Some Inequalities for the Omori-Yau Maximum Principle

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We generalize A. Borb´ely’s condition for the conclusion of the Omori-Yau maximum principle for the Laplace operator on a complete Riemannian manifold to a second-order linear semielliptic operator \( L \) with bounded coefficients and no zeroth order term. Also, we consider a new sufficient condition for the existence of a tamed exhaustion function. From these results, we may remark that the existence of a tamed exhaustion function is more general than the hypotheses in the version of the Omori-Yau maximum principle that was given by A. Ratto, M. Rigoli, and A. G. Setti.

1. Introduction

Let \( (M, g) \) be a smooth complete Riemannian manifold of dimension \( n \). For a smooth real-valued function \( h \) on \( M \), a second-order linear differential operator \( L : C^{\infty}(M) \rightarrow C^{\infty}(M) \) without zeroth-order term can be written as

\[
Lh = \text{Tr}(A \circ \text{Hess}_h) + g(V, \nabla h),
\]

(1)

where \( A \in \Gamma(\text{End}(TM)) \) is self-adjoint with respect to \( g \), \( \text{Hess}_h \in \Gamma(\text{End}(TM)) \) is the Hessian of \( h \) in the form defined by \( \text{Hess}_h(X) = \nabla_X \nabla h \) for \( X \in \Gamma(TM) \), and finally \( V \in \Gamma(TM) \).

In this paper, we will deal with the semielliptic case, that is, \( A \) is positive semidefinite at each point, and we always assume that

\[
\sup_M \text{Tr}(A) + \sup_M |V| < \infty.
\]

(2)

Definition 1. A smooth complete Riemannian manifold \( M \) is said to satisfy the Omori-Yau maximum principle for the Laplace operator \( \Delta \) (the above semielliptic operator \( L \)) if for any \( C^2 \) function \( h : M \rightarrow \mathbb{R} \) which is bounded from above and for any \( \varepsilon > 0 \) there is a point \( x_\varepsilon \in M \) such that \( |h(x_\varepsilon) - \sup_M h| < \varepsilon, \|\nabla h(x_\varepsilon)\| < \varepsilon, \) and \( \Delta h(x_\varepsilon) < \varepsilon \) (\( Lh(x_\varepsilon) < \varepsilon \)).

The Omori-Yau maximum principle is a useful substitute of the usual maximum principle in noncompact settings. For the operator \( \Delta \), Definition 1 is the well-known Omori-Yau maximum principle for the Laplacian, which was first proven by Omori [1] and Yau [2] when the Ricci curvature is bounded below. This was improved upon by Chen and Xin [3] and Ratto et al. [4] when the Ricci curvature decays were slower than a certain decreasing function tending to minus infinity. For instance, we have the following.

Theorem 2 (Ratto-Rigoli-Setti’s condition [4, Theorem 2.3]). Let \( o \in M \) be a fixed point and \( r(x) \) be the distance function from \( o \). Let one assumes that away from the cut locus of \( o \) one has

\[
\text{Ricc}(\nabla r, \nabla r) \geq -(n-1)BG^2(r),
\]

(3)

where \( B > 0 \) is some constant and \( G(t) \) on \([0, \infty)\) satisfies

\[
\int_0^\infty \frac{1}{G(t)} dt = \infty, \quad G(0) = 1, \quad G' \geq 0,
\]

(4)

\[
\sqrt{G^{(2k+1)}(0)} = 0, \quad \forall k \geq 0,
\]

(5)

Then \( M \) satisfies the Omori-Yau maximum principle for the Laplacian \( \Delta \).

Borb´ely [5, Theorem] has given an elegant proof of the validity of the Omori-Yau maximum principle where
the Ricci curvature condition \((3)\) is replaced by the assumption 
\(\Delta r(x) \leq G(r(x))\) without \((4)\) and \((5)\). Also, Bessa et al. 
[6, Theorem 5.6] proved Borbély’s theorem [5, Theorem] for 
the \(f\)-Laplacian \(\Delta_f\) for a selected smooth function \(M\). In 
this paper, we first show that Borbély’s theorem [5, Theorem] 
is also true for our semielliptic operator \(L\) by following his 
method in [5] (see Theorem 5).

To state other results, we need the following definitions.

