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Research Article

A Cubic Set Theoretical Approach to Linear Space

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The main motivation of this paper is to introduce the notion of cubic linear space. This inspiration is received from the structure of cubic sets. The notions of R-intersection, R-union, P-intersection, and P-union of cubic linear spaces are defined and we provide some results on these. We further introduce the notion of internal cubic linear space and external cubic linear space and establish some results on them.

1. Introduction

The notion of fuzzy sets introduced by Zadeh [1] in 1965 laid the foundation for the development of fuzzy Mathematics. This theory has a wide range of application in several branches of Mathematics such as logic, set theory, group theory, semigroup theory, real analysis, measure theory, and topology. After a decade, the notion of interval-valued fuzzy sets was introduced by Zadeh [2] in 1975, as an extension of fuzzy sets, that is, fuzzy sets with interval-valued membership functions. Lubczonok and Murali [3] introduced an interesting theory of flags and fuzzy subspaces of vector spaces. Katsaras and Liu [4] introduced the concepts of fuzzy vector and fuzzy topological vector spaces. Fuzzy bases of vector spaces and fuzzy vector spaces have been studied in [5, 6]. Nanda [7] introduced the notion of fuzzy field and fuzzy linear space over a fuzzy field. Wenxiang and Lu [8] redefined the concepts of fuzzy field and fuzzy linear space. Vijayabalaji et al. [9] introduced the notion of interval-valued fuzzy linear subspace and interval-valued fuzzy n-normed linear space. They have also proved that the intersection of two interval-valued fuzzy linear spaces is again an interval-valued fuzzy linear space. Atanassov [10] introduced the notion of intuitionistic fuzzy sets as a generalization of fuzzy sets.

Jun et al. [11] have introduced a remarkable theory, namely, the theory of cubic sets. This structure is comprised of an interval-valued fuzzy set and a fuzzy set. In the same paper they introduced the notion of cubic subalgebras/ideals in BCK/BCI algebras and investigated some of their properties. Moreover, Jun et al. [12] introduced the notion of cubic subgroups. They also studied images or inverse images of cubic subgroups. Furthermore, Jun et al. [13] introduced the concept of an internal cubit set and an external cubic set. Recently, Yaqoob et al. [14] introduced the notion of cubic KU ideals of KU-algebras.

Attracted by the theory of cubic sets we introduce the notion of cubic linear space. The concept of R-intersection, R-union, P-intersection, and P-union of cubic linear space are introduced and some properties are studied. We prove that the R-intersection of two cubic linear spaces is again a cubic linear space. It is shown by means of counter examples that the R-union, P-intersection, and P-union of two cubic linear spaces need not be a cubic linear space. We also introduce the notions of internal cubic linear space and external cubic linear space. It is established that the R-intersection of two internal (resp., external) cubic linear spaces is again an internal (resp., external) cubic linear space. We conclude the paper by providing examples to show that the P-intersection, P-union, and the R-union of two internal (resp., external) cubic linear spaces are not internal (resp., external) cubic linear spaces.

2. Preliminaries

In the following we provide the essential definitions and results necessary for the development of our theory.

Definition 1 (see [2]). An interval number on $[0,1]$, say $\overline{a}$, is a closed subinterval of $[0,1]$; that is, $\overline{a} = [a^-, a^+]$, where $0 \le a^- \le a^+ \le 1$. Let $D[0,1]$ denote the set of all closed subintervals of $[0,1]$; that is,

$$D[0,1] = \{\overline{a} = [a^-, a^+] : a^- \le a^+ \text{ and } a^-, a^+ \in [0,1]\}. \tag{1}$$

Definition 2 (see [2]). Let $\overline{a_i} = [a_i^-, a_i^+] \in D[0,1]$ for all $i \in \Omega$, Ω, the index set. Define

(a) $\inf^i\{\overline{a_i} : i \in \Omega\} = [\inf_{i\in\Omega} a_i^-, \inf_{i\in\Omega} a_i^+]$;

(b) $\sup^i\{\overline{a_i} : i \in \Omega\} = [\sup_{i\in\Omega} a_i^-, \sup_{i\in\Omega} a_i^+]$.

In particular, whenever $\overline{a} = [a^-, a^+]$, $\overline{b} = [b^-, b^+]$ in $D[0,1]$, one defines

(i) $\overline{a} \le \overline{b}$ if and only if $a^- \le b^-$ and $a^+ \le b^+$;

(ii) $\overline{a} = \overline{b}$ if and only if $a^- = b^-$ and $a^+ = b^+$;

(iii) $\overline{a} < \overline{b}$ if and only if $a^- < b^-$ and $a^+ < b^+$;

(iv) $\min^i\{\overline{a}, \overline{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]$;

(v) $\max^i\{\overline{a}, \overline{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}]$.

Definition 3 (see [2]). Let X be a set. A mapping $\overline{A} : X \to D[0,1]$ is called an interval-valued fuzzy set (briefly, an i-v fuzzy set) of X, where $\overline{A}(x) = [A^-(x), A^+(x)]$, for all $x \in X$, and A^- and A^+ are fuzzy sets in X such that $A^-(x) \le A^+(x)$ for all $x \in X$.

Definition 4 (see [4]). A fuzzy linear space is a pair $\mathbf{V} = (V, \mu)$, where V is a vector space over a field F, and $\mu : V \to [0,1]$ is a mapping satisfying $\mu(\alpha x + \beta y) \ge \mu(x) \wedge \mu(y)$ for any $x, y \in V$, $\alpha, \beta \in F$. Here $\wedge$ stands for intersection.

Definition 5 (see [9]). Let V denote a vector space over a field F. Let $\overline{\mu} : V \to D[0,1]$ be an interval-valued fuzzy subset of V. Then $\overline{\mu}$ is said to be an interval-valued fuzzy linear space if $\overline{\mu}(\alpha x + \beta y) \ge \min^i\{\overline{\mu}(x), \overline{\mu}(y)\}$; $x, y \in V$ and $\alpha, \beta \in F$.

