A\to A$, and $\alpha:X\times X\to[0,\infty)$ be three mappings. A mapping T is called (α,g)-regular if, for all $x,y\in A_0$, such that $\alpha(gx,gy)<1$, there exists $z\in A_0$ such that $\alpha(gx,gz)\ge1$ and $\alpha(gy,gz)\ge1$.

Theorem 21. *Under the hypothesis of Theorem 13, assume that $\theta\in\Theta$ and T is (α,g)-regular. Then for all g-best proximity points x and y of T in A_0 we have that $gx=gy$. In particular, if g is injective on the set of all g-best proximity points of T in A_0, then T has a unique g-best proximity point.*

Proof. Let $x,y\in A_0$ be two g-best proximity points of T in A_0. Since $G(gx,Tx,Tx)=G(gy,Ty,Ty)=\Delta_{AB}^{G}$ and T is a (α,g)-proximal and admissible, we deduce that

$$G(gx,gy,gy)\le G(Tx,Ty,Ty).\tag{73}$$

We always have $\alpha(gx,gy)\ge1$ or $\alpha(gx,gy)<1$. If $\alpha(gx,gy)\ge1$, then we obtain that

$$\begin{aligned}
&G(gx,gy,gy)\\
&\quad\le G(Tx,Ty,Ty)\le\alpha(gx,gy)G(Tx,Ty,Ty)\\
&\quad\le\varphi\left(M^{g}(x,y,y)\right)\\
&\qquad+\theta\Big(G(gy,Tx,Tx)-\Delta_{AB}^{G},\\
&\qquad\quad G(gx,Ty,Ty)-\Delta_{AB}^{G},G(gx,Tx,Tx)-\Delta_{AB}^{G},\\
&\qquad\quad G(gy,Ty,Ty)\Big)\\
&\quad=\varphi\left(M^{g}(x,y,y)\right).
\end{aligned}\tag{74}$$

The last equality holds since $\theta\in\Theta$ and the last two arguments of θ are zero. Note that

$$\begin{aligned}
&\frac{G(gx,Ty,Ty)+G(gy,Tx,Tx)}{2}-\Delta_{AB}^{G}\\
&\quad\le\Big(\big(G(gx,gy,gy)+G(gy,Ty,Ty)\\
&\qquad+G(gy,gx,gx)+G(gx,Tx,Tx)\big)\times(2)^{-1}\Big)-\Delta_{AB}^{G}\\
&\quad=\frac{G(gx,gy,gy)+\Delta_{AB}^{G}+G(gy,gx,gx)+\Delta_{AB}^{G}}{2}-\Delta_{AB}^{G}\\
&\quad=\frac{G(gx,gy,gy)+G(gy,gx,gx)}{2}\\
&\quad=G(gx,gy,gy).
\end{aligned}\tag{75}$$

Hence

$$\begin{aligned}
&M^{g}(x,y,y)\\
&\quad=\max\Big(G(gx,gy,gy),G(gx,Tx,Tx)-\Delta_{AB}^{G},\\
&\qquad G(gy,Ty,Ty)-\Delta_{AB}^{G},\\
&\qquad\frac{G(gx,Ty,Ty)+G(gy,Tx,Tx)}{2}-\Delta_{AB}^{G}\Big).
\end{aligned}\tag{76}$$

Therefore

$$G(gx,gy,gy)\le\varphi\left(M^{g}(x,y,y)\right)=\varphi\left(G(gx,gy,gy)\right)\tag{77}$$

gives the fact that $G(gx,gy,gy)=0$; that is, $gx=gy$.

Now, if $\alpha(gx,gy)<1$, then, by the (α,g)-regularity of T, there exists $z_0\in A_0$ such that $\alpha(gx,gz_0)\ge1$ and $\alpha(gy,gz_0)\ge1$. Based on z_0, we define a sequence $\{z_n\}$ such that $\{gz_n\}$ converges to gx and gy which proves the uniqueness. First, we will prove that $\{gz_n\}$ converges to gx.

