1. Introduction—Preliminaries

In what follows, $X$ is a real reflexive separable locally uniformly convex Banach space with locally uniformly convex dual space $X^*$. The norm of the space $X$, and any other normed spaces herein, will be denoted by $\| \cdot \|$. For $x \in X$ and $x^* \in X^*$, the pairing $\langle x^*, x \rangle$ denotes the value $x^*(x)$. Let $X$ and $Y$ be Banach spaces. For a multivalued mapping $T : X \to Y$, we define the domain $D(T)$ of $T$ by $D(T) = \{x \in X : Tx \neq \emptyset\}$ and the range $R(T)$ of $T$ by $R(T) = \cup_{x \in D(T)} Tx$. We also denote the graph of $T$ by $G(T) = \{(x, Tx) : x \in D(T)\}$.

A mapping $T : X \supset D(T) \to Y$ is "demicontinuous" if it is continuous from the strong topology of $D(T)$ to the weak topology of $Y$. A multivalued mapping $T : X \supset D(T) \to Y$ is "bounded" if it maps bounded subsets of $D(T)$ to bounded subsets of $Y$. It is "compact" if it is strongly continuous and maps bounded subsets of $D(T)$ to relatively compact subset of $Y$. It is "finitely continuous" if it is upper semicontinuous from each finite dimensional subspace $F$ of $X$ to the weak topology of $Y$. It is "quasibounded" if for every $M > 0$ there exists $K(M) > 0$ such that $\|x, w^*\| \leq M$ and $\langle w^*, x \rangle \leq M \|x\|$ imply $\|w^*\| \leq K(M)$. It is "strongly quasibounded" if for every $M > 0$ there exists $K(M) > 0$ such that $\|x, w^*\| \leq G(T)$ with $\|x\| \leq M$ and $\langle w^*, x \rangle \leq M$ imply $\|w^*\| \leq K(M)$. In what follows, a mapping will be called "continuous" if it is strongly continuous.

Let $\psi : [0, \infty) \to [0, \infty)$ be continuous strictly increasing function such that $\psi(0) = 0$ and $\psi(t) \to \infty$ as $t \to \infty$. The duality mapping corresponding to $\psi$ denoted by $I_\psi : X \to 2^{X^*}$ is defined by

$$I_\psi(x) = \{x^* \in X^* : \langle x^*, x \rangle = \|x^*\| \|x\|, \|x^*\| = \psi(\|x\|)\}.$$  

(1)

It is well-known that, for each $x \in X$, the Hahn-Banach Theorem implies $I_\psi(x) \neq \emptyset$. Since $X$ and $X^*$ are locally...
uniformly convex, \( J_\psi \) is single-valued, bounded monotone of type \((S_+)\) and bicontinuous. If \( \psi(t) = t \) for \( t \geq 0 \), then \( J_\psi \) is denoted by \( J \) and is called the normalized duality mapping.

An operator \( T : X \ni D(T) \to 2^{X^*} \) is said to be “monotone” if, for every \( x \in D(T), y \in D(T), \) and every \( u^* \in Tx, v^* \in Ty, \) we have \( \langle u^* - v^*, x - y \rangle \geq 0 \). A monotone mapping \( T : X \ni D(T) \to 2^{X^*} \) is “maximal monotone” if \( B(T + M) = X^* \) for every \( \lambda > 0 \); that is, \( T \) is maximal monotone if and only if \( T \) is monotone and \( \langle u^* - u_0^*, x - x_0 \rangle \geq 0 \) for every \( (x,u^*) \in G(T) \) implies \( x_0 \in D(T) \) and \( u_0^* \in Tx_0 \). If \( T \) is maximal monotone, the operator \( T_\lambda : X \ni T^\lambda x \) is continuous, continuous, maximal monotone and such that \( T_\lambda x \to T^\lambda x \) as \( \lambda \to 0 \). For all \( x \in X \), \( J_\lambda x = \inf \{ \| y^* \| : \langle y^* \rangle = x \} \). The “resolvent” \( J_\lambda : X \ni D(T) \) is maximal monotone, the operator \( J_\lambda : X \ni J_\lambda x \) is continuous, continuous, maximal monotone and such that \( J_\lambda x \to J_\lambda x \) as \( \lambda \to 0 \).

The following definitions are used throughout the paper. In arbitrary Banach space \( X \), Browder and Hess [1] introduced the definitions of pseudomonotone and generalized pseudomonotone operators. The original definition for single-valued pseudomonotone, generalized pseudomonotone, and operators of type \((M)\) with domain all of \( X \), is due to Brézis [2].

**Definition 1.** An operator \( S : X \ni D(S) \to X^* \) is called

(i) “generalized pseudomonotone” if, for each sequence \( \{x_n\} \ni D(S) \) with \( x_n \to x_0 \) and \( Sx_n \to v_0^* \) as \( n \to \infty \) such that \( \lim sup_{n \to \infty} \langle Sx_n, x_n - x_0 \rangle \leq 0 \), then \( x_0 \in D(S) \), \( Sx_0 = v_0^* \), and \( \langle Sx_n, x_n \rangle \to \langle Sx_0, x_0 \rangle \) as \( n \to \infty \).

(ii) “type \((M)\)” if, for each sequence \( \{x_n\} \ni D(S) \) with \( x_n \to x_0 \) and \( Sx_n \to v_0^* \) as \( n \to \infty \) such that \( \lim sup_{n \to \infty} \langle Sx_n, x_n - x_0 \rangle \leq 0 \), then \( x_0 \in D(S) \) and \( Sx_0 = v_0^* \).

(iii) “\(a\)-expansive” if there exists \( \alpha > 0 \) such that \( \| v^* - u^* \| \geq \alpha \| x - y \| \) for all \( x \in D(S), y \in D(S), v^* \in Sx, \) and \( u^* \in Sy \). It is called expansive if \( \alpha = 1 \).

We notice here that the direction of single-valued expansive mapping is due to Nirenberg [3]. In order to enlarge the class of single-valued operators, the multivalued version is introduced in (iii) of Definition 1. It is not hard to notice that every uniformly monotone operator is expansive. Furthermore, in a Hilbert space \( X = H \), if \( T : H \ni D(T) \to 2^H \) is monotone, we see that, for each \( \lambda > 0, T + \lambda I \) is multivalued expansive with domain \( D(T) \).

The following definition gives a larger class of operators of monotone type, which can be found in Kartsatos and Skrypnik [4].

**Definition 2.** Let \( T : X \ni D(T) \to 2^{X^*} \) be maximal monotone and \( A : X \ni D(A) \to X^* \). Let \( L \ni D(M) \cap D(A) \) be a linear subspace of \( X \). Then \( A \) is said to be

(i) “quasibounded with respect to \( T\)” if, for each \( M > 0 \), there exists \( K(M) > 0 \) such that

\[
\langle Au + u^*, u \rangle \leq M,
\]

\[
\| u \| \leq M,
\]

where \( u \in L \) and \( u^* \in Tu \), then \( \| Au \| \leq K(M) \).

(ii) “generalized \((S_+)\) with respect to \( T\)” if, for each \( \{u_n\} \ni L \) with \( u_n^* \in Tu_n, u_n \to u_0 \) in \( X \) and \( Au_n \to h_0^* \) in \( X^* \) as \( n \to \infty \) such that

\[
\lim sup_{n \to \infty} \langle Au_n, u_n - u_0 \rangle \leq 0,
\]

\[
\langle u_n^* + Au_n, u_n \rangle \leq 0
\]

for all \( n \), then \( u_n \to u_0 \in D(A) \) and \( Au_0 = h_0^* \).