Theorem 6 (see [9]). *The intersection of two interval-valued fuzzy linear spaces is again an interval-valued fuzzy linear space.*

Definition 7 (see [11]). Let X be a nonempty set. A cubic set $\mathscr{A}$ is a structure of the form $\mathscr{A} = \{\langle x, \overline{\mu}_A(x), \lambda(x)\rangle : x \in X\}$ and denoted by $\mathscr{A} = \langle \overline{\mu}_A, \lambda\rangle$. $\overline{\mu}_A = [\mu_A^-, \mu_A^+]$ is an interval-valued fuzzy set (briefly, IVF) in X and $\lambda : X \to [0,1]$ is a fuzzy set in X.

Definition 8 (see [13]). Let X be a nonempty set. A cubic set $\mathscr{A} = \langle \overline{\mu}_A, \lambda\rangle$ is said to be an internal cubic set (briefly, ICS) if $\mu_A^-(x) \le \lambda(x) \le \mu_A^+(x)$ for all $x \in X$.

Definition 9 (see [13]). Let X be a nonempty set. A cubic set $\mathscr{A} = \langle \overline{\mu}_A, \lambda\rangle$ is said to be an external cubic set (briefly, ECS) if $\lambda(x) \notin (\mu_A^-(x), \mu_A^+(x))$ for all $x \in X$.

Definition 10 (see [13]). For any $\mathscr{A}_i = \{\langle x, \overline{\mu}_i(x), \lambda_i(x)\rangle \mid x \in X\}$ where $i \in \Lambda$ (index set), one defines

(i) $\cup_{P,i\in\Lambda}\mathscr{A}_i = \{\langle x, (\cup_{i\in\Lambda}\overline{\mu}_i)(x), (\cup_{i\in\Lambda}\lambda_i)(x)\rangle \mid x \in X\}$ (P-union);

(ii) $\cap_{P,i\in\Lambda}\mathscr{A}_i = \{\langle x, (\cap_{i\in\Lambda}\overline{\mu}_i)(x), (\cap_{i\in\Lambda}\lambda_i)(x)\rangle \mid x \in X\}$ (P-intersection);

(iii) $\cup_{R,i\in\Lambda}\mathscr{A}_i = \{\langle x, (\cup_{i\in\Lambda}\overline{\mu}_i)(x), (\cap_{i\in\Lambda}\lambda_i)(x)\rangle \mid x \in X\}$ (R-union);

(iv) $\cap_{R,i\in\Lambda}\mathscr{A}_i = \{\langle x, (\cap_{i\in\Lambda}\overline{\mu}_i)(x), (\cup_{i\in\Lambda}\lambda_i)(x)\rangle \mid x \in X\}$ (R-intersection).

Definition 11 (see [13]). The complement of $\mathscr{A} = \langle \overline{\mu}_A, \lambda\rangle$ is defined to be the cubic set

$$\mathscr{A}^c = \left\{\left\langle x, (\overline{\mu}_A)^c(x), 1 - \lambda(x)\right\rangle \mid x \in X\right\}. \tag{2}$$

Theorem 12 (see [14]). *Let $\mathscr{A} = \langle \overline{\mu}, \lambda\rangle$ be a cubic subset in X, then $\mathscr{A} = \langle \overline{\mu}, \lambda\rangle$ in a cubic KU-ideal of X if and only if for all $\tilde{t} \in D[0,1]$ and $s \in [0,1]$, the set $U(\mathscr{A}; \tilde{t}, s)$ is either empty or a KU-ideal of X.*

3. A Cubic Set Theoretical Approach to Linear Space

In this section, we introduce the notion of cubic linear space as follows.

Definition 13. Let V be a linear space over a field F, $(V, \overline{\mu})$ an interval-valued fuzzy linear space, and (V, η) a fuzzy linear space. A cubic set $\mathscr{A} = \langle \overline{\mu}, \eta\rangle$ in V is called a cubic linear space of V if for all $x, y \in V$ and $\alpha, \beta \in F$,

(i) $\overline{\mu}(\alpha x * \beta y) \ge \min\{\overline{\mu}(x), \overline{\mu}(y)\}$,

(ii) $\eta(\alpha x * \beta y) \le \max\{\eta(x), \eta(y)\}$.

Example 14. Let $V = \{e, x, y, z\}$ be the Klein 4-group defined by the binary operation $*$ as follows:

$*$	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

(3)

Let F be the field GF(2). Let $(0)w = e$, $(1)w = w$ for all $w \in V$. Then V is a linear space over F.

Define an interval-valued fuzzy set $\overline{\mu}$ in V by

$$\begin{aligned} \overline{\mu}(e) &= [0.7, 0.9], \\ \overline{\mu}(x) &= [0.4, 0.5] = \overline{\mu}(y), \\ \overline{\mu}(z) &= [0.6, 0.8]. \end{aligned} \tag{4}$$

Then $\overline{\mu}$ is an interval-valued fuzzy linear space.

Define a fuzzy set η in V by

$$\eta(x) = \begin{cases} 0.5, & \text{if } x = e \\ 0.9, & \text{otherwise.} \end{cases} \tag{5}$$

Note that η is a fuzzy linear space of V.

Hence $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ is a cubic linear space of V.

Theorem 15. *Let $\mathscr{A}_1 = (\overline{\mu}_1, \eta_1)$ and $\mathscr{A}_2 = (\overline{\mu}_2, \eta_2)$ be two cubic linear spaces. Then their R-intersection $(\mathscr{A}_1 \cap \mathscr{A}_2)_R = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cup \eta_2)$ is a cubic linear space.*

Proof. Define $\overline{\mu}_1 \cap \overline{\mu}_2$ as follows

$$\begin{aligned} &(\overline{\mu}_1 \cap \overline{\mu}_2)(\alpha x * \beta y) \\ &\quad = \min\{\overline{\mu}_1(\alpha x * \beta y), \overline{\mu}_2(\alpha x * \beta y)\}. \end{aligned} \tag{6}$$