**Definition 3.** Let \(u\) be a real-valued continuous function on \(M\) and let a point \(p \in M\).

(i) A function \(u\) is called proper, if the set \(\{p : u(p) \leq r\}\) 
is compact for every real number \(r\).

(ii) A function \(v\) defined on a neighborhood \(U_p\) of \(p\) is 
called an upper-supporting function for \(u\) at \(p\), if the conditions 
\(v(p) = u(p)\) and \(v \geq u\) hold in \(U_p\).

**Definition 4.** A proper continuous function \(u : M \to \mathbb{R}\) is 
called a \(\Delta\)-tamed exhaustion, if the following condition holds:

1. \(u \geq 0\).

2. At all points \(p \in M\) it has a \(C^2\) smooth, upper-
supporting function \(v\) at \(p\) defined on an open 
neighborhood \(U_p\) such that \(\|\nabla v\| \leq 1\) and \(\Delta v \leq 1\).

Royden [7] showed that every complete Riemannian 
manifold satisfying Omori-Yau’s condition (i.e., the Ricci 
curvature is bounded from below) admits a \(\Delta\)-tamed exhaustion function. 
Inspired by Royden’s article [7], Kim and Lee [8, 
Theorem 2] proved the Omori-Yau maximum principle for 
the Laplacian \(\Delta\) when there exists a \(\Delta\)-tamed exhaustion function. 
Moreover, they proved that every complete Riemannian 
manifold satisfying Ratto-Rigoli-Setti’s condition admits a \(\Delta\)- 
tamed exhaustion function [8]. Similar to Definition 4, 
we define an \(L\)-tamed exhaustion function (i.e., we replace \(\Delta\) 
with \(L\)) [9, Definition 1.4]. Then, using the existence of an 
\(L\)-tamed exhaustion function, Hong and Sung [9, Theorem 
2.1] generalized the Omori-Yau maximum principle for the 
Laplacian \(\Delta\) to the operator \(L\). In this paper, we give a new 
sufficient condition for the existence of an \(L\)-tamed exhaustion function (see Theorem 6). We prove this result using 
the ideas adapted from [8]. Note that Theorem 6, together with 
[9, Theorem 2.1], implies the maximum principle of Omori 
and Yau for the operator \(L\). As a corollary, we prove that the 
existence of a \(\Delta\)-tamed exhaustion is more general than 
Ratto-Rigoli-Setti’s condition. Unfortunately, for the operator \(L\), 
the relation between Borbély’s condition (or the existence of an \(L\)-
tamed exhaustion) and Ratto-Rigoli-Setti’s condition remains 
for further study.

Now, we formulate our main results. From \((1)\), \(A\) is 
diagonalizable at each point on an orthonormal basis, since \(A\) is 
symmetric. Then one can take a normal coordinate \((x_1, \ldots, x_n)\) 
around \(x_e \in M\) such that \(A\) at \(x_e\) is represented 
as a diagonal matrix. Thus, we have

\[
L h|_{x_e} = \sum_{I} a_{ij}(x_e) \frac{\partial^2 h}{\partial x_i^2} |_{x_e} + \sum_{I} a_i(x_e) \frac{\partial h}{\partial x_i} |_{x_e},
\]  
for a real-valued function \(h\) on \(M\), where each \(a_{ij}(x_e)\) 
is nonnegative; the entries \(a_i(x_e)\) and \(|a(x_e)|\) are bounded 
above as \(x_e\) varies by \((2)\). We introduce a locally defined 
differential operator for convenience as follows:

\[
\tilde{\Delta}_x := a_{11}(x_1) \frac{\partial^2}{\partial x_1^2} + \cdots + a_{nn}(x_n) \frac{\partial^2}{\partial x_n^2},
\]

\[
\tilde{v}^1_x := a_i(x_1) \frac{\partial}{\partial x_1} + \cdots + a_n(x_n) \frac{\partial}{\partial x_n},
\]

\[
\tilde{v}_x := \left( a_{11}(x_1) \frac{\partial}{\partial x_1}, \ldots, a_{nn}(x_n) \frac{\partial}{\partial x_n} \right).
\]

Put \(d_1 = a_i(x_e)\) and \(e_1 = |a_i(x_e)|\) for \(1 \leq l \leq n\). We may assume 
that \(d_1\) and \(e_1\) are the largest of \(|d_1, \ldots, d_n\|\) and \(|e_1, \ldots, e_n|\), 
respectively.