(iii) “generalized pseudomonotone with respect to \( T\)” if, for each \( \{u_n\} \ni L \) with \( u_n^* \in Tu_n, u_n \to u_0 \) in \( X \) and \( Au_n \to h_0^* \) in \( X^* \) as \( n \to \infty \) such that

\[
\lim sup_{n \to \infty} \langle Au_n, u_n - u_0 \rangle \leq 0,
\]

\[
\langle u_n^* + Au_n, u_n \rangle \leq 0
\]

for all \( n \), then \( u_n \in D(A) \), \( Au_0 = h_0^* \), and \( \langle Su_n, u_n \rangle \to \langle Su_0, u_0 \rangle \) as \( n \to \infty \).

(iv) “of \((M)\) with respect to \( T\)” if, for each \( \{u_n\} \ni L \) with \( u_n^* \in Tu_n, u_n \to u_0 \) in \( X \) and \( Au_n \to h_0^* \) in \( X^* \) as \( n \to \infty \) such that

\[
\lim sup_{n \to \infty} \langle Au_n, u_n - u_0 \rangle \leq 0,
\]

\[
\langle u_n^* + Au_n, u_n \rangle \leq 0
\]

for all \( n \), then \( u_n \in D(A) \) and \( Au_0 = h_0^* \).

By Definition 2, it is not difficult to see that \( 0 \in D(T) \) and \( A \) is quasibounded implying that \( A \) is quasibounded with respect to \( T \). Furthermore, it follows that the class of generalized \((S_+)\) operators with respect to \( T \) includes the class of operators of type \((S_+)\).

For basic definitions and further properties of mappings of monotone type, the reader is referred to Barbu [5], Brézis et al. [6], Brézis [2], Browder and Hess [1], Pascali and Sburlan [7], Browder [8], and Zeidler [9]. For results concerning perturbations of maximal monotone operators by bounded and everywhere defined pseudomonotone type operators, the reader is referred to Browder and Hess [1], Brézis [2], Browder [10], Brézis and Nirenberg [11], Kenmochi [12–14], Guan et al. [15], Le [16], Guan and Kartsatos [15, 17],
and Kartsatos and Skrypnik [4] and the references therein. For recent degree theory and applications for solvability of operator inclusions involving bounded pseudomonotone perturbations of maximal monotone operators under general coercivity and Leray-Schauder type boundary conditions, we cite the paper due to Asfaw and Kartsatos [18]. Existence results concerning noncoercive operators of the type \( T + S \), where \( T : X \ni D(T) \to 2^X \) is maximal monotone and \( S : X \to 2^X \) is bounded pseudomonotone, can be found in the paper due to Asfaw [19]. For applications of the theory of perturbed monotone type operators to variational and hemivariational inequality problems, the reader is referred to the papers due to Carl and Le [20], Carl et al. [21], Carl [22], and Carl and Motreanu [23] and the references therein.

For a separable reflexive Banach space \( X \) and a nonempty, closed, and convex subset \( K \) of \( X \), Asfaw and Kartsatos [24] gave existence results for locally defined operators of the type \( T + S \), where \( T : X \ni D(T) \to 2^X \) is maximal monotone and \( S : K \to X^* \) is demicontinuous and generalized pseudomonotone under coercivity condition on \( S \).

The main contribution of the paper is to obtain surjectivity results for noncoercive and not everywhere defined operators of the type

\[
(i) \quad T + S, \quad \text{where} \quad S : X \ni D(S) \to X^* \quad \text{is quasibounded, demicontinuous, and generalized pseudomonotone such that}
\]

\[
(a) \quad \text{there exists a real reflexive separable Banach space } W \ni D(S), \text{dense and continuously embedded in } X;
\]
\[
(b) \quad \text{there exists } d \geq 0 \text{ such that } \langle v^* + Sx, x \rangle \geq -d\|x\|^2 \text{ for all } x \in D(T) \cap D(S) \text{ and } v^* \in TX;
\]
\[
(c) \quad \text{there exist } \alpha > d \text{ and } \mu \geq 0 \text{ such that } \|v^* + Sx\| \geq \alpha\|x\| - \mu \text{ for all } x \in D(T) \cap D(S) \text{ and } v^* \in TX,
\]

\[
(ii) \quad L + S, \quad \text{where} \quad S : X \ni D(S) \to X^* \quad \text{is quasibounded, demicontinuous of type } (M) \text{ with } D(L) \subseteq D(S) \text{ such that} (b) \text{ and } (c) \text{ of } (i) \text{ are satisfied.}
\]

In Section 2, we proved surjectivity results for \( T + S \) and \( L + S \) satisfying conditions (i) and (ii), respectively. In Theorem 6, we provide a surjectivity result for operators of the type \( T + S \), where \( T \) and \( S \) satisfy condition (i). Theorem 6 is new and improves the existing surjectivity results for an operator \( S \), which is single-valued, everywhere defined, bounded, and coercive pseudomonotone. In particular, for a single-valued pseudomonotone operator \( S \), Theorem 6 improves the surjectivity results due to Browder and Hess [1], Kenmochi [12–14], Le [16], Guan and Kartsatos [17], Asfaw and Kartsatos [18], and Asfaw [19, 25] because the results in these references require \( S \) to be everywhere defined, bounded, and coercive while Theorem 6 used \( S \) to be densely defined, quasibounded, and noncoercive. Moreover, Browder (cf. Zeidler [9, Theorem 32. A, pages 866–872]) gave the main theorem for perturbations of maximal monotone operator by a single-valued, bounded, demicontinuous, and coercive operator \( S \) with \( D(S) = C \), a nonempty, closed, and convex subset of \( X \). In view of this, Theorem 6 gives an analogous result, where \( D(S) \) is dense in \( X \), possibly, neither closed nor convex, and \( S \) is weakly coercive. It is also known, due to Browder and Hess [1], that every pseudomonotone operator \( S \) from \( X \) into \( X^* \) with \( D(S) = X \) is generalized pseudomonotone. It is also true that \( S \) is demicontinuous provided that it is bounded, single-valued, and everywhere defined. Consequently, the arguments used in the proof of Theorem 6 give analogous conclusion if \( S : X = D(S) \to X^* \) is bounded pseudomonotone and \( T \) and \( S \) satisfy the given hypotheses. As a consequence of Corollary 7, a partial positive answer for Nirenberg’s problem on surjectivity of densely defined demicontinuous generalized pseudomonotone expansive mapping is provided. In addition, Theorem 8 provides surjectivity result for operators of the type \( T + S \), where \( T \) and \( S \) satisfy condition (ii). As a result of Theorem 8, a new characterization of linear maximal monotone operator is proved when the space \( X \) is separable. It is well known due to Brézis (cf. Zeidler [9, Theorem 32. L, pages 897–899]) that a linear monotone operator \( L \) is maximal monotone if and only if \( L \) is closed and densely defined and the adjoint operator \( L^* \) is monotone. An interesting result in the present paper is that a linear monotone operator \( L \) is maximal monotone if and only if \( L \) is closed and densely defined, provided that \( X \) is separable. This result weakens the monotonicity condition on \( L^* \) used by Brézis (cf. Zeidler [9, Theorem 32. L, pages 897–899]). To the best of the author’s knowledge, Theorem 8 is a new result and Corollary 9 improves the well-known result of Brézis. In Section 3, we demonstrate the applicability of the results by proving existence of weak solution in \( L^p(0,T; W_0^{1,p}(\Omega)) \) of a nonlinear parabolic problem, where \( p > 1 \) and \( \Omega \) is a nonempty, bounded, and open subset of \( \mathbb{R}^N \).

The following important lemma is due to Brézis et al. [6].