Indeed, $Tz_0\in TA_0\subseteq B_0$ implies that $s_0\in A_0$ such that $G(s_0,Tz_0,Tz_0)=\Delta_{AB}^{G}$, and, for $s_0\in A_0\subseteq gA_0$, there is

$z_1 \in A_0$ verifying $gz_1 = s_0$. Therefore, $G(gz_1, Tz_0, Tz_0) = \Delta_{AB}^G$. Following the similar arguments, there exists a sequence $\{z_n\} \subseteq A_0$ such that $G(gz_{n+1}, Tz_n, Tz_n) = \Delta_{AB}^G$ for all $n \geq 0$. In particular, $gz_{n+1} \in A_0$ and $Tz_n \in B_0$. We claim that

$$\alpha(gx, gz_n) \geq 1 \quad \forall n \geq 0. \tag{78}$$

If $n = 0$, $\alpha(gx, gz_0) \geq 1$ by the choice of z_0. Suppose that $\alpha(gx, gz_n) \geq 1$ for some $n \geq 0$. As T is $(\alpha; g)$-proximal and admissible, so we have

$$\begin{aligned} &\alpha(gx, gz_n) \geq 1, \\ &G(gx, Tx, Tx) = \Delta_{AB}^G, \\ &G(gz_{n+1}, Tz_n, Tz_n) = \Delta_{AB}^G \\ &\qquad x, z_n, z_{n+1} \in A_0, \end{aligned} \tag{79}$$

which imply that $\alpha(gx, gz_{n+1}) \geq 1$. Hence (78) holds. For all $n \geq 0$, we have

$$\begin{aligned} &\frac{G(gx, Tz_n, Tz_n) + G(gz_n, Tx, Tx)}{2} - \Delta_{AB}^G \\ &\quad \leq \Big((G(gx, gz_{n+1}, gz_{n+1}) + G(gz_{n+1}, Tz_n, Tz_n) \\ &\qquad + G(gz_n, gx, gx) + G(gx, Tx, Tx)) \times (2)^{-1}\Big) - \Delta_{AB}^G \\ &\quad = \frac{G(gx, gz_{n+1}, gz_{n+1}) + \Delta_{AB} + G(gz_n, gx, gx) + \Delta_{AB}}{2} \\ &\qquad - \Delta_{AB}^G \\ &\quad = \frac{G(gx, gz_{n+1}, gz_{n+1}) + G(gz_n, gx, gx)}{2} \\ &\quad \leq \max(G(gx, gz_n, gz_n), G(gx, gz_{n+1}, gz_{n+1})), \end{aligned} \tag{80}$$

which implies that

$$\begin{aligned} &M^g(x, z_n, z_n) \\ &\quad = \max\Big(G(gx, gz_n, gz_n), G(gx, Tx, Tx) - \Delta_{AB}^G, \\ &\qquad G(gz_n, Tz_n, Tz_n) - \Delta_{AB}^G, \\ &\qquad \frac{G(gx, Tz_n, Tz_n) + G(gz_n, Tx, Tx)}{2} - \Delta_{AB}^G\Big) \\ &\quad \leq \max(G(gx, gz_n, gz_n), G(gx, gz_{n+1}, gz_{n+1})). \end{aligned} \tag{81}$$

By weak P-property of the first kind,

$$\begin{aligned} &G(gx, Tx, Tx) = \Delta_{AB}, \\ &G(gz_{n+1}, Tz_n, Tz_n) = \Delta_{AB}^G \\ &\qquad x, z_n, z_{n+1} \in A_0, \end{aligned} \tag{82}$$

imply that $G(gx, gz_{n+1}, gz_{n+1}) \leq G(Tx, Tz_n, Tz_n)$.