Then we have the following.

**Theorem 5.** Let \(o \in M\) be a fixed point and \(r(x)\) be the 
distance function from \(o\.\) Assume that for all \(x \in M\)

\[
\tilde{\Delta}_x r(x) \leq G(r(x)),
\]

where \(r\) is smooth, \(r(x) > 1\), and \(G(t)\) on \([0, \infty)\) satisfies

\[
\int_0^\infty \frac{dt}{G(t)} = \infty, \quad G \geq 1, \quad G' \geq 0.
\]

Then \(M\) satisfies the Omori-Yau maximum principle for the 
operator \(L\).

**Theorem 6.** Let \(o \in M\) be a fixed point and \(r(x)\) be the 
distance function from \(o\.\) Assume that for all \(x \in M\)

\[
\tilde{\Delta}_x r(x) \leq G(r(x)),
\]

where \(r\) is smooth, \(r(x) > 1\), and \(G(t)\) on \([0, \infty)\) satisfies

\[
\lim_{t \to -\infty} \frac{t \sqrt{G(\sqrt{t})}}{\sqrt{G(t)}} \leq + \infty.
\]

Then \(M\) admits an \(L\)-tamed exhaustion function.

**Remark 7.** By [5, Corollary] and Theorem 6, Ratto-Rigoli-
Setti’s condition without \(\sqrt{G(2k-1)}(0) = 0 \forall k \geq 0\) implies 
the existence of a \(\Delta\)-tamed exhaustion function. Therefore, 
the existence of a \(\Delta\)-tamed exhaustion function for the 
conclusion of the Omori-Yau maximum principle for the Laplacian 
\(\Delta\) is more general than the hypothesis in Theorem 2.

There are some other sufficient conditions under which 
the Omori-Yau maximum principle for the Laplacian \(\Delta\) holds 
[10–12]. Also, [13] deals with the general setting of 
semielliptic operators (trace type operators). Recently, Bessa 
and Pessoa [14, Theorem 1] present a sufficient condition 
for the conclusion of the Omori-Yau maximum principle.
for a second-order linear semi-elliptic operator with bounded first-order coefficients and no zeroth-order term. However, they will not consider the existence of a tamed exhaustion as sufficient conditions for the conclusion of the Omori-Yau maximum principle.

2. Proof of Theorem 5

The proof is similar to the method in [5]. Let $U = \sup h$. We may assume that $h < U$ at every point of $M$; otherwise, $h$ has its maximum at some point and that point directly satisfies the Omori-Yau maximum principle for a second-order linear semi-elliptic operator.

Define the function $F(t)$ as

$$F(t) = e^{\int_1^{1/(\rho(t))} ds}.$$ (13)

Then

$$F' = \frac{F}{G}.$$ (14)

Since $G \geq 1$ on $[0, \infty)$, we have $F \geq 1$, and $F' > 0$. Hence the function $F$ is strictly increasing, and $\lim_{s \to \infty} F(t) = \infty$. Since the set $\{x \in M : r(x) \leq 1\}$ is compact, we have

$$U - \sup \{h(x) : r(x) \leq 1\} > 0.$$ (15)

For any positive constant $\epsilon < \min\{1, U - \sup\{h(x) : r(x) \leq 1\}\}$, we define the function $h_\lambda : M \to \mathbb{R}^+$ as

$$h_\lambda(x) = \lambda F(r(x)) + U - \epsilon.$$ (16)

Then

$$h_\lambda(x) > h(x) \quad \text{if} \quad r(x) \leq 1, \quad \lambda > 0.$$ (17)

Because, for all $x \in M$, $F(r(x)) \geq 1$ and $U > h(x)$. If $\lambda > \epsilon$, then we have

$$h_\lambda(x) > h(x), \quad \forall x \in M.$$ (18)

Define $\lambda_0$ as

$$\lambda_0 = \inf \{\lambda : h_\lambda(x) > h(x), \quad \forall x \in M\}.$$ (19)

Then, clearly, $\lambda_0 > 0$. Furthermore, we can obtain $h_{\lambda_0}(x) \geq h(x)$ for all $x \in M$; that is, there is a point $x_\lambda \in M$ such that $h_{\lambda_0}(x_\lambda) = h(x_\lambda)$. Assume that to the contrary $h_{\lambda_0}(x) > h(x)$ for all $x \in M$. Then we will show that there is a constant $\lambda'$ with $\lambda_0 > \lambda'$ such that $h_{\lambda'}(r(x)) > h(x)$ for all $x \in M$. This is a contradiction to the definition of $\lambda_0$.