**Lemma 3.** Let \( B \) be a maximal monotone set in \( X \times X^* \). If \( (u_n, u_n^*) \in B \) such that \( u_n \rightharpoonup u, u_n^* \rightharpoonup u^* \) as \( n \to \infty \), and either

\[
\limsup_{n,m \to \infty} \langle u_n^* - u_m^*, u_n - u_m \rangle \leq 0
\]

or

\[
\limsup_{n \to \infty} \langle u_n^* - u^*, u_n - u \rangle \leq 0,
\]

then \( (u, u^*) \in B \) and \( \langle u_n^*, u_n \rangle \to \langle u^*, u \rangle \) as \( n \to \infty \).
(i) $F$ has a fixed point in $U$

or

(ii) there exist $x \in \partial_\circ U$ and $\lambda \in (0, 1)$ such that $x = \lambda F(x)$, where $\partial_\circ U$ is the boundary of $U$ with respect to the subspace topology on $C$.

2. Main Results

In this section, we prove the following new surjectivity result for maximal monotone perturbation of densely defined noncoercive generalized pseudomonotone operator in separable reflexive Banach spaces.

**Theorem 6.** Let $T : X \ni D(T) \to 2^{X^*}$ be maximal monotone with $0 \in T(0)$ and $S : X \ni D(S) \to X^*$ quasibounded demicontinuous generalized pseudomonotone. Suppose $W \subseteq D(S)$ is a real reflexive separable Banach space dense and continuously embedded in $X$. Assume, further, that there exist $\mu \geq 0, d \geq 0$, and $\alpha > d$ satisfying

$$
\langle j^*_Q(T\lambda + S + \delta^1_J)Qx_n - j^*_Q(T\lambda + S + \delta^1_J)Qx_0, w \rangle \to 0 \text{ as } n \to \infty.
$$

(18)

for all $x \in D(S)$ and either

(i) $\|j^* + Sx\| \geq \alpha \|x\| - \mu \quad \forall x \in D(T) \cap D(S), \quad v^* \in Tx$

or

(ii) there exists $\phi : [0, \infty) \to (-\infty, \infty)$ such that $\phi(t) \to \infty$ as $t \to \infty$ and

$$
\|j^* + Sx\| \geq \phi(\|x\|)\|x\| \quad \forall x \in D(T) \cap D(S), \quad v^* \in Tx.
$$

(10)

Then $T + S$ is surjective.

**Proof.** Let $\lambda > 0$ be fixed temporarily and $T_\lambda$ the Yosida approximant of $T$. For each $\epsilon > 0$, by using the inner product condition on $S$ and monotonicity of $T_\lambda$ ($T_\lambda(0) = 0$ for all $\lambda > 0$), we see that

$$
\langle T_\lambda x + Sx + \delta \|x\| jx - f^*, x \rangle \\
\geq \delta \|x\|^3 - d \|x\|^2 - \|f^*\|\|x\| \\
= \|x\|^3 \left[ \delta - \frac{d}{\|x\|^2} - \frac{\|f^*\|}{\|x\|^2} \right] > 0
$$

(11)

for all $x \in D(S) \cap \partial B_\mathcal{H}(0)$ for some $R_\epsilon > 0$. Let $G_\delta = B_\mathcal{H}(0)$. Let $H$ be a real separable Hilbert space and $Q : H \to W$ a compact injection such that $Q(H)$ is dense in $W$ guaranteed by Lemma 4. Let $j : W \to X$ be the natural injection and let $Q^* : W^* \to H^*$ and $j^* : X^* \to W^*$ be adjoint of $Q$ and $j$, respectively. It follows that $\psi = jQ : H \to X$ is a compact operator. Let $U = Q^{-1}(G_\delta \cap W)$. First we show that $G_\delta \cap W$ is open in $W$; that is, $W \setminus (G_\delta \cap W) = W \cap (X \setminus G_\delta)$ is closed in $W$. To this end, let $\{x_n\}$ be a sequence in $W \cap (X \setminus G_\delta)$ such that $x_n \to x_0$ in $W$ as $n \to \infty$. Since $W$ is continuously embedded in $X$, we get $x_n \to x_0$ in $X$ as $n \to \infty$. Since $X \setminus G_\delta$ is closed in $X$, it follows that $x_0 \in X \setminus G_\delta$; that is, $x_0 \in W \cap (X \setminus G_\delta)$. This shows that $W \cap (X \setminus G_\delta)$ is closed in $W$; that is, $G_\delta \cap W$ is open in $W$. The continuity of $Q$ implies that $U$ is open in $H$. Since $W$ is continuously embedded in $X$, it follows that

$$
\overline{G_\delta \cap W}^W \subseteq \overline{G_\delta \cap W}^X \subseteq \overline{G_\delta}.
$$

(12)

where the closures are taken with respect to the spaces $W$ and $X$, respectively. Since $\overline{G_\delta \cap W}^W \subseteq W$, we obtain that

$$
(\overline{G_\delta \cap W} \cup \partial_W(G_\delta \cap W)) = \overline{G_\delta \cap W}^W \subseteq \overline{G_\delta \cap W} \subseteq (\overline{G_\delta \cap W} \cup (\partial G_\delta \cap W)).
$$

(13)

Since the sets $G_\delta \cap W$ and $\partial_W(G_\delta \cap W)$ are disjoint, we conclude that

$$
\partial_W(G_\delta \cap W) \subseteq \partial G_\delta \cap W.
$$

(14)

For each $\lambda > 0$, let $T_\lambda$ be the Yosida approximant of $T$. Let $j^*_Q(T_\lambda + S + \delta^1_J)Qx_n \to j^* Qx_0$ as $n \to \infty$. Since $Q$ is continuous from $H$ into $X$, we have $Qx_n \to Qx_0$ as $n \to \infty$. Since $x_n \in \overline{U}$, the sequence $\{Qx_n\}$ lies in $W$. Since $x_n \in H$ for all $n$ and $x_0 \in \overline{U}$, it follows that $Qx_n \in W$ and $Qx_0 \in W$ for all $n$. Since $S$ and $T_\lambda$ are demicontinuous, it follows that $(T_\lambda + S)Qx_n \to (T_\lambda + S)Qx_0$ as $n \to \infty$. By the density of $W$ in $X$, it is known that $j^*$ is defined from $W^*$ into $H$. As a result, for each $w \in W$, we see that

$$
\langle j^*_Q(T_\lambda + S + \delta^1_J)Qx_n - j^*_Q(T_\lambda + S + \delta^1_J)Qx_0, w \rangle \to 0 \text{ as } n \to \infty.
$$

(17)

for all $n$. However, the right side expression goes to 0 as $n \to \infty$; that is, for each $w \in W$, it follows that

$$
\langle j^* (T_\lambda + S + \delta^1_J)Qx_n - j^* (T_\lambda + S + \delta^1_J)Qx_0, w \rangle \to 0 \text{ as } n \to \infty.
$$

