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Research Article

Norms and Spread of the Fibonacci and Lucas RSFMLR Circulant Matrices

Wenai Xu and Zhaolin Jiang

Department of Mathematics, Linyi University, Linyi, Shandong 276000, China

Correspondence should be addressed to Zhaolin Jiang; jzh1208@sina.com

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Circulant type matrices have played an important role in networks engineering. In this paper, firstly, some bounds for the norms and spread of Fibonacci row skew first-minus-last right (RSFMLR) circulant matrices and Lucas row skew first-minus-last right (RSFMLR) circulant matrices are given. Furthermore, the spectral norm of Hadamard product of a Fibonacci RSFMLR circulant matrix and a Lucas RSFMLR circulant matrix is obtained. Finally, the Frobenius norm of Kronecker product of a Fibonacci RSFMLR circulant matrix and a Lucas RSFMLR circulant matrix is presented.

1. Introduction

Circulant type matrices have been put on the firm basis with the work in [1–4] and so on. Circulant type matrices have significant applications in networks systems. In [5], some preliminary results on the dynamical behaviours of some specific nonmonotone Boolean automata networks which are called xor circulant networks were showed. In [6], the authors proposed a special class of the feedback delay network using circulant matrices. In [7], the impact of interior symmetries on the multiplicity of the eigenvalues of the Jacobian matrix at a fully synchronous equilibrium for the coupled cell systems associated with homogeneous networks was analyzed by Aguiar and Ruan, which was based on the circulant adjacency matrices of the networks induced by these interior symmetries. Exploiting the circulant structure of the channel matrices, the realistic near fast fading scenarios with circulant frequency selective channels were analysed by Eghbali et al. in [8]. The existence of doubly periodic travelling waves in cellular networks involving the discontinuous Heaviside step function by circulant matrix was studied by Wang and Cheng in [9].

The Fibonacci and Lucas sequences F_n and L_n are defined by the recurrence relations [10, 11]:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2, \tag{1}$$

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2. \tag{2}$$

If we start from $n = 0$, then Fibonacci and Lucas sequences are given by

$$\begin{array}{cccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ F_n & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & \cdots \\ L_n & 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & \cdots \end{array} \tag{3}$$

In [10], their Binet forms are given by

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \cos(\pi n)\left(\frac{1+\sqrt{5}}{2}\right)^{-n}\right], \\ L_n &= \left(\frac{1+\sqrt{5}}{2}\right)^n + \cos(\pi n)\left(\frac{1+\sqrt{5}}{2}\right)^{-n}. \end{aligned} \tag{4}$$

The following sum formulations for the Fibonacci and Lucas numbers are well known [11]:

$$\sum_{s=1}^{n-1} F_s^2 = F_n F_{n-1}, \tag{5}$$

$$\sum_{s=1}^{n-1} L_s^2 = L_n L_{n-1} - 2, \tag{6}$$

$$\begin{aligned} \sum_{s=1}^{n} F_s F_{s-1} &= \begin{cases} F_n^2, & n \text{ even}, \\ F_n^2 - 1, & n \text{ odd}, \end{cases} \\ \sum_{s=1}^{n} L_s L_{s-1} &= \begin{cases} L_n^2 - 4, & n \text{ even}, \\ L_n^2 + 1, & n \text{ odd}. \end{cases} \end{aligned} \tag{7}$$