Let $\lambda_0 > \lambda_1$. Because $\lim_{s \to \infty} F(r) = \infty$, there is a sufficiently large positive number $r_0$ such that $h_{\lambda_1}(x) > U > h(x)$ for $r(x) > r_0$. Also, because the set $\{x \in M : r(x) \leq r_0\}$ is compact, the statement $h_{\lambda_1}(x) > h(x)$ for all $x \in M$ implies that there is a constant $\lambda_2$ with $\lambda_0 > \lambda_2$ such that $h_{\lambda_2}(x) > h(x)$ for $r(x) \leq r_0$. Now, let $\lambda' = \max(\lambda_1, \lambda_2)$. Then, for $\lambda_0 > \lambda'$, we have $h_{\lambda'}(x) > h(x)$ for all $x \in M$. Moreover, by (17) and $\lambda_0 > 0$, we have $r(x) > 1$.

Next, we have to show that $h_{\lambda_0}$ is smooth at $x_\lambda$. Since $h_{\lambda_0}(x) = \lambda F(r(x)) + U - \epsilon$, it is enough to show that $r$ is smooth at $x_\lambda$. To avoid confusion, the point $o$ in the statement of Theorem 5, is switched to $p$. Note that $r$ is a Lipschitz function and is smooth on $M \setminus \{p, C_p]\$, where $C_p$ is the cut locus of $p$. Suppose that $x_\lambda \in C_p$. Then we have two possibilities (Petersen [15, Lemma 8.2]): either there are two distinct minimizing geodesic segments $y_1, y_2 : [0, t_0] \to M$ joining $p$ to $x_\lambda$, or there is a geodesic segment $y : [0, t_0] \to M$ from $p$ to $x_\lambda$ along which $x_\lambda$ is conjugate to $p$. Notice that

$$t_0 = r(y_i(t_0)) = r(x_\lambda) \quad \text{for} \quad i = 1 \text{ or } 2.$$ (20)

We consider the first case. Let $w = y'_1(t_0)$ and $v = y'_2(t_0)$. Since $y_1$ and $y_2$ are distinct segments, we have $w \neq v$. For $i = 1$ or 2, the functions $t \to r(y_i(t))$ are differentiable on $(0, t_0)$ and they have a left-derivative at $t_0$. Note that $h$ is $C^2$ smooth on $M$. From the definition of $\lambda_0$, $h_{\lambda_0} \geq h$, and $h_{\lambda_0}(x_\lambda) = h(x_\lambda)$ we obtain

$$\lim_{s \to 0^+} \frac{h_{\lambda_0}(y_1(t_0 + s)) - h_{\lambda_0}(y_2(t_0))}{s} \geq D_jh(x_\lambda),$$ (21)

where $D_jh(x_\lambda)$ denotes the directional derivative of $h$ at the point $x_\lambda$ in the direction of $v$. Furthermore, since $h_{\lambda_0}$ has a directional derivative at $x_\lambda$ in the direction of $-v$, we have

$$-\lambda_0 F'(r(x_\lambda)) = -\lambda_0 F'(r(x_\lambda)) = D_jh_{\lambda_0}(x_\lambda) \geq D_jh(x_\lambda) = -D_jh(x_\lambda).$$ (22)

This yields

$$D_jh(x_\lambda) \geq \lambda_0 F'(r(x_\lambda)).$$ (23)

Hence, by (21) and (23), we get the following inequality:

$$\lim_{s \to 0^+} \frac{h_{\lambda_0}(y_1(t_0 + s)) - h_{\lambda_0}(y_2(t_0))}{s} \geq \lambda_0 F'(r(x_\lambda)).$$ (24)

Note that $(h_{\lambda_0}(y_1))' = \lambda_0 F'(r(y_1))r'(y_1)$ and $r(y_2(t_0)) = r(x_\lambda)$. Recall that $\lambda_0 > 0$. Then, from (24), we can get

$$\lim_{s \to 0^+} \frac{r(y_2(t_0 + s)) - r(y_2(t_0))}{s} \geq 1.$$ (25)