(18)
On the other hand, by the density of $W$ in $X$, for each $x \in X$, we get
\begin{equation}
\langle j^* (T_\lambda + S + \delta J_1) Q x_n - j^* (T_\lambda + S + \delta J_1) Q x_0, x \rangle 
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty;
\end{equation}
that is, $j^* (T_\lambda + S + \delta J_1) Q x_n \rightarrow j^* (T_\lambda + S + \delta J_1) Q x_0$ as $n \rightarrow \infty$. Since $Q^*$ is compact linear, which is completely continuous and $(jQ)^* = Q^* j^*$, we arrive at $\psi^* (T_\lambda + S + \delta J_1) Q x_n \rightarrow \psi^* (T_\lambda + S + \delta J_1) Q x_0$ as $n \rightarrow \infty$. This shows that the mapping $C_{\epsilon, \lambda}^\delta$ is continuous. Following similar argument as above, it is not difficult to show that $C_{\epsilon, \lambda}^\delta$ maps any bounded subset of $U$ into relatively compact subset of $H$. As a result, we conclude that $C_{\epsilon, \lambda}^\delta$ is a compact operator. Fix $\epsilon > 0$. In order to use Lemma 5, it is enough to show that (i) of Lemma 5 does not hold; that is, for all $\mu \in (0, 1)$ and $x \in \partial I_\mu U$, we have $x \neq \mu C_{\epsilon, \lambda}^\delta (x)$. Suppose this is false; that is, there exist $x_0 \in \partial I_\mu U$ and $\mu_0 \in (0, 1)$ such that $x_0 = \mu_0 C_{\epsilon, \lambda}^\delta (x_0)$. This yields
\begin{equation}
\epsilon x_0 + \mu_0 \psi^* (T_\lambda + S + \delta J_1) \psi x_0 - f^* = 0.
\end{equation}
We notice here that the continuity of $Q$, property of $Q^{-1}$, and definition of boundary of an open set imply that
\begin{equation}
\partial I_\mu U = \partial H Q^{-1} (G_\delta \cap W) \subseteq Q^{-1} (\partial W (G_\delta \cap W))
\end{equation}
holds. Since $x_0 \in \partial I_\mu U$, it follows that $Q x_0 \in \partial G \cap W$. By (11) and (20), we get
\begin{equation}
\frac{\epsilon}{\mu_0} \| x_0 \|^2 = - \langle \psi^* (T_\lambda + S + \delta J_1) Q x_0 - f^*, x_0 \rangle
\end{equation}
\begin{equation}
= - \langle \psi^* (T_\lambda + S + \delta J_1) Q x_0 - f^*, Q x_0 \rangle \leq 0,
\end{equation}
which implies $x_0 = 0$. But this is impossible because $0 \in G_\delta \cap W$. Therefore, by applying Lemma 5, for each $\epsilon > 0$, $\lambda > 0$, and $\delta > 0$, we conclude that the compact operator $C_{\epsilon, \lambda}^\delta$ has a fixed point $x_\epsilon \in U$; that is,
\begin{equation}
\epsilon x_\epsilon + \psi^* (T_\lambda + S + \delta J_1) \psi x_\epsilon - f^* = 0.
\end{equation}
Therefore, for each $\epsilon_n \downarrow 0^+$, there exists $x_n \in U$ such that
\begin{equation}
\epsilon_n x_n + \psi^* (T_\lambda + S + \delta J_1) \psi x_n - f^* = 0
\end{equation}
for all $n$. Since $G_\delta$ is bounded, the sequence $\{\psi x_n\}$ is bounded. Since $T_\lambda$ and $J_1$ are bounded, it follows that the sequence $\{(T_\lambda + \delta J_1)\psi x_n\}$ is bounded. Since $W = \overline{Q(H)}^X$ and $W$ is continuously embedded, we see that $W = \overline{Q(H)}^W \subseteq \overline{Q(H)}^X$, where the closures are with respect to the norms in $W$ and $X$, respectively. As a result, the density of $W$ in $X$ implies that $X = \overline{Q(H)}^X$. By using (11), the monotonicity of $T_\lambda$ and $J_1$, and property of $\psi^*$, we obtain that
\begin{equation}
\langle \psi x_n, \psi x_n \rangle
\end{equation}
\begin{equation}
= - \epsilon_n \| x_n \|^2 - \langle T_\lambda \psi x_n + \delta J_1 \psi x_n - (T_\lambda (0) + \delta J_1 (0)) \psi x_n, \psi x_n \rangle
\end{equation}
\begin{equation}
+ \langle T_\lambda (0) + \delta J_1 (0) + f^*, \psi x_n \rangle
\end{equation}
\begin{equation}
\leq \| T_\lambda (0) + \delta J_1 (0) + f^* \| \| \psi x_n \| \leq (|T_\lambda (0)| + \| f^* \|) \| Q x_n \|
\end{equation}
for all $n$. Since $Q x_n \in W$, it follows that $\psi x_n = jQ x_n = Q x_n$ for all $n$. Consequently, we obtain that
\begin{equation}
\langle SQ x_n, Q x_n \rangle \leq (|T_\lambda (0)| + \| f^* \|) \| Q x_n \|
\end{equation}
for all $n$. Since $\{Q x_n\}$ is bounded and $S$ is quasibounded, we conclude that $\{SQ x_n\}$ is bounded. Consequently, by using (24), it is not difficult to see that $\{\epsilon_n \| x_n \|^2\}$ is bounded. If the sequence $\{x_n\}$ is bounded, then $\epsilon_n x_n \rightarrow 0$ as $n \rightarrow \infty$. Otherwise, by using the boundedness of $\{\epsilon_n \| x_n \|^2\}$, we assume without loss of generality that $\epsilon_n x_n \rightarrow 0$ as $n \rightarrow \infty$, $Q x_n \rightarrow x_0$, and $SQ x_n \rightarrow v_0^\ast$ as $n \rightarrow \infty$. Since $Q(H)^X = X$, by choosing a sequence $\{z_m = Q y_m\}$ such that $z_m \rightarrow x_0$ as $m \rightarrow \infty$ and using (24) together with the monotonicity of $T_\lambda + \delta J_1$, we get
\begin{equation}
\langle SQ x_n, Q x_n \rangle = \langle SQ x_n, Q y_m - Q y_m + Q x_n \rangle
\end{equation}
\begin{equation}
= \langle SQ x_n, Q y_m \rangle + \langle SQ x_n, Q x_n - Q y_m \rangle
\end{equation}
\begin{equation}
= \langle SQ x_n, Q y_m \rangle + \langle f^*, Q x_n - Q y_m \rangle
\end{equation}
\begin{equation}
- \langle (T_\lambda + \delta J_1) Q x_n - (T_\lambda + \delta J_1) Q y_m, Q x_n - Q y_m \rangle
\end{equation}
\begin{equation}
- \langle (T_\lambda + \delta J_1) Q y_m, Q x_n - Q y_m \rangle
\end{equation}
\begin{equation}
eq \epsilon_n \langle x_n, x_n - y_m \rangle \leq \langle SQ x_n, Q y_m \rangle
\end{equation}
\begin{equation}
- \langle (T_\lambda + \delta J_1) Q y_m, Q x_n - Q y_m \rangle + \epsilon_n \langle x_n, y_m \rangle
\end{equation}
\begin{equation}+ \langle f^*, Q x_n - Q y_m \rangle
\end{equation}
for all $n$ and $m$. Fixing $m$ and letting $n \rightarrow \infty$ in (27), we obtain that
\begin{equation}
\limsup_{n \rightarrow \infty} \langle SQ x_n, Q x_n \rangle
\end{equation}
\begin{equation}
\leq \langle v_0^\ast, Q y_m \rangle
\end{equation}
\begin{equation}
- \langle (T_\lambda + \delta J_1) Q y_m - f^*, x_0 - Q y_m \rangle.
\end{equation}
Since $T_\lambda + \delta J_1$ is demicontinuous, letting $m \rightarrow \infty$, we arrive at
\begin{equation}
\limsup_{n \rightarrow \infty} \langle SQ x_n, Q x_n \rangle \leq \langle v_0^\ast, x_0 \rangle;
\end{equation}
that is,
\begin{equation}
\limsup_{n \rightarrow \infty} \langle SQ x_n, Q x_n - x_0 \rangle \leq 0.
\end{equation}
Since $S$ is generalized pseudomonotone, we conclude that $x_0 \in D(S)$, $Sx_0 = v_0$, and $\langle SQx_n, Qx_n \rangle \to \langle v_0, x_0 \rangle$ as $n \to \infty$. For any $y \in Q(H)$, applying the monotonicity of $T_\lambda + \delta I_1$, we arrive at