Lately, some authors studied the problems of the norms of some special matrices [11–21]. The author [11] found upper and lower bounds for the spectral norms of Toeplitz matrices such that $a_{ij} \equiv F_{i-j}$ and $b_{i-j} \equiv L_{i-j}$. In [13], the authors obtain upper and lower bounds for the spectral norms of matrices $A = C_r(F_{k,0}, F_{k,1}, \ldots, F_{k,n-1})$ and $B = C_r(L_{k,0}, L_{k,1}, \ldots, L_{k,n-1})$, where $\{F_{k,n}\}_{n\in N}$ and $\{L_{k,n}\}_{n\in N}$ are k-Fibonacci and k-Lucas sequences, respectively, and they also give the bounds for the spectral norms of Kronecker and Hadamard products of these special matrices, respectively [14]. Solak and Bozkurt [16] have found out upper and lower bounds for the spectral norms of Cauchy-Toeplitz and Cauchy-Hankel matrices. Solak [18–20] has defined $A = [a_{ij}]$ and $B = [b_{ij}]$ as $n \times n$ circulant matrices, where $a_{ij} \equiv F_{(\mathrm{mod}(j-i,n))}$ and $b_{ij} \equiv L_{(\mathrm{mod}(j-i,n))}$; then he has given some bounds for the A and B matrices concerned with the spectral and Euclidean norms.

In this paper, we define two kinds of special matrices as follows.

A Fibonacci row skew first-minus-last right (RSFMLR) circulant matrix is defined as a square matrix of the form

$$\begin{pmatrix} F_0 & F_1 & \cdots & F_{n-1} \\ -F_{n-1} & F_0 - F_{n-1} & F_1 & F_{n-2} \\ \vdots & -F_{n-1} - F_{n-2} & \ddots & \vdots \\ -F_2 & \ddots & \ddots & F_1 \\ -F_1 & -F_2 - F_1 & \cdots & F_0 - F_{n-1} \end{pmatrix}. \tag{8}$$

A Lucas row skew first-minus-last right (RSFMLR) circulant matrix is defined as a square matrix of the form

$$\begin{pmatrix} L_0 & L_1 & \cdots & L_{n-1} \\ -L_{n-1} & L_0 - L_{n-1} & L_1 & L_{n-2} \\ \vdots & -L_{n-1} - L_{n-2} & \ddots & \vdots \\ -L_2 & \ddots & \ddots & L_1 \\ -L_1 & -L_2 - L_1 & \cdots & L_0 - L_{n-1} \end{pmatrix}. \tag{9}$$

Obviously, the RSFMLR circulant matrix is determined by its first row, and RSFMLR circulant matrix is a x^n+x+1 circulant matrix [22].

We define $\Theta_{(-1,-1)}$ as the basic RSFMLR circulant matrix; that is,

$$\begin{aligned} \Theta_{(-1,-1)} &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -1 & -1 & 0 & \cdots & 0 \end{pmatrix}_{n\times n} \\ &= \mathrm{RSFMLRcircfr}\,(0, 1, 0, \ldots, 0)\,. \end{aligned} \tag{10}$$

It is easily verified that $g(x) = x^n + x + 1$ has no repeated roots in its splitting field and $g(x) = x^n + x + 1$ is both the minimal polynomial and the characteristic polynomial of the matrix $\Theta_{(-1,-1)}$. In addition, $\Theta_{(-1,-1)}$ is nonderogatory and satisfies $\Theta^j_{(-1,-1)} = \mathrm{RFMLRcircfr}(\underbrace{0,\ldots,0}_{j}, 1, \underbrace{0,\ldots,0}_{n-j-1})$ and $\Theta^n_{(-1,-1)} = -I_n + \Theta_{(-1,-1)}$.

As we all know, letting $A = \mathrm{RSFMLRcircfr}(a_0, a_1, \ldots, a_{n-1})$ be a RSFMLR circulant matrix with the first row $(a_0, a_1, \ldots, a_{n-1})$, it is clear that

$$\begin{aligned} A &= \mathrm{RSFMLRcircfr}\,(a_0, a_1, \ldots, a_{n-1}) \\ &= \sum_{i=0}^{n-1} a_i \Theta^i_{(-1,-1)}. \end{aligned} \tag{11}$$

Thus, A is a RSFMLR circulant matrix if and only if $A = f(\Theta_{(-1,-1)})$ for some polynomial $f(x)$. The polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^i$ will be called the representer of the RSFMLR circulant matrix A. By (11), it is clear that A is a RSFMLR circulant matrix if and only if A commutes with $\Theta_{(-1,-1)}$; that is, $A\Theta_{(-1,-1)} = \Theta_{(-1,-1)}A$.