The inequality (25) will lead to a contradiction. Since $y_1$ and $y_2$ are different segments, by connecting from the point $y_1(t_0 - s)$ to the point $y_2(t_0 + s)$ with a geodesic segment, there is a constant $c$ with $0 < c < 1$ such that, for a sufficiently small $s > 0$, the distance $d(y_1(t_0 - s), y_2(t_0 + s)) < c2s$. Thus there is a constant $c'$ with $0 < c' < 1$ depending only on the angle of $v$ and $w$ such that

$$r(y_2(t_0 + s)) < t_0 + c's,$$ (26)

for a sufficiently small $s > 0$. Note that $r(y_2(t_0)) = t_0$. By plugging (26) to (25), we have a contradiction.
From now, let us consider the second case. Since $\gamma$ is distance minimizing between $p$ and $x_e$, $r$ is smooth at $\gamma(t)$ for $0 < t < t_o$. Let $m(t) = \Delta r(\gamma(t))$. Then $m(t)$ is also smooth for $0 < t < t_o$. Because $\gamma(t_o)$ is conjugate to $p = \gamma(0)$ along $\gamma$, by a simple calculation, we get

$$\lim_{t \to t_o} m(t) = -\infty. \quad (27)$$

Because $\lambda_0 F'(r(x_e)) > 0$, by (23), we get $D_j h(x_e) > 0$; that is, $\nabla h(x_e) \neq 0$. Hence the level surface $H = \{ x \in M : h(x) = h(x_e) \}$ is a $C^2$ smooth hypersurface near $x_e$. Denote by $H_t$ the parallel surface to $H$ and passing through the point $\gamma(t_o - s)$ for some $s > 0$. Since $H$ is $C^2$ smooth near $x_e$, the surface $H_t$ is also $C^2$ smooth near $\gamma(t_o - s)$ for a sufficiently small $s > 0$. Therefore, by (27), for some sufficiently small $s$, the trace of the second fundamental form of $H_t$ at $\gamma(t_o - s)$ in the direction of $\gamma'(t_o - s)$ is greater than $m(t_o - s)$, where $m(t_o - s)$ is the trace of the second fundamental form of the geodesic sphere $B(p, t_o - s)$$ at $\gamma(t_o - s)$ with respect to the normal vector $\gamma'(t_o - s)$. This implies that there has to be a point $q_s \in H_t$ sufficiently close to $\gamma(t_o - s)$, which lies inside $B(p, t_o - s)$; that is,

$$r(q_s) < t_o - s. \quad (28)$$

Since $H_t$ is parallel to $H$, we also have a point on $q \in H$ such that the distance $d(q_s, q) = s$. By (28), we have

$$r(q) < t_o = r(x_e). \quad (29)$$

Since $F$ is strictly increasing, we get

$$h_{q_s}(q) = \lambda_0 F(r(q)) + U - \epsilon < \lambda_0 F(r(x_e)) + U - \epsilon = h_{x_e}(x_e) = h(x_e) = h(q). \quad (30)$$

This is a contradiction to the fact that $h_{q_s}(x_e) \geq h(x_e)$ for all $x \in M$. Therefore, the function $r$ must be smooth at $x_e$.

By the definition of $F, F \geq 1, G \geq 1, \text{ and } G' \geq 0$, we have

$$0 < F' = \frac{F}{G}, \quad \quad \quad \quad F'' = \frac{F'}{G} - \frac{FG'}{G^2} = \frac{F}{G} - \frac{FG'}{G^2} \leq \frac{F}{G}. \quad (31)$$

Because $\lambda_0 > 0, F \geq 1, \text{ and } h(x_e) = \lambda_0 F(r(x_e)) + U - \epsilon < U$, we have

$$0 < -\lambda_0 F(r(x_e)) + \epsilon = U - h(x_e) < \epsilon. \quad (32)$$

Hence

$$\lambda_0 < \frac{\epsilon}{F(r(x_e))} \leq \epsilon. \quad (33)$$

Recall notations (6) and (7). Since

$$h_{x_e}(x) \geq h(x), \quad \forall x \in M,$$

$$h_{x_e}(x_e) = h(x_e), \quad (34)$$

we have

$$\forall h_{x_e}(x_e) = \nabla h(x_e),$$

$$Lh_{x_e}(x_e) \geq Lh(x_e). \quad (35)$$

Note that $||\nabla r|| = 1$. By (31), (33), and $G \geq 1$, the first equality of (35) yields

$$\nabla h(x_e) = \frac{\lambda_0 F'(r(x_e)) \nabla r(x_e)}{F(r(x_e)) G(r(x_e))} \leq \epsilon. \quad (36)$$