$$
\langle T_\lambda y + \delta I_1 y - f^*, y - x_0 \rangle = \lim_{n \to \infty} \langle T_\lambda y + \delta I_1 y - f^*, y - Qx_n \rangle 
$$

for all $n$. Assume without loss of generality that $\|y\| \to \infty$ as $n \to \infty$. By (31),

$$
\liminf_{n \to \infty} \langle T_\lambda y + \delta I_1 y - f^*, y - Qx_n \rangle 
$$

for all $n$. As a result, we arrive at

$$
\liminf_{n \to \infty} \langle T_\lambda y + \delta I_1 y - f^*, y - Qx_n \rangle 
$$

for all $n$. Consequently, we have

$$
\limsup_{n \to \infty} \langle T_\lambda y + \delta I_1 y - f^*, y - Qx_n \rangle 
$$

for all $n$. By the density of $Q(H)$ in X and the continuity of $T_\lambda + \delta I_1$, we conclude that

$$
\langle T_\lambda y + \delta I_1 y - f^*, y - x_0 \rangle \geq 0
$$

for all $y \in Q(H)$. By the density of $Q(H)$ in X and the continuity of $T_\lambda + \delta I_1$, we conclude that

$$
\langle T_\lambda y + \delta I_1 y - f^*, y - x_0 \rangle \geq 0
$$

for all $y \in X$. Since, for any $y \in X$, $x_t = x_0 + (1-t)y \in X$ for all $t \in [0, 1]$, using $x_t$ in place of $y$, we obtain that

$$
\langle (T_\lambda + \delta I_1) x_t + Sx_0 - f^*, (1-t)(y - x_0) \rangle \geq 0
$$

for all $t \in [0, 1]$; that is,

$$
\langle (T_\lambda + \delta I_1) x_t + Sx_0 - f^*, y - x_0 \rangle \geq 0
$$

for all $t \in [0, 1]$. Since $T_\lambda + \delta I_1$ is continuous and $x_t \to x_0$ as $t \to 1^-$, we have $T_\lambda x_t + \delta I_1 x_t \to T_\lambda x_0 + \delta I_1 x_0$ as $t \to 1^-$. Letting $t \to 1^-$, we arrive at

$$
\langle (T_\lambda + \delta I_1) x_0 + Sx_0 - f^*, y - x_0 \rangle \geq 0
$$

for all $y \in X$. Since $y \in X$ is arbitrary, setting $y + x_0$ in place of $y$ yields

$$
\langle (T_\lambda + \delta I_1) x_0 + Sx_0 - f^*, y \rangle \geq 0
$$

for all $y \in X$. Therefore, for each $\lambda > 0$ (by fixing $\delta > 0$ temporarily), we see that there exists $x_\lambda \in D(S) \cap \bar{G}_\delta$ such that $T_\lambda x_\lambda + \delta I_1 x_\lambda + Sx_\lambda = f^*$. Thus, for each $\lambda_n \downarrow 0^+$, there exists $y_n \in D(S) \cap \bar{G}_\delta$ such that

$$
T_\lambda y_n + \delta I_1 y_n + Sx_\lambda - f^* = 0
$$

for all $n$. Since $\bar{G}_\delta$ and $I_1$ are bounded, it follows that $\{y_n\}$ and $\{I_1 y_n\}$ are bounded. Since $S$ is quasi-bounded, it is not hard to see that $\{Sx_n\}$ is bounded, which implies the boundedness of $\{T_\lambda y_n\}$. Assume without loss of generality that $y_n \to y_0$, $Sx_n \to v_0$ and $T_\lambda y_n \to u_\delta$, as $n \to \infty$. Since $S + \delta I_1$ is generalized pseudomonotone with domain $D(S)$, it follows that

$$
\liminf_{n \to \infty} \langle Sx_n + \delta I_1 y_n, y_n - y_0 \rangle \geq 0.
$$

Consequently, from (40), we arrive at

$$
\limsup_{n \to \infty} \langle T_\lambda y_n, y_n - y_0 \rangle \leq 0.
$$

Let $I_{\lambda_n}$ be the Yosida resolvent of $T$. It is well known that $I_{\lambda_n} y_n \in D(T), I_{\lambda_n} y_n = x_\lambda - \lambda_n I^{-1}(T_\lambda y_n)$, and $T_{\lambda_n} y_n \in T(I_{\lambda_n} y_n)$ for all $n$. Since $y_n \to y_0$ and $\{I_{\lambda_n} y_n\}$ is bounded, it follows that $I_{\lambda_n} y_n \to y_0$ as $n \to \infty$. Thus, we have

$$
\langle T_{\lambda_n} y_n, I_{\lambda_n} y_n - y_0 \rangle = \langle T_{\lambda_n} y_n, I_{\lambda_n} y_n - y_n \rangle
$$

$$
+ \langle T_{\lambda_n} y_n, y_n - y_0 \rangle
$$

$$
- \langle T_{\lambda_n} y_n, y_n - y_0 \rangle
$$

$$
\leq \langle T_{\lambda_n} y_n, y_n - y_0 \rangle
$$

for all $n$. Consequently, we have

$$
\lim_{n \to \infty} \langle T_{\lambda_n} y_n, I_{\lambda_n} y_n - y_0 \rangle \leq \lim_{n \to \infty} \langle T_{\lambda_n} y_n, y_n - y_0 \rangle
$$

$$
\leq 0.
$$

By the maximality of $T$, applying Lemma 3, we obtain $x_0 \in D(T), v_0^* = T x_0$, and $\langle T_{\lambda_n} y_n, I_{\lambda_n} y_n \rangle \to \langle v_0^*, x_0 \rangle$ as $n \to \infty$, which implies

$$
\limsup_{n \to \infty} \langle Sx_n, y_n - y_0 \rangle \leq 0.
$$

The generalized pseudomonotonicity of $S$ implies $y_0 \in D(S)$ and $Sx_\lambda = -v_\lambda$. As a result, letting $n \to \infty$ in (40), we conclude that $v_0^* + Sx_\lambda = f^*$. This implies that, for each $\delta_n \downarrow 0^+$, there exist $z_n \in D(T) \cap D(S)$ and $v_n^* \in T z_n$ such that

$$
v_n^* + S z_n + \delta I_1 z_n = f^*
$$

for all $n$. Next we will show that $\{z_n\}$ is bounded. Assume without loss of generality that $\|z_n\| \to \infty$ as $n \to \infty$. By
the inner product condition on $S$ and monotonicity of $T$ with $0 \in T(0)$, we get
\[ \delta_n \|z_n\|^3 \leq d \|z_n\|^2 + \|f^*\| \|z_n\| \tag{47} \]
for all $n$; that is, dividing this inequality by $\|z_n\|$ for all large $n$, we get $\delta_n \|z_n\|^2 \leq d \|z_n\| + \|f^*\|$ for all large $n$. By using condition (i) and (46), we get that
\[ -\mu + \alpha \|z_n\|^2 \leq \|f^* + Sx\| \leq \delta_n \|z_n\|^2 + \|f^*\| \tag{48} \]
for all $n$. This gives $(\alpha - d)\|z_n\| \leq 2\|f^*\| + \mu$ for all $n$. Consequently, the boundedness of $\{z_n\}$ follows. Since $S$ is quasibounded and $0 \in D(T)$, it is not hard to see that $\{Sz_n\}$ is bounded. Consequently, the boundedness of $\{v^*_n\}$ follows.