In addition to the algebraic properties that can be easily derived from the representation (11), we mention that RSFMLR circulant matrices have very nice structure. The product of two RSFMLR circulant matrices is a RSFMLR circulant matrix and A^{-1} is a RSFMLR circulant matrix too.

Let $A = (a_{ij})$ be an $n \times n$ matrix. The Euclidean (or Frobenius) norm, the spectral norm, the maximum column sum matrix norm, and the maximum row sum matrix norm of the matrix A are, respectively [11],

$$\|A\|_F = \left(\sum_{i,j=1}^{n} \left|a_{ij}\right|^2\right)^{1/2}, \tag{12}$$

$$\|A\|_2 = \left(\max_{1\le i\le n} \lambda_i(A^* A)\right)^{1/2}, \tag{13}$$

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} \left|a_{ij}\right|, \tag{14}$$

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} \left|a_{ij}\right|, \tag{15}$$

where A^* denotes the conjugate transpose of A. The following inequality holds:

$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F\,. \tag{16}$$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. The Hadamard product of A and B is defined by $A \circ B = [a_{ij}b_{ij}]$. If $\|\cdot\|$ is any norm on $n \times m$ matrices, then [18, 23]

$$\|A \circ B\| \le \|A\| \cdot \|B\|\,. \tag{17}$$

Kronecker product of A and B is given to be [18]

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}. \tag{18}$$

Then [18]

$$\|A \otimes B\|_F = \|A\|_F \|B\|_F\,. \tag{19}$$

Let $A = (a_{ij})$ be an $n \times n$ matrix with eigenvalues λ_i, $i = 1, 2, \ldots, n$. The spread of A is defined as [24, 25]

$$s(A) = \max_{i,j} \left|\lambda_i - \lambda_j\right|. \tag{20}$$

An upper bound for the spread due to Mirsky [24] states that

$$s(A) \le \sqrt{2\|A\|_F^2 - \frac{2}{n}|\operatorname{tr} A|^2}, \tag{21}$$

where $\|A\|_F$ denotes the Frobenius norm of A and $\operatorname{tr} A$ is the trace of A.

2. Norms and Spread of Fibonacci RSFMLR Circulant Matrices

Theorem 1. *Let $A = RSFMLRcircfr(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix, where $\{F_i\}_{0\le i\le n-1}$ denote Fibonacci numbers given by (1); then two kinds of norms of A are given by*

$$\|A\|_1 = \|A\|_\infty = 2\left(F_{n+1} - 1\right). \tag{22}$$

Proof. The matrix A is of the form (8), by (14), (15); then we have

$$\begin{aligned}\|A\|_1 &= \max_{1\le j\le n}\sum_{i=1}^{n}\left|a_{ij}\right| \\ &= \sum_{i=1}^{n-1} F_i + F_{n+1} - F_2, \\ \|A\|_\infty &= \max_{1\le i\le n}\sum_{j=1}^{n}\left|a_{ij}\right| \\ &= \sum_{i=1}^{n-1} F_i + F_{n+1} - F_2.\end{aligned} \tag{23}$$

Since the Fibonacci sequences F_n are defined by the recurrence relations (1), then we obtain

$$F_{n-1} = F_n - F_{n-2} \quad n \ge 2. \tag{24}$$

To sum up, we can get

$$\sum_{s=1}^{n-1} F_s = F_{n+1} - F_2. \tag{25}$$

Then

$$\|A\|_1 = \|A\|_\infty = 2\left(F_{n+1} - 1\right), \tag{26}$$

which completes the proof. □

Theorem 2. *Let $A = RSFMLRcircfr(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix, where $\{F_i\}_{0\le i\le n-1}$ denote Fibonacci numbers given by (1); then*

$$\begin{gathered}\sqrt{\frac{\Gamma}{n}} \le \|A\|_2, \\ \|A\|_2 \le 2\left(F_{n+1} - 1\right),\end{gathered} \tag{27}$$

where

$$\begin{aligned}\Gamma = {} & F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ & + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2.\end{aligned} \tag{28}$$