Also, by (2), (31), (33), (36), $G \geq 1, \text{ and } \Delta_{x_e} r \leq G$, the second inequality of (35) yields

$$Lh(x_e) \leq Lh_{x_e}(x_e) = \sum_i a_i \frac{\partial^2}{\partial x_i^2} h_{x_e} \bigg|_{x_e}$$

$$+ \sum_i b_i \frac{\partial}{\partial x_i} h_{x_e} \bigg|_{x_e} \leq \lambda_0 \left( F'(r(x_e)) \Delta_{x_e} r(x_e) \right)$$

$$+ F''(r(x_e))(\nabla x_e \cdot \nabla r(x_e)) + e \epsilon \epsilon \quad (37)$$

$$< \frac{\epsilon}{F(r(x_e)) G(r(x_e))} \left( F(r(x_e)) \right) \left( G(r(x_e)) \right)$$

$$+ d_1 \frac{F(r(x_e))}{G(r(x_e))} + e \epsilon \leq (1 + d_1 + e_1).$$

If we replace $\epsilon$ with $e(1 + d_1 + e_1)$, then the above inequality, (32), and (36) show that the point $x_e$ satisfies the conditions in Definition 1.

3. Proof of Theorem 6

The proof is similar to the method in [8]. Let $o \in M$ be a fixed point and $r(x)$ be the distance function from $o$. Define a function $u : M \to \mathbb{R}$ by

$$u(x) = \int_0^{r(x)^2} G(s)^{-1} ds. \quad (38)$$

Assume that a smooth complete Riemannian manifold satisfies assumption (10). Then we will prove that $u$ is an $L$-tamed exhaustion function. We consider two cases.

First Case. Assume that $o$ has no cut points in $M$.

By the definition, the function $u$ is an exhaustion function for $M$. We have to show that, for certain positive constants $C$ and $C_1, ||\nabla u|| < C$ and $L u < C_1$ outside a ball of a certain radius with center $x_e$. Let $\phi(t) = \exp(|\int_0^t G(s)^{-1} ds|)$ and $B(x_e, r) = \{ x \in M \mid \text{dist}(x, x_e) < r \}$. Then $u(x) = \log \phi(r(x)^2)$. By a direct calculation, one gets

$$\nabla u = \nabla \log \phi(r^2) = 2 \nabla r \frac{\phi'(r^2)}{\phi(r^2)} = 2 r \nabla \phi(r^2). \quad (39)$$
By (12), there is a positive constant $C$ such that
\[
\frac{r^2 G(r)}{G(r^2)} = r^2 G(r) G(r^2)^{-1} < \frac{C}{4}.
\] (40)

Then, for $r > 1$, we obtain
\[
r G(r) G(r^2)^{-1} < r^2 G(r) G(r^2)^{-1} < \frac{C}{4}.
\] (41)

Moreover, by (11), we have
\[
\sup_{[0,\infty)} G(r)^{-1} = \left( \inf_{[0,\infty)} G(r) \right)^{-1} < 1.
\] (42)

By plugging (41) to (39), we have
\[
\|\nabla u\| < \frac{1}{2} \|\nabla r\| C G(r)^{-1}.
\] (43)

Note that $\|\nabla r\| = 1$. Applying (42) gives
\[
\|\nabla u\| < \frac{C}{2}.
\] (44)

By (2) and (44), one gets
\[
\|\bar{\nabla}_{x_e} u\| < e_1 \frac{C}{2}.
\] (45)

By assumption (II), we have
\[
\left( \frac{\phi'(r^2)}{\phi(r^2)} \right)' \left( G(r^2)^{-1} \right)' = -G(r^2)^{-2} G'(r^2) \leq 0.
\] (46)