Now,

$$\begin{aligned} &(\overline{\mu}_1 \cap \overline{\mu}_2)(\alpha x * \beta y) \\ &\quad = \min\{\overline{\mu}_1(\alpha x * \beta y), \overline{\mu}_2(\alpha x * \beta y)\} \\ &\quad \geq \min\{\min[\overline{\mu}_1(x), \overline{\mu}_1(y)], \min[\overline{\mu}_2(x), \overline{\mu}_2(y)]\} \\ &\quad = \min\{\min[\overline{\mu}_1(x), \overline{\mu}_2(x)], \min[\overline{\mu}_1(y), \overline{\mu}_2(y)]\} \\ &\quad = \min\{(\overline{\mu}_1 \cap \overline{\mu}_2)(x), (\overline{\mu}_1 \cap \overline{\mu}_2)(y)\} \\ &\quad \Longrightarrow (\overline{\mu}_1 \cap \overline{\mu}_2)(\alpha x * \beta y) \\ &\quad \geq \min\{(\overline{\mu}_1 \cap \overline{\mu}_2)(x), (\overline{\mu}_1 \cap \overline{\mu}_2)(y)\}. \end{aligned} \tag{7}$$

Also define $\eta_1 \cup \eta_2$ by

$$\begin{aligned} &(\eta_1 \cup \eta_2)(\alpha x * \beta y) \\ &\quad = \max\{\eta_1(\alpha x * \beta y), \eta_2(\alpha x * \beta y)\}. \end{aligned} \tag{8}$$

So,

$$\begin{aligned} &(\eta_1 \cup \eta_2)(\alpha x * \beta y) \\ &\quad = \max\{\eta_1(\alpha x * \beta y), \eta_2(\alpha x * \beta y)\} \\ &\quad \leq \max\{\max[\eta_1(x), \eta_1(y)], \max[\eta_2(x), \eta_2(y)]\} \\ &\quad = \max\{\max[\eta_1(x), \eta_2(x)], \max[\eta_1(y), \eta_2(y)]\} \\ &\quad = \max\{(\eta_1 \cup \eta_2)(x), (\eta_1 \cup \eta_2)(y)\} \\ &\quad \Longrightarrow (\eta_1 \cup \eta_2)(\alpha x * \beta y) \\ &\quad \leq \max\{(\eta_1 \cup \eta_2)(x), (\eta_1 \cup \eta_2)(y)\}. \end{aligned} \tag{9}$$

Thus $(\mathscr{A}_1 \cap \mathscr{A}_2)_R = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cup \eta_2)$ is a cubic linear space. □

Remark 16. (i) Let $\mathscr{A}_1 = (\overline{\mu}_1, \eta_1)$ and $\mathscr{A}_2 = (\overline{\mu}_2, \eta_2)$ be two cubic linear spaces. Then their R-union $(\mathscr{A}_1 \cup \mathscr{A}_2)_R = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be a cubic linear space.

(ii) Let $\mathscr{A}_1 = (\overline{\mu}_1, \eta_1)$ and $\mathscr{A}_2 = (\overline{\mu}_2, \eta_2)$ be two cubic linear spaces. Then their P-intersection $(\mathscr{A}_1 \cap \mathscr{A}_2)_P = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be a cubic linear space.

(iii) Let $\mathscr{A}_1 = (\overline{\mu}_1, \eta_1)$ and $\mathscr{A}_2 = (\overline{\mu}_2, \eta_2)$ be two cubic linear spaces. Then their P-union $(\mathscr{A}_1 \cup \mathscr{A}_2)_P = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cup \eta_2)$ need not be a cubic linear space.

Proof. We will prove the above three statements by means of an example.

(i) Let $V = \{e, x, y, z\}$ be the Klein 4-group as in Example 14.

Let F be the field GF(2). Let $(0)w = e$, $(1)w = w$ for all $w \in V$. Then V is a linear space over F.

Define two interval-valued fuzzy sets $\overline{\mu}_1$ and $\overline{\mu}_2$ as follows:

$$\begin{aligned} \overline{\mu}_1(e) &= [0.6, 0.9], \\ \overline{\mu}_1(x) &= [0.3, 0.4] = \overline{\mu}_1(y), \\ \overline{\mu}_1(z) &= [0.5, 0.8], \\ \overline{\mu}_2(e) &= [0.5, 0.8], \\ \overline{\mu}_2(x) &= [0.4, 0.7], \\ \overline{\mu}_2(y) &= [0.2, 0.3] = \overline{\mu}_2(z). \end{aligned} \tag{10}$$

Observe that $\overline{\mu}_1$ and $\overline{\mu}_2$ are interval-valued fuzzy linear spaces of V.

Define $\overline{\mu}_1 \cup \overline{\mu}_2$ by $(\overline{\mu}_1 \cup \overline{\mu}_2)(x) = \max\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}$ for all $x, y \in V$.

So, $(\overline{\mu}_1 \cup \overline{\mu}_2)(e) = [0.6, 0.9]$, $(\overline{\mu}_1 \cup \overline{\mu}_2)(x) = [0.4, 0.7]$, $(\overline{\mu}_1 \cup \overline{\mu}_2)(y) = [0.3, 0.4]$, $(\overline{\mu}_1 \cup \overline{\mu}_2)(z) = [0.5, 0.8]$.

Thus $\overline{\mu}_1 \cup \overline{\mu}_2$ is an interval-valued fuzzy subset of V.

When $\alpha = \beta = 1$ we have

$$\begin{aligned} &(\overline{\mu}_1 \cup \overline{\mu}_2)(x * z) \\ &\quad \geq \min\{(\overline{\mu}_1 \cup \overline{\mu}_2)(x), (\overline{\mu}_1 \cup \overline{\mu}_2)(z)\} \\ &\quad \Longrightarrow (\overline{\mu}_1 \cup \overline{\mu}_2)(y) \geq \min\{[0.4, 0.7], [0.5, 0.8]\} \\ &\quad = [0.4, 0.7]. \end{aligned} \tag{11}$$

But $(\overline{\mu}_1 \cup \overline{\mu}_2)(y) = [0.3, 0.4] \geq [0.4, 0.7]$, which is absurd.

This shows that the union of two interval-valued fuzzy linear spaces need not be an interval-valued fuzzy linear space.