For all $n \geq 0$, we have

$$\begin{aligned} &G(gx, gz_{n+1}, gz_{n+1}) \\ &\quad \leq G(Tx, Tz_n, Tz_n) \\ &\quad \leq \varphi(M^g(x, z_n, z_n)) \\ &\qquad + \theta\big(G(gz_n, Tx, Tx) - \Delta_{AB}^G, G(gx, Tz_n, Tz_n) - \Delta_{AB}, \\ &\qquad\quad G(gx, Tx, Tx) - \Delta_{AB}^G, G(gz_n, Tz_n, Tz_n)\big) \\ &\quad \leq \varphi(M^g(x, z_n, z_n)) \\ &\quad \leq \varphi(\max(G(gx, Tz_n, Tz_n), G(gx, gz_{n+1}, gz_{n+1}))). \end{aligned} \tag{83}$$

Suppose that there is $n_0 \in \mathbb{N}$ such that $gz_{n_0} = gx$. In this case, we have

$$\begin{aligned} &G(gx, gz_{n+1}, gz_{n_0+1}) \\ &\quad \leq \varphi\big(\max\big(G(gx, Tz_{n_0}, Tz_{n_0}), G(gx, gz_{n_0+1}, gz_{n_0+1})\big)\big), \end{aligned} \tag{84}$$

but this is possible only when $G(gx, gz_{n_0+1}, gz_{n_0+1})$; that is, $gz_{n_0+1} = gx$. Following the similar arguments, we have $gz_n = gx$ for all $n \geq n_0$. Hence $\{gz_n\}$ converges to gx.

Suppose that $gz_n \neq gx$ for all $n \geq 0$; that is, $G(gx, gz_n, gz_n) > 0$ for all $n \geq 0$. Suppose that

$$\begin{aligned} &\max(G(gx, gz_n, gz_n), G(gx, gz_{n+1}, gz_{n+1})) \\ &\quad = G(gx, gz_{n+1}, gz_{n+1}) \end{aligned} \tag{85}$$

for some n. Then (83) would yield

$$\begin{aligned} &G(gx, gz_{n+1}, gz_{n+1}) \\ &\quad \leq \varphi(\max(G(gx, gz_n, gz_n), G(gx, gz_{n+1}, gz_{n+1}))) \\ &\quad = \varphi(G(gx, gz_{n+1}, gz_{n+1})) \\ &\quad < G(gx, gz_{n+1}, gz_{n+1}), \end{aligned} \tag{86}$$

which is a contradiction. Therefore, $\max(G(gx, gz_n, gz_n), G(gx, gz_{n+1}, gz_{n+1})) = G(gx, gz_n, gz_n)$; that is, for all $n \geq 0$,

$$\begin{aligned} G(gx, gz_{n+1}, gz_{n+1}) &\leq \varphi(M^g(x, z_n, z_n)) \\ &= \varphi\big(G\big(gx, gz_{n,gz_n}\big)\big). \end{aligned} \tag{87}$$

Recursively, for all $n \geq 0$,

$$\begin{aligned} &G(gx, gz_n, gz_n) \\ &\quad \leq \varphi(G(gx, gz_{n-1}, gz_{n-1})) \\ &\quad \leq \varphi^2(G(gx, gz_{n-2}, gz_{n-2})) \leq \cdots \leq \varphi^n(G(gx, gz_0, gz_0)). \end{aligned} \tag{88}$$

Fix $\epsilon > 0$ arbitrary and consider $t_0 = G(gx, gz_0, gz_0) > 0$. Since $\varphi \in \Psi$, the series $\sum_{n\geq 1} \varphi^n(t_0)$ converges. In particular, there exists $m_0 \in \mathbb{N}$ such that $\sum_{k=m_0^\infty} \varphi^n(t_0) < \epsilon$. More precisely, $\varphi^n(t_0) < \epsilon$ for all $n \geq m_0$. Therefore, if $n \geq m_0$, we have that

$$G(gx, gz_n, gz_n) \leq \varphi^n(G(gx, gz_0, gz_0)) = \varphi^n(t_0) < \epsilon. \tag{89}$$

This means that $\{gx_n\}$ converges to gx. Similarly, it can be shown that $\{gx_n\}$ converges to gy and this completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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