Proof. Since $F_{n+2} = F_{n+1} + F_n$ and $F(0) = 0$ given by (1), the matrix A is of the form

$$\begin{pmatrix} F_0 & F_1 & F_{n-2} & F_{n-1} \\ -F_{n-1} & -F_{n-1} & \cdots & F_{n-2} \\ \vdots & -F_{n-1} - F_{n-2} & \ddots & \vdots \\ -F_2 & \vdots & \ddots & F_1 \\ -F_1 & -F_3 & -F_{n-1} - F_{n-2} & -F_{n-1} \end{pmatrix}. \tag{29}$$

We know that $(1/\sqrt{n})\|A\|_F \le \|A\|_2 \le \|A\|_F$ from equivalent norms. By (5), we can get

$$\begin{aligned}\|A\|_F^2 &= n\sum_{i=0}^{n-1}F_i^2 + \sum_{i=1}^{n-1} iF_i^2 + 2\sum_{i=1}^{n-2} iF_iF_{i+1} \\ &= n\sum_{i=0}^{n-1}F_i^2 + \sum_{k=1}^{n-1}\sum_{i=n-k}^{n-1}F_i^2 + 2\sum_{k=1}^{n-2}\sum_{i=n-k-1}^{n-2}F_iF_{i+1} \\ &= n\sum_{i=0}^{n-1}F_i^2 + \sum_{k=1}^{n-1}\left(\sum_{i=0}^{n-1}F_i^2 - \sum_{i=0}^{n-k-1}F_i^2\right) \\ &\quad + 2\sum_{k=1}^{n-2}\left(\sum_{i=0}^{n-2}F_iF_{i+1} - \sum_{i=0}^{n-k-2}F_iF_{i+1}\right) \\ &= F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ &\quad + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2.\end{aligned} \tag{30}$$

Then

$$\frac{1}{\sqrt{n}}\|A\|_F = \sqrt{\frac{\Gamma}{n}}, \tag{31}$$

where

$$\begin{aligned}\Gamma = {} & F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ & + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2.\end{aligned} \tag{32}$$

We have

$$\sqrt{\frac{\Gamma}{n}} \le \|A\|_2. \tag{33}$$

On the other hand, suppose that

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{34}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Then

$$A = \sum_{i=0}^{n-1} F_i M_1^i - \sum_{i=1}^{n-2} F_{n-i-1} M_2^i + F_{n-1} M_3. \tag{35}$$

We can get

$$\begin{aligned} \|A\|_2 &= \left\| \sum_{i=0}^{n-1} F_i M_2^i + \sum_{i=1}^{n-2} F_{n-i-1} M_2^i + F_{n-1} M_3 \right\|_2 \\ &\le \sum_{i=0}^{n-1} F_i \|M_1\|_2^i - \sum_{i=1}^{n-2} F_{n-i-1} \|M_2\|_2^i + F_{n-1} \|M_3\|_2 . \end{aligned} \tag{36}$$

Furthermore,

$$M_1^H M_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$M_2^H M_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{37}$$

$$M_3^H M_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

We obtain

$$\|M_1\|_2 = \|M_2\|_2 = \|M_3\|_2 = 1. \tag{38}$$

The other result is obtained as follows:

$$\begin{aligned} \|A\|_2 &\le \sum_{i=1}^{n-1} F_i \|M_1\|_2^i + \sum_{i=1}^{n-2} F_{n-i-1} \|M_2\|_2^i + F_{n-1} \|M_3\|_2 \\ &= 2\sum_{i=0}^{n-1} F_i = 2\left(F_{n+1} - 1\right), \end{aligned} \tag{39}$$

which completes the proof. □

Theorem 3. *Let $A = RSFMLRcircfr(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix, where $\{F_i\}_{0\le i\le n-1}$ denote Fibonacci numbers given by (1); then the bound for the spread of A is*

$$s(A) \le \sqrt{\tau_1(n) - \frac{2}{n}\tau_2(n)}, \tag{40}$$

where

$$\begin{aligned} \tau_1(n) &= 2\left(F_1^2 + (2n-1)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \right. \\ &\quad \left. + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2\right), \\ \tau_2(n) &= \left[(n-1)F_{n-1}\right]^2. \end{aligned} \tag{41}$$