Now define two fuzzy sets η_1 and η_2 in V by

$$\begin{aligned} \eta_1(e) &= 0.3, \\ \eta_1(x) &= \eta_1(y) = 0.9, \\ \eta_1(z) &= 0.7, \\ \eta_2(e) &= 0.4, \\ \eta_2(x) &= 0.5, \\ \eta_2(y) &= \eta_2(z) = 0.85. \end{aligned} \tag{12}$$

We observe that η_1 and η_2 are fuzzy linear spaces over V.

Define $\eta_1 \cap \eta_2$ by $(\eta_1 \cap \eta_2)(x) = \min\{\eta_1(x), \eta_2(y)\}$.

Then $(\eta_1 \cap \eta_2)(e) = 0.3$, $(\eta_1 \cap \eta_2)(x) = 0.5$, $(\eta_1 \cap \eta_2)(y) = 0.85$, and $(\eta_1 \cap \eta_2)(z) = 0.7$.

So $(\eta_1 \cap \eta_2)$ is fuzzy subset of V.

When $\alpha = \beta = 1$ we have

$$\begin{aligned}&(\eta_1 \cap \eta_2)(x * z)\\&\quad \le \max\{(\eta_1 \cap \eta_2)(x), (\eta_1 \cap \eta_2)(z)\}\\&\quad \Longrightarrow (\eta_1 \cap \eta_2)(y) \le \max\{0.5, 0.7\} = 0.7.\end{aligned} \tag{13}$$

But $(\eta_1 \cap \eta_2)(y) = 0.85 \le 0.7$, which is absurd.

So, the intersection of two fuzzy linear spaces need not be a fuzzy linear space.

Hence the R-union $(\mathscr{A}_1 \cup \mathscr{A}_2)_R = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be a cubic linear space.

(ii) Let $\overline{\mu}_1, \overline{\mu}_2, \eta_1$ and η_2 be as in (i).

Define $\overline{\mu}_1 \cap \overline{\mu}_2$ by $(\overline{\mu}_1 \cap \overline{\mu}_2)(x) = \min\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}$ for all $x, y \in V$.

So, $(\overline{\mu}_1 \cap \overline{\mu}_2)(e) = [0.5, 0.8]$, $(\overline{\mu}_1 \cap \overline{\mu}_2)(x) = [0.3, 0.4]$, and $(\overline{\mu}_1 \cap \overline{\mu}_2)(y) = [0.2, 0.3] = (\overline{\mu}_1 \cap \overline{\mu}_2)(z)$.

One can verify that $\overline{\mu}_1 \cap \overline{\mu}_2$ is an interval-valued fuzzy linear space.

Also by (i), $\eta_1 \cap \eta_2$ is not a fuzzy linear space.

Hence the P-intersection $(\mathscr{A}_1 \cap \mathscr{A}_2)_P = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cap \eta_2)$ is not a cubic linear space.

(iii) Let $\overline{\mu}_1, \overline{\mu}_2, \eta_1$, and η_2 be as in (i).

Define $(\eta_1 \cup \eta_2)$ by $(\eta_1 \cup \eta_2)(x) = \max\{\eta_1(x), \eta_2(y)\}$.

Then $(\eta_1 \cup \eta_2)(e) = 0.4$, $(\eta_1 \cup \eta_2)(x) = (\eta_1 \cup \eta_2)(y) = 0.9$, and $(\eta_1 \cup \eta_2)(z) = 0.85$.

By verification it can be seen that $\eta_1 \cup \eta_2$ is a fuzzy linear space.

By (i), $\overline{\mu}_1 \cup \overline{\mu}_2$ is not an interval-valued fuzzy linear space.

Thus the P-union $(\mathscr{A}_1 \cup \mathscr{A}_2)_P = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cup \eta_2)$ is not a cubic linear space. □

Definition 17. Let $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ be a cubic linear space of V. Define $U(\mathscr{A}; \widetilde{t}, s) = \{x \in V \mid \overline{\mu}(x) \ge \widetilde{t} \text{ and } \eta(x) \le s\}$, where $\widetilde{t} \in D[0,1]$, $s \in [0,1]$, called the cubic level set of $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$.

Theorem 18. *Let V be a linear space over a field F. A cubic set $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ is a cubic linear space of V if and only if for all $\widetilde{t} \in D[0,1]$ and $s \in [0,1]$, the set $U(\mathscr{A}; \widetilde{t}, s)$ is either empty or a linear space of V over a field F.*

Proof. Assume that $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ is a cubic linear space of V over a field F, let $\widetilde{t} \in D[0,1]$ and $s \in [0,1]$ be such that $U(\mathscr{A}; \widetilde{t}, s) \ne \phi$, and let $x, y \in V$ be such that $x, y \in U(\mathscr{A}; \widetilde{t}, s)$; then $\overline{\mu}(x) \ge \widetilde{t}$, $\overline{\mu}(y) \ge \widetilde{t}$ and $\eta(x) \le s$, $\eta(y) \le s$.

Therefore,

$$\begin{aligned}&\overline{\mu}(\alpha x * \beta y) \ge \min\{\overline{\mu}(x), \overline{\mu}(y)\} \ge \min\{\widetilde{t}, \widetilde{t}\} = \widetilde{t}\\&\qquad \Longrightarrow \overline{\mu}(\alpha x * \beta y) \ge \widetilde{t}.\end{aligned} \tag{14}$$

Moreover,

$$\begin{aligned}&\eta(\alpha x * \beta y) \le \max\{\eta(x), \eta(y)\} \le \max\{s, s\} = s\\&\qquad \Longrightarrow \eta(\alpha x * \beta y) \le s,\\&\qquad\qquad \text{so that } \alpha x * \beta y \in U(\mathscr{A}; \widetilde{t}, s).\end{aligned} \tag{15}$$

Therefore, $U(\mathscr{A}; \widetilde{t}, s)$ is a linear space over a field F.

Conversely, suppose that $U(\mathscr{A}; \widetilde{t}, s)$ is a linear space V over a field F and let $x, y \in V$ and $\alpha, \beta \in F$ be such that $\overline{\mu}(\alpha x * \beta y) < \min\{\overline{\mu}(x), \overline{\mu}(y)\}$, $\eta(\alpha x * \beta y) > \max\{\eta(x), \eta(y)\}$.