Assuming that $z^*_n \to z^*_0$, $v^*_n \to v^*_0$, and $Sz_n \to S_0$ as $n \to \infty$ and using the arguments used in the first half of the proof of Theorem 6, we conclude that $z_0 \in D(T) \cap D(S)$, $v^*_0 \in Tz_0$, $S_0 = h_0$, and $v^*_0 + S_0 = f^*$; that is, for each $f^* \in X^*$, the inclusion problem $Tu + Su \ni f^*$ is solvable in $D(T) \cap D(S)$. Since $f^* \in X^*$ is arbitrary, we obtain the surjectivity of $T + S$. The proof using condition (ii) can be completed following similar arguments. The details are omitted here. This completes the proof.

It is worth mentioning that Theorem 6 is a new result because the perturbed operator $T + S$ is noncoercive and $S$ is densely defined such that $D(S)$ contains a dense real separable reflexive Banach space. Under the conditions on $T + S$, the result was unknown earlier even for coercive operator $T + S$. The analog of Theorem 6 for single multivalued, finitely continuous, coercive, and quasibounded generalized pseudomonotone operator $S$ such that $D(S)$ contains a dense linear subspace is due to Brodver and Hess [1]. If $S$ is quasimonotone with weakly closed graph or graph of $T + S$ is weakly closed and $S$ is monotone of type $(M)$, the arguments used in the proof of Theorem 6 can be easily carried out to conclude the surjectivity of $T + S$. The reader is referred to Gupta [28] for a result for $T + S$, where graph of $S$ is assumed to be weakly closed and $S : X \supseteq D(S) \to 2^{X^*}$ is quasibounded, finitely continuous coercive operator of type $(M)$ such that $D(S)$ contains a dense linear subspace. Theorem 6 improves and gives unifications of the existing surjectivity results due to Le [16], Asfaw and Kartosatos [18, 24], Asfaw [19], and Kenmochi [12–14] for maximal monotone perturbations of coercive bounded pseudomonotone operators with domain, all of $X$. In addition, it can be easily seen that the proof of Theorem 6 can go through if the quasiboundedness of $S$ is omitted and $T$ is assumed to be strongly quasibounded with $0 \in T(0)$. Another observation is that the condition $\langle v^* + Sx, x \rangle \geq -d \|x\|^2$ for all $x \in D(T) \cap D(S)$ and $v^* \in Tx$ can be replaced by a stronger condition $\langle v^* + Sx, x \rangle \geq -d \|x\|^2$ for all $x \in D(T) \cap D(S)$ and $v^* \in Tx$, and the weak coercivity condition (i) can be relaxed to satisfy $\|Tx + Sx\| \to \infty$ as $\|x\| \to \infty$. On the other hand, one can easily see that if weak coercivity condition on $T + S$ is automatically satisfied if $T + S$ is $\alpha$-expansive. Consequently, the following corollary is immediate.

**Corollary 7.** Let $T : X \supseteq D(T) \to 2^{X^*}$ be strongly quasibounded maximal monotone with $0 \in T(0)$ and let $S : X \supseteq D(S) \to X^*$ be demicontinuous generalized pseudomonotone. Suppose $W \subseteq D(S)$ is a real reflexive separable Banach space and continuously embedded in $X$. Assume, further, that $T + S$ is $\alpha$-expansive and $\alpha > d \geq 0$ such that
\[ \langle Lx + Sx, x \rangle \geq -d \|x\|^2 \]
for all $x \in D(S)$. Then $T + S$ is surjective.

**Proof.** Since $0 \in T(0)$ and $T$ is monotone, by the condition on $S$, it follows that $\langle v^* + Sx, x \rangle \geq -d \|x\|^2$ for all $x \in D(T) \cap D(S)$. Furthermore, by the expansiveness of $T + S$, for some $u_0 \in D(T) \cap D(S)$ and $v^*_0 \in Tu_0$, we arrive at
\[ \|v^* + Sx\| \geq \|v^* + Sx - (v^*_0 + Su_0)\| - \|v^*_0 + Su_0\| \]
\[ \geq \alpha \|x - u_0\| - \|v^*_0 + Su_0\| \]
\[ \geq \alpha \|x\| - A_0, \]
where $A_0 = \alpha \|u_0\| + \|v^*_0 + Su_0\|$, for all $x \in D(T) \cap D(S)$ and $v^* \in Tx$. This shows that $T + S$ satisfies conditions of Theorem 6. By applying similar arguments as in the last part of the proof of Theorem 6 and using the strong quasiboundedness of $T$ instead of quasiboundedness of $S$, we conclude that $T + S$ is surjective. The details are omitted here.

It is worth noticing here that Corollary 7 gives a partial positive answer for Nirenberg’s problem on the surjectivity of expansive mapping in a real separable reflexive Banach space. More precisely, Corollary 7 gives surjectivity of densely defined demicontinuous generalized pseudomonotone expansive mapping. To the best of the authors’ knowledge, this result was unknown. For related surjectivity results for continuous expansive mappings in a real Hilbert space, we cite the papers by Kartosatos [29] and Xiang [30]. For range results for single continuous quasimonotone expansive mapping defined from arbitrary reflexive Banach space into its dual space $X^*$, the reader is referred to the paper due to Asfaw [25].

The content of the following theorem addresses the solvability of operator equations involving operators of the type $L + S$, where $L : X \supseteq D(L) \to X^*$ is linear, densely defined, monotone, and closed, and $S : X \supseteq D(S) \to X^*$ is quasibounded demicontinuous of type $(M)$ such that $D(L)$ lies in $D(S)$.

**Theorem 8.** Let $L : X \supseteq D(L) \to X^*$ be closed, densely defined, and linear monotone, and let $S : X \supseteq D(S) \to X^*$ be quasibounded demicontinuous of type $(M)$ with respect to $L$ such that $D(L)$ lies in $D(S)$. Assume, further, that there exist $\mu \geq 0$ and $\alpha > d \geq 0$ such that
\[ \langle Lx + Sx, x \rangle \geq -d \|x\|^2 \]
for all $x \in D(L)$ and either
\[ \|Lx + Sx\| \geq \alpha \|x\| - \mu \quad \forall x \in D(L) \]
(i)
or

\[(ii) \text{ there exists } \phi : [0, \infty) \to (-\infty, \infty) \text{ such that } \phi(t) \to \infty \text{ as } t \to \infty \text{ and} \]

\[\|Lx + Sx\| \geq \phi(\|x\|) \|x\| \quad \forall x \in D(L). \quad (52)\]

Then \(L + S\) is surjective.

**Proof.** Fix \(f^* \in X^*\). Let \(Y = D(L)\) and let \(\|\cdot\|_Y\) be the graph norm on \(Y\) given by

\[\|x\|_Y = \|x\|_X + \|Lx\|_{X^*}, \quad x \in Y. \quad (53)\]

It is well-known that \(Y\) equipped with the graph norm becomes a real reflexive separable Banach space. By Lemma 4, let \(H\) be a Hilbert space and let \(Q : H \to Y\) be a compact injection such that \(Q(H)\) is dense in \(Y\). Let \(j : Y \to X\) be the natural embedding of \(Y\) into \(X\) and let \(j^* : X^* \to Y^*\) be its adjoint. It follows that \(\psi = jQ\) is a compact injection from \(H\) into \(X\).

By using the graph norm on \(Y\), it follows that \(Y\) is dense and continuously embedded in \(X\). Moreover, by the inner product condition on \(L + S\), for each \(\delta > 0\), there exists \(R_\delta > 0\) such that

\[\langle Lx + Sx + \delta j_1x, x \rangle \geq \delta \|x\|^3 - d \|x\|^2 \]

\[= \|x\|^3 \left[ \delta - d \frac{\|x\|^2}{\|x\|^2} \right] > 0 \quad (54)\]

for all \(x \in D(L) \cap D(S) \cap \partial B_{R_\delta}(0)\). Let \(G_\delta = B_{R_\delta}(0)\). By using the arguments used in the first half of the proof of Theorem 6, we see that \(G_\delta \cap Y\) is open in \(Y\) and \(\partial (G_\delta \cap Y) \subseteq \partial G_\delta \cap Y\). Let \(U = \psi^{-1}(G_\delta \cap Y)\). Since \(j : Y \to X\) and \(Q : H \to Y\) are continuous, it follows that \(U\) is open in \(H\).