Proof. The trace of A is $\operatorname{tr} A = nF_0 + (n-1)F_{n-1}$. By Theorem 2 and inequation (21), we have

$$s(A) \le \sqrt{2\|A\|_F^2 - \frac{2}{n}\operatorname{tr} A^2}, \tag{42}$$

where

$$\begin{aligned} \|A\|_F^2 &= F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ &\quad + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2, \\ \operatorname{tr} A &= nF_0 - (n-1)F_{n-1}. \end{aligned} \tag{43}$$

We can get

$$s(A) \le \sqrt{\tau_1(n) - \frac{2}{n}\tau_2(n)}, \tag{44}$$

where

$$\begin{aligned} \tau_1(n) &= 2\left(F_1^2 + (2n-1)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \right. \\ &\quad \left. + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2\right), \\ \tau_2(n) &= \left[(n-1)F_{n-1}\right]^2, \end{aligned} \tag{45}$$

which completes the proof. □

3. Norms and Spread of Lucas RSFMLR Circulant Matrices

Theorem 4. *Let $B = RSFMLRcircfr(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix, where $\{L_i\}_{0\le i\le n-1}$ denote Lucas numbers given by (2); then two kinds of norms of B are given by*

$$\|B\|_1 = \|B\|_\infty = 2\left(L_{n+1} - 3\right) + 2. \tag{46}$$

Proof. The matrix B is of the form (9), by (14), (15); then we get

$$\begin{aligned}\|B\|_1 &= \max_{1\le j\le n}\sum_{i=1}^{n}\left|b_{ij}\right| = \sum_{i=1}^{n-1}L_i + L_0 + L_{n+1} - L_2,\\ \|B\|_\infty &= \max_{1\le i\le n}\sum_{j=1}^{n}\left|b_{ij}\right| = \sum_{i=1}^{n-1}L_i + L_0 + L_{n+1} - L_2.\end{aligned} \tag{47}$$

Since the Lucas sequences L_n are defined by the recurrence relations (2), then we obtain

$$L_{n-1} = L_n - L_{n-2} \quad n \ge 2. \tag{48}$$

To sum up, we can get

$$\sum_{s=1}^{n-1}L_s = L_{n+1} - L_2. \tag{49}$$

Then

$$\|B\|_1 = \|B\|_\infty = 2\left(L_{n+1} - 3\right) + 2, \tag{50}$$

which completes the proof. □

Theorem 5. *Let $B = RSFMLRcircfr(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix, where $\{L_i\}_{0\le i\le n-1}$ denote Lucas numbers given by (2); then*

$$\begin{aligned}\sqrt{\frac{\Pi}{n}} &\le \|B\|_2,\\ \|B\|_2 &\le 2\left(L_{n+1} - 2\right),\end{aligned} \tag{51}$$

where

$$\begin{aligned}\Pi = {} & L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2}\\ & + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1}\\ & + (2n-4)L_{n-1}L_{n-2} + 2n + 2.\end{aligned} \tag{52}$$

Proof. Since $L_{n+2} = L_{n+1} + L_n$ and $L(0) = 2$, the matrix B is of the form

$$\begin{pmatrix} L_0 & L_1 & L_{n-2} & L_{n-1}\\ -L_{n-1} & L_0 - L_{n-1} & \cdots & L_{n-2}\\ \vdots & -L_{n-1} - L_{n-2} & \ddots & \vdots\\ -L_2 & \vdots & \ddots & L_1\\ -L_1 & -L_3 & -L_{n-1} - L_{n-2} & L_0 - L_{n-1}\end{pmatrix}. \tag{53}$$