Taking $\widetilde{\beta_1} = (1/2)\{\overline{\mu}(\alpha x * \beta y) + \min\{\overline{\mu}(x), \overline{\mu}(y)\}\}$ and $\beta_2 = (1/2)\{\eta(\alpha x * \beta y) + \max\{\eta(x), \eta(y)\}\}$, we have $\widetilde{\beta_1} \in D[0,1]$, $\beta_2 \in [0,1]$.

Also $\overline{\mu}(\alpha x * \beta y) < \widetilde{\beta_1} < \min\{\overline{\mu}(x), \overline{\mu}(y)\}$, $\eta(\alpha x * \beta y) > \beta_2 > \max\{\eta(x), \eta(y)\}$.

It follows that $x, y \in U(\mathscr{A}; \widetilde{\beta_1}, \beta_2)$ and $\alpha x * \beta y \notin U(\mathscr{A}; \widetilde{\beta_1}, \beta_2)$.

This is a contradiction and hence $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ is a cubic linear space of V over a field F. □

Definition 19. Let $\mathscr{A} = \langle \overline{\mu}_1, \eta_1 \rangle$ and $\mathscr{B} = \langle \overline{\mu}_2, \eta_2 \rangle$ be two cubic linear spaces of V over the field F. The Cartesian product of cubic linear spaces $\mathscr{A}$ and $\mathscr{B}$ is denoted by $\mathscr{A} \times \mathscr{B} = \langle \overline{\mu}_1 \times \overline{\mu}_2, \eta_1 \times \eta_2 \rangle$ defined as

(i) $(\overline{\mu}_1 \times \overline{\mu}_2)(\alpha x * \beta y) = \min\{\overline{\mu}_1(x), \overline{\mu}_2(y)\}$,

(ii) $(\eta_1 \times \eta_2)(\alpha x * \beta y) = \max\{\eta_1(x), \eta_2(y)\}$, for all $x, y \in V$ and $\alpha, \beta \in F$.

Theorem 20. *Let $(\overline{\mu}_1, \eta_1)$ and $(\overline{\mu}_2, \eta_2)$ be cubic linear spaces of V_1 and V_2 over the field F. Then $(\overline{\mu}_1 \times \overline{\mu}_2, \eta_1 \times \eta_2)$ is a cubic linear space of $V_1 \times V_2$ over F.*

Proof. Let $\overline{\mu} = \overline{\mu}_1 \times \overline{\mu}_2$, $x = (x_1, x_2)$, $y = (y_1, y_2) \in V_1 \times V_2$, and $\alpha, \beta \in F$. Consider

$$\begin{aligned}&\text{(i) } \overline{\mu}(\alpha x * \beta y)\\&\quad = (\overline{\mu}_1 \times \overline{\mu}_2)(\alpha x_1 * \beta y_1, \alpha x_2 * \beta y_2)\\&\quad = \min_{j=1,2} \overline{\mu}_j(\alpha x_j * \beta y_j)\\&\quad \ge \min_{j=1,2}\{\min[\overline{\mu}_j(x_j), \overline{\mu}_j(y_j)]\}\\&\quad = \min\left\{\left[\min_{j=1,2}\overline{\mu}_j(x_j)\right], \left[\min_{j=1,2}\overline{\mu}_j(y_j)\right]\right\}\\&\quad = \min\{\overline{\mu}(x), \overline{\mu}(y)\}.\end{aligned} \tag{16}$$

Let $\eta = \eta_1 \times \eta_2$. Then

$$\begin{aligned}&\text{(ii) } \eta(\alpha x * \beta y)\\&\quad = (\eta_1 \times \eta_2)(\alpha x_1 * \beta y_1, \alpha x_2 * \beta y_2)\\&\quad = \max_{j=1,2} \eta_j(\alpha x_j * \alpha y_j)\\&\quad \le \max_{j=1,2}\{\max[\eta_j(\alpha x_j), \eta_j(\beta y_j)]\}\\&\quad = \max\left\{\left[\max_{j=1,2}\eta_j(x_j)\right], \left[\max_{j=1,2}\eta_j(y_j)\right]\right\}\\&\quad = \max\{\eta(x), \eta(y)\}.\end{aligned} \tag{17}$$

Therefore $(\overline{\mu}_1 \times \overline{\mu}_2, \eta_1 \times \eta_2)$ is a cubic linear space of $V_1 \times V_2$ over F. □

4. Internal and External Cubic Linear Spaces

In this section, we introduce the notion of internal and external cubic linear spaces and establish some of their properties.

Definition 21. Let V be a linear space over a field F. A cubic set $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ in V is called an internal cubic linear space (briefly, ICLS) if $(\overline{\mu})^-(\alpha x * \beta y) \leq \eta(\alpha x * \beta y) \leq (\overline{\mu})^+(\alpha x * \beta y)$ for all $x, y \in V$ and $\alpha, \beta \in F$. It is denoted by

$$\mathscr{A}^I = \langle \overline{\mu}, \eta \rangle . \tag{18}$$

Example 22. Let $V = \{e, x, y, z\}$ be the Klein 4-group as in Example 14.

Define an interval-valued fuzzy set $\overline{\mu}$ in V by

$$\begin{aligned} \overline{\mu}(e) &= [0.55, 0.9], \\ \overline{\mu}(x) &= [0.4, 0.76] = \overline{\mu}(y), \\ \overline{\mu}(z) &= [0.5, 0.8]. \end{aligned} \tag{19}$$

Then $\overline{\mu}$ is an interval-valued fuzzy linear space.

Define a fuzzy set η in V by

$$\eta(x) = \begin{cases} 0.6, & \text{if } x = e \\ 0.7, & \text{otherwise.} \end{cases} \tag{20}$$

It is easy to verify that η is a fuzzy linear space of V.

When $\alpha = \beta = 1$, we have $(\overline{\mu})^-(\alpha x * \beta y) \leq \eta(\alpha x * \beta y) \leq (\overline{\mu})^+(\alpha x * \beta y)$ for all $x, y \in V$. So, $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ is an ICLS.