By the arguments used in the proof of Theorem 6, using \(Y\) in place of \(X\) and the closed convex subset \(U\) of \(H\), it follows that the mapping \(C_\varepsilon : U \to H\) defined by

\[C_\varepsilon (v) = -\varepsilon^{-1} (\psi (L + S + \delta J_1) f - f^*), \quad v \in U \quad (55)\]

is compact. In addition, we see that

\[\partial_h U = \partial_h \psi^{-1}(\partial G_\delta \cap Y) \subseteq \psi^{-1}(\partial G_\delta \cap Y) \subseteq \psi^{-1}(\partial G_\delta \cap Y) \subseteq \psi^{-1}(\partial G_\delta \cap Y) \subseteq \psi^{-1}(\partial G_\delta \cap Y). \quad (56)\]

Following the argument as in the proof of Theorem 6, it is not difficult to see that \(x \neq \lambda C_\varepsilon (x)\) for all \(x \in \partial U\) and all \(\lambda \in (0, 1)\). Consequently, by Lemma 5, we obtain that, for each \(\varepsilon > 0\), \(C_\varepsilon\) has a fixed point in \(U\). Therefore, for each \(\varepsilon, n \downarrow 0^+\), there exists \(x_n \in U\) such that

\[\varepsilon_n x_n + \psi^* (L + S + \delta J_1) x_n = \psi^* f^* \quad (57)\]

for all \(n\); that is,

\[\langle \varepsilon_n x_n, x \rangle + \langle Q^* (j^* (L + S + \delta J_1) j Q x_n), x \rangle = \langle Q^* f^*, x \rangle \quad \forall x \in H. \quad (58)\]

Since \(j : Y \to X\) and \(Q : H \to Y\), by the definition of \(Q^*\) and \(f^*\), we see that

\[\langle \varepsilon_n x_n, x \rangle + \langle (L + S + \delta J_1) Q x_n, Q x \rangle = \langle f^*, Q x \rangle \quad \forall x \in H. \quad (59)\]

Since \(\psi x_n \in G_\delta \cap Y\) and \(G_\delta\) is bounded in \(X\), it follows that the sequence \(\{x_n\} = \{Q x_n\}\) is bounded in \(X\). From (57), using the monotonicity \(L(L(0) = 0), \text{ boundedness of } \{Q x_n\}, \text{ and quasiboundedness of } S\), we get the boundedness of the sequence \(\{Q x_n\}\). This gives

\[\varepsilon_n \|x_n\|^2 = -\langle Q^* (L + S + \delta J_1) Q x_n - Q^* f^*, x_n \rangle \]

\[\leq -\langle Q^* \psi x_n + \delta J_1 Q x_n - f^*, Q x_n \rangle \]

\[\leq \| Q^* \psi x_n + \delta J_1 Q x_n - \psi^* f^* \| Q x_n \| \quad \forall n. \quad (60)\]

As a result, we get the boundedness of \(\{\varepsilon_n \|x_n\|^2\}\). If \(\{x_n\}\) is bounded, then \(\varepsilon_n x_n \to 0\) as \(n \to \infty\). If \(\{x_n\}\) is unbounded, by passing into a subsequence, we see that

\[\varepsilon_n \|x_n\|^2 = \varepsilon_n \frac{\|x_n\|^2}{\|x_n\|} \to 0 \quad \text{as } n \to \infty. \quad (61)\]

In all cases, we assume without loss of generality that \(\varepsilon_n x_n \to 0\) as \(n \to \infty\). As a result, we get

\[\langle LQ x_n, Q x \rangle = -\langle \varepsilon_n x_n, x \rangle - \langle Q^* \psi x_n + \delta J_1 Q x_n, Q x \rangle \]

\[+ \langle f^*, Q x \rangle \]

\[\leq \varepsilon_n \|x_n\| + \| Q^* \psi x_n + \delta J_1 Q x_n + f^* \| Q x \| \]

\[\leq \mu \| Q x \|, \quad (62)\]

where \(\mu\) is an upper bound for the sequence \(\{\|\varepsilon_n x_n + Q^* \psi x_n + \delta J_1 Q x_n + f^*\|\}\). Since \(Q(H)\) is dense in \(Y\), for each \(y \in Y\), there exists a sequence \(\{y_m\}\) in \(H\) such that \(Q y_m \to y\) as \(n \to \infty\). This gives

\[\langle LQ x_n, y \rangle = \lim_{m \to \infty} \langle LQ x_n, Q y_m \rangle \leq \lim_{m \to \infty} \mu \| Q y_m \| \leq \mu \| y \| \quad (63)\]

By similar argument, the density of \(Y\) in \(X\) implies that

\[\langle LQ x_n, x \rangle \leq \mu \| x \| \quad \forall x \in X. \quad (64)\]

By using the uniform boundedness principle, we conclude that \(\{LQ x_n\}\) is bounded. Assume without loss of generality that \(Q x_n \to x_0\) in \(X\), \(S Q x_n \to v_0\), and \(L Q x_n \to h^*\) in \(X^*\) as \(n \to \infty\). Since \(L\) is closed linear, it follows that \(x_0 \in Y\) and \(h^* = L x_0\). By following the arguments used in the first half of the proof of Theorem 6 along with (57), we get

\[\lim_{n \to \infty} \sup \langle S Q x_n - f^*, Q x_n - x_0 \rangle \leq 0. \quad (65)\]

On the other hand, from (58), by using \(x_n \in H\) in place of \(x\), we see that

\[\langle LQ x_n + SQ x_n - f^*, Q x_n \rangle \leq 0 \quad \forall n. \quad (66)\]
Since $S$ is of type $(M)$ with respect to $L$, it follows that $S - f^*$ is also of type $(M)$ with respect to $L$, which yields $x_n \in D(S)$ and $v^*_n = Sx_n$. Finally, letting $n \to \infty$ in (57), we get $\psi^*(L + S + \delta I)x_n = \psi^* f^*$; that is, $Q^{-j'}(L + S + \delta I)x_n = Q^{-j'} f^*$. Since $Q(H)$ and $Y$ are dense in $Y$ and $X$, respectively, it follows that $j'$ and $Q^*$ are one to one. Therefore, we arrive at $Lx_n + Sx_n + \delta Ix_n = f^*$. Consequently, for each $\delta_n \downarrow 0^+$, there exists $y_n \in D(L)$ such that

$$
Ly_n + S y_n + \delta_n I y_n = f^* \hspace{1cm} \forall n.
$$

(67)

Since $L$ is closed and $S$ is of type $(M)$, by weak coercivity condition on $L + S$, the same arguments used in the second half of the proof of Theorem 6 can be carried over to conclude the solvability of the equation $Lx + Sx \ni f^*$ in $D(L)$. Since $f^* \in X^*$ is arbitrary, we conclude that $L + S$ is surjective. The details are omitted here.

The following corollary gives a characterization of linear maximal monotone operator in separable reflexive Banach space.

**Corollary 9.** Let $X$ be a real separable reflexive Banach space and let $L : X \supseteq D(L) \to X^*$ be linear operator. Then the following two statements are equivalent:

(i) $L$ is maximal monotone,

(ii) $L$ is monotone, densely defined, and closed.

**Proof.** The proof of (i) implies (ii) follows by the well-known result due to Brézis (cf. Zeidler [9, Theorem 32.1, page 897]).