We know that $(1/\sqrt{n})\|B\|_F \le \|B\|_2 \le \|B\|_F$ from equivalent norms. By (6), we can get

$$\begin{aligned}\|B\|_F^2 &= n\sum_{i=0}^{n-1}L_i^2 + \sum_{i=1}^{n-1}iL_i^2 + 2\sum_{i=1}^{n-2}iL_iL_{i+1} - 4(n-1)L_{n-1}\\ &= n\sum_{i=0}^{n-1}L_i^2 + \sum_{k=1}^{n-1}\sum_{i=n-k}^{n-1}L_i^2 + 2\sum_{k=1}^{n-2}\sum_{i=n-k-1}^{n-2}L_iL_{i+1}\\ &\quad - 4(n-1)L_{n-1}\\ &= n\sum_{i=0}^{n-1}L_i^2 + \sum_{k=1}^{n-1}\left(\sum_{i=0}^{n-1}L_i^2 - \sum_{i=0}^{n-k-1}L_i^2\right)\\ &\quad + 2\sum_{k=1}^{n-2}\left(\sum_{i=0}^{n-2}L_iL_{i+1} - \sum_{i=0}^{n-k-2}L_iL_{i+1}\right)\\ &\quad - 4(n-1)L_{n-1}\\ &= L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2}\\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1}\\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2.\end{aligned} \tag{54}$$

Then

$$\frac{1}{\sqrt{n}}\|B\|_F = \sqrt{\frac{\Pi}{n}}. \tag{55}$$

We have

$$\sqrt{\frac{\Pi}{n}} \le \|B\|_2, \tag{56}$$

where

$$\begin{aligned}\Pi = {} & L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2}\\ & + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1}\\ & + (2n-4)L_{n-1}L_{n-2} + 2n + 2.\end{aligned} \tag{57}$$

On the other hand, supposing that

$$\begin{aligned}M_1 &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \ddots & 0 & 0\\ 0 & 0 & 0 & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ -1 & 0 & 0 & \cdots & 0 & 0\end{pmatrix},\\ M_2 &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ddots & 0 & 0\\ 0 & 0 & 0 & \cdots & 1 & 0\end{pmatrix},\\ M_3 &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & -1 & 0 & \cdots & 0 & 0\\ 0 & 0 & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -1 & 0\\ 0 & 0 & 0 & \cdots & 0 & -1\end{pmatrix},\end{aligned} \tag{58}$$

then

$$B = \sum_{i=0}^{n-1}L_iM_1^i - \sum_{i=1}^{n-2}L_{n-i-1}M_2^i + L_{n-1}M_3. \tag{59}$$

We obtain

$$\begin{aligned}\|B\|_2 &= \left\|\sum_{i=0}^{n-1} L_i M_2^i + \sum_{i=1}^{n-2} L_{n-i-1} M_2^i + L_{n-1} M_3\right\|_2 \\ &\le \sum_{i=0}^{n-1} L_i \|M_1\|_2^i - \sum_{i=1}^{n-2} L_{n-i-1} \|M_2\|_2^i + L_{n-1} \|M_3\|_2 .\end{aligned} \tag{60}$$

We have

$$\begin{aligned}M_1^H M_1 &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \\ M_2^H M_2 &= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \\ M_3^H M_3 &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.\end{aligned} \tag{61}$$

We get

$$\|M_1\|_2 = \|M_2\|_2 = \|M_3\|_2 = 1. \tag{62}$$

The other result is obtained as follows:

$$\begin{aligned}\|B\|_2 &\le \sum_{i=1}^{n-1} L_i \|M_1\|_2^i + \sum_{i=1}^{n-2} L_{n-i-1} \|M_2\|_2^i + L_{n-1} \|M_3\|_2 \\ &= 2\sum_{i=0}^{n-1} L_i = 2\left(L_{n+1} - 2\right),\end{aligned} \tag{63}$$

which completes the proof. □

Theorem 6. *Let $B = RSFMLRcircfr(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix, where $\{L_i\}_{0\le i\le n-1}$ denote Lucas numbers given by (2); then*

$$s(B) \le \sqrt{\kappa_1(n) - \frac{2}{n}\kappa_2(n)}, \tag{64}$$

where

$$\begin{aligned}\kappa_1(n) &= 2\Big(L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2\Big), \\ \kappa_2(n) &= \left[nL_0 - (n-1)L_{n-1}\right]^2 .\end{aligned} \tag{65}$$