Definition 23. Let V be a linear space over a field F. A cubic set $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ in V is called an external cubic linear space (briefly, ECLS) if $\eta(\alpha x * \beta y) \notin ((\overline{\mu})^-(\alpha x * \beta y), (\overline{\mu})^+(\alpha x * \beta y))$ for all $x, y \in V$ and $\alpha, \beta \in F$. It is denoted by $\mathscr{A}^E = \langle \overline{\mu}, \eta \rangle$.

Example 24. Let $V = \{e, x, y, z\}$ be the Klein 4-group as in Example 14.

Define an interval-valued fuzzy set $\overline{\mu}$ in V by

$$\begin{aligned} \overline{\mu}(e) &= [0.7, 0.85], \\ \overline{\mu}(x) &= [0.4, 0.5] = \overline{\mu}(y), \\ \overline{\mu}(z) &= [0.6, 0.8]. \end{aligned} \tag{21}$$

Then $\overline{\mu}$ is an interval-valued fuzzy linear space.

Define a fuzzy set η in V by

$$\eta(x) = \begin{cases} 0.9, & \text{if } x = e \\ 0.95, & \text{otherwise.} \end{cases} \tag{22}$$

Observe that η is a fuzzy linear space of V.

So, $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ is a cubic linear space of V.

For all $\alpha > 0$ and $\beta > 0$, we have $\eta(\alpha x * \beta y) \notin ((\overline{\mu})^-(\alpha x * \beta y), (\overline{\mu})^+(\alpha x * \beta y))$.

Thus, $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ is an ECLS.

Theorem 25. *Let $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ be a cubic linear space of V which is not an ECLS. Then there exist $x, y \in V$ such that $\eta(\alpha x * \beta y) \in ((\overline{\mu})^-(\alpha x * \beta y), (\overline{\mu})^+(\alpha x * \beta y))$.*

Proof. The proof is straightforward. □

Theorem 26. *Let $\mathscr{A} = \langle \overline{\mu}, \eta \rangle$ be a cubic linear space of V. If $\mathscr{A}$ is both an ICLS and an ECLS, then $(\forall x, y \in V)$ $(\eta(\alpha x * \beta y) \in U(\overline{\mu}(\alpha x * \beta y)) \cup L(\overline{\mu}(\alpha x * \beta y)))$, where $U(A) = \{(\overline{\mu})^+(\alpha x * \beta y) \mid x, y \in V\}$ and $L(A) = \{(\overline{\mu})^-(\alpha x * \beta y) \mid x, y \in V\}$.*

Proof. Assume that $\mathscr{A}$ is both an ICLS and an ECLS. Using Definitions 21 and 23, we have $(\overline{\mu})^-(\alpha x * \beta y) \leq \eta(\alpha x * \beta y) \leq (\overline{\mu})^+(\alpha x * \beta y)$ and $\eta(\alpha x * \beta y) \notin ((\overline{\mu})^-(\alpha x * \beta y), (\overline{\mu})^+(\alpha x * \beta y))$ for all $x, y \in V$. Thus $\eta(\alpha x * \beta y) = (\overline{\mu})^-(\alpha x * \beta y)$ or $\eta(\alpha x * \beta y) = (\overline{\mu})^+(\alpha x * \beta y)$, and so $\eta(\alpha x * \beta y) \in U(\mu(\alpha x * \beta y)) \cup L(\mu(\alpha x * \beta y))$. □

Theorem 27. *Let ${\mathscr{A}_1}^I = (\overline{\mu}_1, \eta_1)$ and ${\mathscr{A}_2}^I = (\overline{\mu}_2, \eta_2)$ be two ICLSs. Then their R-intersection $(\mathscr{A}_1 \cap \mathscr{A}_2)^I{}_R = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cup \eta_2)$ is again an ICLS.*

Proof. Since ${\mathscr{A}_1}^I$ is an ICLS in V, we have $(\overline{\mu}_1)^-(\alpha x * \beta y) \leq \eta_1(\alpha x * \beta y) \leq (\overline{\mu}_1)^+(\alpha x * \beta y)$ for all $x, y \in V$ and $\alpha, \beta \in F$.

Since ${\mathscr{A}_2}^I$ is an ICLS in V, we have $(\overline{\mu}_2)^-(\alpha x * \beta y) \leq \eta_2(\alpha x * \beta y) \leq (\overline{\mu}_2)^+(\alpha x * \beta y)$ for all $x, y \in V$ and $\alpha, \beta \in F$.

This implies that

$$\begin{aligned} (\overline{\mu}_1 \cap \overline{\mu}_2)^-(\alpha x * \beta y) &\leq (\eta_1 \cup \eta_2)(\alpha x * \beta y) \\ &\leq (\overline{\mu}_1 \cap \overline{\mu}_2)^+(\alpha x * \beta y). \end{aligned} \tag{23}$$

Hence $(\mathscr{A}_1 \cap \mathscr{A}_2)^I{}_R = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cup \eta_2)$ is an ICLS. □

Theorem 28. *Let ${\mathscr{A}_1}^E = (\overline{\mu}_1, \eta_1)$ and ${\mathscr{A}_2}^I = (\overline{\mu}_2, \eta_2)$ be two ECLSs. Then their R-intersection $(\mathscr{A}_1 \cap \mathscr{A}_2)^E{}_R = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cup \eta_2)$ is again an ECLS.*

Proof. Since ${\mathscr{A}_1}^E$ is an ECLS in V, we have $\eta_1(\alpha x * \beta y) \notin ((\overline{\mu}_1)^-(\alpha x * \beta y), (\overline{\mu}_1)^+(\alpha x * \beta y))$ for all $x, y \in V$ and $\alpha, \beta \in F$.

Since ${\mathscr{A}_2}^E$ is an ECLS in V, we have $\eta_2(\alpha x * \beta y) \notin ((\overline{\mu}_2)^-(\alpha x * \beta y), (\overline{\mu}_2)^+(\alpha x * \beta y))$ for all $x, y \in V$ and $\alpha, \beta \in F$.