Because of the above inequality, $\|\bar{\nabla}_{x_e} r\| \leq d_1$, (41), and (42), we have for $r > 1$
\[
\bar{\Delta}_{x_e} u = \bar{\Delta}_{x_e} \log \phi(r^2)
\]
\[
= 4r^2 \left( \frac{\phi'(r^2)}{\phi(r^2)} \right)' \|\bar{\nabla}_{x_e} r\|^2
\]
\[
+ 2G(r^2)^{-1} \left( \|\bar{\nabla}_{x_e} r\|^2 + r\bar{\Delta}_{x_e} r \right)
\]
\[
\leq 2G(r^2)^{-1} \left( \|\bar{\nabla}_{x_e} r\|^2 + r\bar{\Delta}_{x_e} r \right)
\]
\[
\leq 2r G(r^2)^{-1} (d_1^2 r^2 + \bar{\Delta}_{x_e} r).
\]

By our assumption (10), there exits $r_0 > 1$ such that
\[
\bar{\Delta}_{x_e} u < \frac{C}{2} d_1^2 + \frac{C}{2} \text{ on } M \setminus B(x_e, r_0).
\] (48)

Thus, by (45) and (48), we have
\[
Lu = \bar{\Delta}_{x_e} u + \bar{\nabla}_{x_e}^1 u < \frac{C}{2} (d_1^2 + 1 + e_1)
\] (49)

on $M \setminus B(x_e, r_0)$.

If we replace $(C/2)(d_1^2 + 1 + e_1)$ with $C_1$, then $u$ satisfies the additional conditions for an $L$-tamed exhaustion function.

**Second Case.** Assume that the cut locus of $\omega$ is nonempty.

Let $x_e$ be a cut point of $\omega$ and let $f(r) = \log(r^2)$ for $t > 0$. We choose a point $\bar{x}_e$ outside of cut locus of $\omega$ such that $\text{dist}(x_e, \bar{x}_e) < 1$ and $r(x_e) > r(\bar{x}_e)$. Denote by $B(y, r) = \{x \in M \mid \text{dist}(x, y) < r\}$. Take $\eta, \delta > 0$ such that $B(x_e, \eta) \cap B(\bar{x}_e, \delta) = \emptyset$ and $B(\bar{x}_e, \delta)$ does not have cut point of $\omega$.

Now, we present several functions to find an upper-supporting function for $u$.

For a neighborhood $U \subset B(x_e, \eta)$, we define a smooth map $T : U \to B(\bar{x}_e, \delta)$ with $T_{x_e}(x_e) = \bar{x}_e$, and it is translation sending $x_e$ to $\bar{x}_e$ in a coordinate chart including both $B(x_e, \eta)$ and $B(\bar{x}_e, \delta)$ and satisfying $r(T(x)) \geq r(x)$. Also, we define a $C^2$ function $\lambda$ such that $\lambda(x_e) = 1$, $\nabla\lambda(x_e) = 0$, $\Delta\lambda(x_e) = 0$, and
\[
\lambda(x) r(T(x)) \geq r(x) + r(\bar{x}_e) - r(x_e) \quad \text{on } U.
\] (50)

Since $r(\bar{x}_e) > r(x_e)$ and $r \geq 0$, we get $\lambda(x) > 0$. Finally, for $x \in U$, we define a function
\[
\begin{align*}
H(x) &= \begin{cases} 
N(x) + \left( \frac{1}{2} \right) F''(r(x_e)) \lambda(x) (T(x) - r(\bar{x}_e))^2 & \text{when } F''(r(x_e)) > 0, \\
N(x) - \left( \frac{1}{2} \right) F''(r(\bar{x}_e)) (T(x) - r(\bar{x}_e))^2 & \text{when } F''(r(x_e)) < 0, \\
N(x) + \left( \frac{1}{2} \right) Q(r(x_e)) (T(x) - r(\bar{x}_e))^2 & \text{when } F''(r(x_e)) = 0,
\end{cases}
\end{align*}
\] (51)
where \( N(x) = -F'(\hat{r}(x)) (r(T(x)) - r(\hat{x})) + F'(r(x)) (\lambda(x) r(T(x)) - r(\hat{x})) \) and \( Q(r(x)) = \sup_{t \in (r(x) - 1, r(x) + 1)} |F''(t)| \).