Next we prove (ii) implies (i). Let $\lambda > 0$. It is sufficient to show that $R(L + \lambda J) = X^*$. To this end, we will use Theorem 8. Since $L$ is linear and monotone, that is, $(Lx, x) \geq 0$ for all $x \in D(L)$, and $J$ is monotone, it follows that $(Lx + \lambda Jx, x) \geq \lambda \|x\|^2$ for all $x \in D(L)$. Therefore, for each $\lambda > 0$, it follows that

$$
\|Lx + \lambda Jx\| \geq \lambda \|x\| \hspace{1cm} \forall x \in D(L).
$$

(68)

By using $J$ in place of $S$ in Theorem 8, we conclude that $R(L + \lambda J) = X^*$ for any $\lambda > 0$. Thus, $L$ is maximal monotone.

It is worth noticing that Brézis proved (i) in arbitrary reflexive Banach space provided that $L^*$ is monotone and (ii) holds. As a result, Corollary 9 is an improvement of the result of Brézis when $X$ is separable. It is important to mention that Gupta [28] gave surjectivity result for graph weakly closed maximal monotone perturbations of quasibounded, finitely continuous multivalued coercive operator $S$ of type $(M)$ such that $D(S)$ contains a dense linear subspace of $X$. However, the result in Theorem 8 is for noncoercive operator $S$ along with weak coercivity of $L + S$. It is also important to mention here that the results in Theorems 6 and 8 are new even in the case where the operator $S$ is coercive but not everywhere defined. In conclusion, Theorems 6 and 8 gave improvements over the existing theory for maximal monotone perturbations of coercive and everywhere defined operators of pseudomonotone type.

### 3. Example and Discussion

In this section, we demonstrate the existence of weak solution in $X = L^p(0, T; W^{1,p}_0(\Omega))$ for the parabolic problem of the type

$$
\frac{du}{dt} - \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, t, u, \nabla u) = f(x, t) \hspace{1cm} (x, t) \in Q
$$

(69)

$$
u(x, t) = 0 \hspace{1cm} (x, t) \in \partial\Omega \times (0, T)
$$

$$
u(x, 0) = 0 \hspace{1cm} x \in \Omega,
$$

where $p > 1$, $p'$ is conjugate exponent of $p$, $Q = \Omega \times (0, T), \Omega$ is a nonempty, bounded, and open subset of $\mathbb{R}^N$, and $f \in L^{p'}(Q)$ such that the following conditions are satisfied:

(A1) $a_i(x, t, s, \xi) \hspace{1cm} (i = 1, 2, \ldots, N)$ satisfies the Carathéodory conditions; that is, for each $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, the function $(x, t) \mapsto a_i(x, t, s, \xi)$ is measurable and, for almost all $(x, t) \in \Omega \times (0, T)$, the function $(s, \xi) \mapsto a_i(x, t, s, \xi)$ is continuous.

(A2) there exists positive constants $\mu_1$ and $\mu_2$ such that

$$
\sum_{i=1}^N (a_i(x, t, s, \xi) - a_i(x, t, s, \eta)) (\xi_i - \eta_i)
$$

$$
\geq \mu_1 |\xi - \eta|^p + \mu_2 \left[|s|^p + |\xi|^{p-1}\right] + g(x, t)
$$

(70)

for all $(x, t, s, \xi) \in \Omega \times (0, T) \times \mathbb{R} \times \mathbb{R}^N$, where $\xi = (\xi_i) \in \mathbb{R}^N$ and $\eta = (\eta_i) \in \mathbb{R}^N$, $g \in L^2(\Omega)$, $p_0 < p - 1 + 2/p - 1/N$, and $p < N$. Let $X = L^2(0, T; V), V = W^{1,p}_0(\Omega)$, and $L : X \supseteq D(L) \to X^*$ be defined by $Lu = u'$, where $u'$ is understood in the sense of distributions; that is,

$$
\int_0^T u'(t) \psi(t) dt = -\int_0^T u(t) \psi'(t) dt,
$$

(71)

$$\psi \in C_0^{\infty}(0, T),
$$

where $D(L) = \{u \in X : u' \in X^*, u(0) = 0\}$. We notice that

$$
\langle Lu, \phi \rangle = \int_0^T \langle u'(t), \phi(t) \rangle d, t
$$

(72)

$$
u \in D(L), \phi \in X.
$$

Let $A : X \supseteq D(A) \to X^*$ be defined by

$$
\langle Au, \phi \rangle = \sum_{i=1}^N \int_0^T \int_Q a_i(x, t, u, \nabla u) \frac{\partial \phi(x, t)}{\partial x_i} dx dt,
$$

(73)

$$\phi \in X, u \in D(A),
$$

where

$$
D(A) = \{u \in X : u \in L^p(\Omega)\}, \quad \bar{p} = \frac{p_0 p}{p - 1}.
$$

(74)
It is known from Kartsatos and Skrypnik [4] that \( A \) is quasibounded demicontinuous generalized \((S_+)\) with respect to \( L \) such that \( D(A) \) contains \( D(L) \). Moreover, it is well-known that \( L \) is linear, closed, and densely defined maximal monotone. The operator \( A \) is densely defined, that is, not everywhere defined, and coercive. Since \( p_0 < p - 1 \) in \((A_2)\), operator \( A \) may be unbounded. Therefore, by Theorem 8 using \( A \) in place of \( S \), for each \( f \in L^p(Q) \), we conclude that the equation \( Au + Lu = f^* \) is solvable in \( D(L) \), where \( f^* : X \to \mathbb{R} \) is given by \( \langle f^*, \phi \rangle = \int_Q f(x, t)\phi(x, t)d\mathcal{H}^d \). Therefore, the parabolic problem \((69)\) admits at least one weak solution in \( D(L) \).

Since \( \Omega \) is bounded and \( g \in L^p(Q) \), it is well known that \( A \) is bounded, continuous, everywhere defined, and coercive provided that \( p_0 = p - 1 \) in \((A_2)\). More precisely, these conditions on \( A \) are satisfied if condition \((A_3)\) is replaced by \((A_3)\): there exists positive constants \( \mu_1 \) and \( \mu_2 \) such that

\[
\sum_{i=1}^N \left( a_i(x, t, s, \xi) - a_i(x, t, s, \eta) \right) \xi_i - \eta_i \geq \mu_1 |\xi - \eta|^p,
\]

\[
|a_i(x, t, s, \xi)| \leq \mu_2 \left( |s|^{p-1} + |\xi|^{p-1} \right) + g(x, t)
\]

for all \((x, t, s, \xi) \in \Omega \times (0, T) \times \mathbb{R}^N \), where \( \xi = (\xi_i) \in \mathbb{R}^N \) and \( \eta = (\eta_i) \in \mathbb{R}^N \), \( g \in L^p(Q) \), and \( 1 < p < N \).

Abstract existence results concerning nonlinear parabolic problems of the type in \((69)\) under conditions \((A_1)\) and \((A_3)\) have been intensively studied by many researchers. For some of the basic and relevant references, the reader is referred to the papers by Browder and Hess [1], Brézis [2], Le [16], Kenmochi [12–14], Guan and Kartsatos [17], Asfaw and Kartsatos [18], Asfaw and Panagiotopoulos [35] and the references therein. The method of sub-supersolution is employed in the papers by Carl and Le [20], Carl et al. [21], Carl [22], Carl and Motreanu [23], and Le [36, 37] to study existence and properties of solution(s) for evolution inclusion problems of the type

\[
u \in Y : -\Delta u + B(u) \ni g^*, \quad g^* \in Y^*, \text{ where } Y = W^{1,p}_0(\Omega) \text{ and } A \text{ and } B \text{ are possibly noncoercive and densely defined and satisfy conditions of either Theorem 6 or Theorem 8.}
\]

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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**References**