This implies that

$$\begin{aligned} &(\eta_1 \cup \eta_2) \\ &\quad \notin \left((\overline{\mu}_1 \cap \overline{\mu}_2)^-(\alpha x * \beta y), (\mu_1 \cap \mu_2)^+(\alpha x * \beta y)\right). \end{aligned} \tag{24}$$

Hence $(\mathscr{A}_1 \cap \mathscr{A}_2)^E{}_R = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cup \eta_2)$ is an ECLS. □

Remark 29. (i) Let ${\mathscr{A}_1}^I = (\overline{\mu}_1, \eta_1)$ and ${\mathscr{A}_2}^I = (\overline{\mu}_2, \eta_2)$ be two ICLSs. Then their P-intersection $(\mathscr{A}_1 \cap \mathscr{A}_2)^I{}_P = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be an ICLS.

(ii) Let ${\mathscr{A}_1}^E = (\overline{\mu}_1, \eta_1)$ and ${\mathscr{A}_2}^E = (\overline{\mu}_2, \eta_2)$ be two ECLSs. Then their P-intersection $(\mathscr{A}_1 \cap \mathscr{A}_2)^E{}_P = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be an ECLS.

(iii) Let ${\mathscr{A}_1}^I = (\overline{\mu}_1, \eta_1)$ and ${\mathscr{A}_2}^I = (\overline{\mu}_2, \eta_2)$ be two ICLSs. Then their P-union $(\mathscr{A}_1 \cup \mathscr{A}_2)^I{}_P = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cup \eta_2)$ need not be an ICLS.

(iv) Let ${\mathscr{A}_1}^I = (\overline{\mu}_1, \eta_1)$ and ${\mathscr{A}_2}^I = (\overline{\mu}_2, \eta_2)$ be two ECLSs. Then their P-union $(\mathscr{A}_1 \cup \mathscr{A}_2)^E{}_P = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cup \eta_2)$ need not be an ECLS.

(v) Let ${\mathscr{A}_1}^I = (\overline{\mu}_1, \eta_1)$ and ${\mathscr{A}_2}^I = (\overline{\mu}_2, \eta_2)$ be two ICLSs. Then their R-union $(\mathscr{A}_1 \cup \mathscr{A}_2)^I{}_R = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be an ICLS.

(vi) Let ${\mathscr{A}_1}^I = (\overline{\mu}_1, \eta_1)$ and ${\mathscr{A}_2}^I = (\overline{\mu}_2, \eta_2)$ be two ECLSs. Then their R-union $(\mathscr{A}_1 \cup \mathscr{A}_2)^I{}_R = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be an ECLS.

Proof. We will prove the above six statements by means of an example.

(i) Let $V = \{e, x, y, z\}$ be the Klein 4-group as in Example 14.

Define two interval-valued fuzzy sets $\overline{\mu}_1$ and $\overline{\mu}_2$ as follows:

$$\begin{aligned} &\overline{\mu}_1(e) = [0.6, 0.95], \\ &\overline{\mu}_1(x) = [0.4, 0.8] = \overline{\mu}_1(y), \\ &\overline{\mu}_1(z) = [0.5, 0.9], \\ &\overline{\mu}_2(e) = [0.63, 1], \\ &\overline{\mu}_2(x) = [0.5, 0.9], \\ &\overline{\mu}_2(y) = \overline{\mu}_2(z) = [0.3, 0.8]. \end{aligned} \tag{25}$$

Observe that $\overline{\mu}_1$ and $\overline{\mu}_2$ are interval-valued fuzzy linear space of V.

Define $(\overline{\mu}_1 \cap \overline{\mu}_2)(x) = \min\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}$ for all $x, y \in V$.

Therefore, $(\overline{\mu}_1 \cap \overline{\mu}_2)(e) = [0.6, 0.95]$, $(\overline{\mu}_1 \cap \overline{\mu}_2)(x) = [0.4, 0.8]$, and $(\overline{\mu}_1 \cap \overline{\mu}_2)(y) = (\overline{\mu}_1 \cap \overline{\mu}_2)(z) = [0.3, 0.8]$.

By routine calculations it can be seen that $\overline{\mu}_1 \cap \overline{\mu}_2$ is an interval-valued fuzzy linear space of V.

Now define two fuzzy sets η_1 and η_2 in V by

$$\begin{aligned} &\eta_1(e) = 0.65, \\ &\eta_1(x) = 0.76 = \eta_1(y), \\ &\eta_1(z) = 0.7, \\ &\eta_2(e) = 0.68, \\ &\eta_2(x) = 0.72, \\ &\eta_2(y) = \eta_2(z) = 0.75. \end{aligned} \tag{26}$$

We observe that η_1 and η_2 are fuzzy linear spaces over V.

Define $(\eta_1 \cap \eta_2)(x) = \min\{\eta_1(x), \eta_2(y)\}$.

Then $(\eta_1 \cap \eta_2)(e) = 0.65$, $(\eta_1 \cap \eta_2)(x) = 0.72$, $(\eta_1 \cap \eta_2)(y) = 0.75$, and $(\eta_1 \cap \eta_2)(z) = 0.7$.

So $\eta_1 \cap \eta_2$ is fuzzy set of V.

Here we note that $(\overline{\mu}_1)^-(\alpha x * \beta y) \le \eta_1(\alpha x * \beta y) \le (\overline{\mu}_1)^+(\alpha x * \beta y)$ for all $x, y \in V$ and

$$\begin{aligned} &(\overline{\mu}_2)^-(\alpha x * \beta y) \\ &\quad \le \eta_2(\alpha x * \beta y) \le (\overline{\mu}_2)^+(\alpha x * \beta y) \quad \forall x, y \in V, \\ &(\eta_1 \cap \eta_2)(x * z) \\ &\quad \le \max\{(\eta_1 \cap \eta_2)(x), (\eta_1 \cap \eta_2)(z)\} \\ &\quad \Longrightarrow (\eta_1 \cap \eta_2)(y) \le \max\{0.72, 0.7\} = 0.72. \end{aligned} \tag{27}$$

But $(\eta_1 \cap \eta_2)(y) = 0.75 \le 0.72$, which is absurd.

So, the intersection of two ICLSs need not be an ICLS.

That is, the P-intersection of two ICLSs $(\mathscr{A}_1 \cap \mathscr{A}_2)^I{}_P = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be an ICLS.