Let \( u(x) = F(r \circ T(x)) - F(r(\hat{x})) + H(x) \). Then one gets \( u(x) = F(r(x)) = u(x) \). Because of the fact \( F'(r(x)) \nabla r(x) = \nabla u(x) = G(r(x)^2) 2r(x) \nabla r(x) \) and the inequality (41), we get

\[
0 < F'(r(x)) = G\left( r(x) \right)^2 < 2r(x) < \frac{C}{2} G\left( r(x) \right)^{-1}. \tag{52}
\]

Moreover, we have two inequalities; that is, for \( x \in \mathcal{E} \),

the first order term of \( v(x) - u(x) = F'(r(x)) \)

the second order term of \( v(x) - u(x) = H(x) - N(x) \)

\[
\leq 0. \tag{53}
\]

Hence \( v \) is an upper-supporting function for \( u \) at the point \( x_n \).

Since \( \nabla H|_{x_n} = \nabla N|_{x_n} \), \( \nabla (\lambda(x) r(T(x))) = 0, \lambda(x) = 1, \) and \( \nabla (r \circ T) = 1 \), we have

\[
\|\nabla u|_{x_n}\| \leq \left| F'(r(x_n)) \right| + \left( \|\nabla \lambda|_{x_n}\| r(\hat{x}) + |\lambda(x)| \|\nabla (r \circ T)|_{x_n}\| \right) \leq \left| F'(r(x_n)) \right| = \|\nabla u|_{x_n}\| < \frac{C}{2}. \tag{54}
\]

By our assumption (2), the above inequality implies that

\[
\|\nabla u|_{x_n}\| < \frac{C}{2}. \tag{55}
\]

Notice that

\[
\tilde{A}_{x_n} (r \circ T(x))_{x_n} = \|D T\| \tilde{A}_{x_n} r_{x_n} = n \tilde{A}_{x_n} r_{x_n}, \tag{56}
\]

where \( \dim M = n \). By a simple calculation, we have

\[
F''(r(x)) \nabla r(x) = 2G\left( r(x) \right)^{-1} \left( -2r(x)^2 G\left( r(x) \right)^{-1} + 1 \right) \nabla r(x) \tag{57}
\]

and hence

\[
F''(r(x)) \leq 2G\left( r(x) \right)^{-1} \left( -2r(x)^2 G\left( r(x) \right)^{-1} + 1 \right) \tag{58}
\]

Using \( \|\nabla (r \circ T)\| = 1, \|\nabla u|_{x_n}\| \leq d_1, \) (52), (56), and (58), we have

\[
\tilde{A}_{x_n} |_{x_n} \leq d_1^2 F''(r(\hat{x})) + F'(r(\hat{x})) \tilde{A}_{x_n} (r \circ T)|_{x_n} + \tilde{A}_{x_n} H|_{x_n} \leq \left\{
\begin{array}{l}
F'(r(x_n)) \tilde{A}_{x_n} (r \circ T)|_{x_n} + d_1^2 (F''(r(\hat{x})) + F''(r(x_n))) \quad \text{if } F''(r(x_n)) > 0, \\
F'(r(x_n)) \tilde{A}_{x_n} (r \circ T)|_{x_n} + d_1^2 (F''(r(\hat{x})) + F''(r(x_n))) \quad \text{if } F''(r(x_n)) < 0, \\
F'(r(x_n)) \tilde{A}_{x_n} (r \circ T)|_{x_n} + d_1^2 (F''(r(\hat{x})) + Q(r(x))) \quad \text{if } F''(r(x_n)) = 0,
\end{array}
\right. \tag{59}
\]

\[
< \left( \frac{1}{2} \right) CG\left( r(x_n) \right)^{-1} n \tilde{A}_{x_n} r_{x_n} + 4d_1^2 G\left( r(x) \right)^{-1}. \tag{60}
\]

Let \( 2a \) be the distance to the closest cut point of \( o \). Because the point \( x_n \) is a cut point of \( o \), by (41) and (42), we get

\[
2a G\left( r(x_n) \right)^{-1} < r(x_n) G\left( r(x_n) \right)^{-1} \leq \frac{C}{4} G\left( r(x) \right)^{-1} \leq \frac{C}{4}, \tag{51}
\]

\[
G\left( r(x_n) \right)^{-1} < \frac{C}{8a}. \tag{62}
\]

By plugging (62) to (60), our assumption (10) tells us that, for \( r > 1, \)

\[
L \leq \frac{C}{2} \left( n + \frac{d_1^2}{a} + e_1 \right). \tag{64}
\]

So \( u \) satisfies the conditions for an \( L \)-tamed exhaustion function.

Altogether, we can conclude that \( u \) must be an \( L \)-tamed exhaustion function for \( M \).
Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References