Proof. The trace of B is $\operatorname{tr} B = nL_0 + (n-1)L_{n-1}$. By Theorem 5 and by inequation (21), we have

$$s(B) \le \sqrt{2\|B\|_F^2 - \frac{2}{n}\operatorname{tr} B^2}, \tag{66}$$

where

$$\begin{aligned}\|B\|_F^2 &= L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2, \\ \operatorname{tr} B &= nL_0 - (n-1)L_{n-1}.\end{aligned} \tag{67}$$

We obtain

$$s(B) \le \sqrt{\kappa_1(n) - \frac{2}{n}\kappa_2(n)}, \tag{68}$$

where

$$\begin{aligned}\kappa_1(n) &= 2\Big(L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2\Big), \\ \kappa_2(n) &= \left[nL_0 - (n-1)L_{n-1}\right]^2 ,\end{aligned} \tag{69}$$

which completes the proof. □

Corollary 7. *Let $A = RSFMLRcircfr(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix and let $B = RSFMLRcircfr(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix, where $\{F_i\}_{0\le i\le n-1}$ and $\{L_i\}_{0\le i\le n-1}$ denote Fibonacci numbers and Lucas numbers, respectively; then the spectral norm of Hadamard product of A and B satisfies the following inequality:*

$$\|A \circ B\|_2 \le 4\left(F_{n+1} - 1\right) \times \left(L_{n+1} - 2\right). \tag{70}$$

Proof. The proof is trivial by Theorems 2 and 5; we obtain

$$\|A\|_2 \le 2\left(F_{n+1} - 1\right), \qquad \|B\|_2 \le 2\left(L_{n+1} - 2\right). \tag{71}$$

By inequation (17), we have

$$\|A \circ B\|_2 \le 4\left(F_{n+1} - 1\right) \times \left(L_{n+1} - 2\right), \tag{72}$$

which completes the proof. □

Corollary 8. *Let $A = RSFMLRcircfr(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix and let $B = RSFMLRcircfr(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix, where $\{F_i\}_{0\le i\le n-1}$ and $\{L_i\}_{0\le i\le n-1}$ denote Fibonacci numbers and Lucas numbers, respectively; then the Frobenius norm of Kronecker product of A and B is*

$$\|A \otimes B\|_F = \sqrt{\Gamma} \times \sqrt{\Pi}, \tag{73}$$

where

$$\begin{aligned}\Gamma &= F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ &\quad + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2, \\ \Pi &= L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2.\end{aligned} \tag{74}$$

Proof. Since the proof is trivial by Theorems 2 and 5, we obtain

$$\begin{aligned}\|A\|_F^2 &= F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ &\quad + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2, \\ \|B\|_F^2 &= L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2.\end{aligned} \tag{75}$$

By (19), then

$$\|A \otimes B\|_F = \sqrt{\Gamma} \times \sqrt{\Pi}, \tag{76}$$

where

$$\begin{aligned}\Gamma &= F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ &\quad + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2, \\ \Pi &= L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2,\end{aligned} \tag{77}$$

which completes the proof. □

4. Conclusion

In this study, we define matrices of the following forms: let $A = \mathrm{RSFMLRcircfr}(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix and let $B = \mathrm{RSFMLRcircfr}(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix. Firstly, we get lower and upper bounds for the spectral norms of these matrices. Upper bounds for the spread of the matrix A and the matrix B are given. Afterwards, we obtain some corollaries related to norms of Hadamard and Kronecker products of these matrices. Based on the existing problems in [26–28], we will explore solving these problems by circulant matrices technology.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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