(ii) Let $V = \{e, x, y, z\}$ be the Klein 4-group as in Example 14.

Define two interval-valued fuzzy sets $\overline{\mu}_1$ and $\overline{\mu}_2$ as follows:

$$\begin{aligned} &\overline{\mu}_1(e) = [0.9, 1], \\ &\overline{\mu}_1(x) = [0.5, 0.6] = \overline{\mu}_1(y), \\ &\overline{\mu}_1(z) = [0.6, 0.7], \\ &\overline{\mu}_2(e) = [0.8, 0.9], \\ &\overline{\mu}_2(x) = [0.7, 0.8], \\ &\overline{\mu}_2(y) = [0.3, 0.5] = \overline{\mu}_2(z). \end{aligned} \tag{28}$$

Observe that $\overline{\mu}_1$ and $\overline{\mu}_2$ are interval-valued fuzzy linear space of V.

Define $(\overline{\mu}_1 \cap \overline{\mu}_2)(x) = \min\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}$ for all $x, y \in V$.

Therefore, $(\overline{\mu}_1 \cap \overline{\mu}_2)(e) = [0.8, 0.9]$, $(\overline{\mu}_1 \cap \overline{\mu}_2)(x) = [0.5, 0.6]$, and $(\overline{\mu}_1 \cap \overline{\mu}_2)(y) = [0.3, 0.5] = (\overline{\mu}_1 \cap \overline{\mu}_2)(z)$.

By routine calculations it can be seen that the intersection of two interval-valued fuzzy linear spaces is again an interval-valued fuzzy linear space.

Now define two fuzzy sets η_1 and η_2 in V by

$$\begin{aligned} &\eta_1(e) = 0.3, \\ &\eta_1(x) = 0.5, \\ &\eta_1(y) = \eta_1(z) = 0.9, \\ &\eta_2(e) = 0.4, \\ &\eta_2(x) = \eta_2(y) = 0.85, \\ &\eta_2(z) = 0.6. \end{aligned} \tag{29}$$

We observe that η_1 and η_2 are fuzzy linear spaces over V.

Here we note that $\eta_1(\alpha x * \beta y) \notin ((\overline{\mu}_1)^-(\alpha x * \beta y), (\overline{\mu}_1)^+(\alpha x * \beta y))$ for all $x, y \in V$ and

$$\eta_2(\alpha x * \beta y) \notin \left((\overline{\mu}_2)^-(\alpha x * \beta y), (\overline{\mu}_2)^+(\alpha x * \beta y)\right) \quad \forall x, y \in V. \tag{30}$$

Now define $(\eta_1 \cap \eta_2)(x) = \min\{\eta_1(x), \eta_2(y)\}$.

Then, $(\eta_1 \cap \eta_2)(e) = 0.3$, $(\eta_1 \cap \eta_2)(x) = 0.5$, $(\eta_1 \cap \eta_2)(y) = 0.85(\eta_1 \cap \eta_2)(z) = 0.6$.

So $(\eta_1 \cap \eta_2)$ is fuzzy subset of V. Consider

$$\begin{aligned} &(\eta_1 \cap \eta_2)(x * z) \\ &\quad \le \max\{(\eta_1 \cap \eta_2)(x), (\eta_1 \cap \eta_2)(z)\} \\ &\quad \Longrightarrow (\eta_1 \cap \eta_2)(y) \le \max\{0.5, 0.6\} = 0.6. \end{aligned} \tag{31}$$

But $(\eta_1 \cap \eta_2)(y) = 0.85 \le 0.6$, which is absurd.

So, the intersection of two ECLSs need not be an ECLS.

That is, P-intersection of two ECLSs $(\mathscr{A}_1 \cap \mathscr{A}_2)^E{}_P = (\overline{\mu}_1 \cap \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be an ECLS.

(iii) Let $V = \{e, x, y, z\}$ be the Klein 4-group as in Example 14.

Define two interval-valued fuzzy sets $\overline{\mu}_1$ and $\overline{\mu}_2$ as follows:

$$\begin{aligned} &\overline{\mu}_1(e) = [0.6, 0.95], \\ &\overline{\mu}_1(x) = [0.4, 0.8] = \overline{\mu}_1(y), \\ &\overline{\mu}_1(z) = [0.5, 0.85], \\ &\overline{\mu}_2(e) = [0.63, 1], \\ &\overline{\mu}_2(x) = [0.5, 0.9], \\ &\overline{\mu}_2(y) = \overline{\mu}_2(z) = [0.3, 0.8]. \end{aligned} \tag{32}$$

Observe that $\overline{\mu}_1$ and $\overline{\mu}_2$ are interval-valued fuzzy linear space of V.

Define $(\overline{\mu}_1 \cup \overline{\mu}_2)(x) = \max\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}$ for all $x, y \in V$.

Therefore, $(\overline{\mu}_1 \cup \overline{\mu}_2)(e) = [0.63, 1]$, $(\overline{\mu}_1 \cup \overline{\mu}_2)(x) = [0.5, 0.9]$, $(\overline{\mu}_1 \cup \overline{\mu}_2)(y) = [0.4, 0.8]$, and $(\overline{\mu}_1 \cup \overline{\mu}_2)(z) = [0.5, 0.85]$.

Thus $\overline{\mu}_1 \cup \overline{\mu}_2$ is an i-v fuzzy subset of V. Consider

$$\begin{aligned} &(\overline{\mu}_1 \cup \overline{\mu}_2)(x * z) \\ &\quad \geq \min\{(\overline{\mu}_1 \cup \overline{\mu}_2)(x), (\overline{\mu}_1 \cup \overline{\mu}_2)(z)\} \\ &\quad \Longrightarrow (\overline{\mu}_1 \cup \overline{\mu}_2)(y) \geq \min\{[0.5, 0.9], [0.5, 0.85]\} \\ &\quad = [0.5, 0.85]. \end{aligned} \tag{33}$$

But $(\overline{\mu}_1 \cup \overline{\mu}_2)(y) = [0.4, 0.8] \geq [0.5, 0.85]$, which is absurd.

This shows that the union of two interval-valued fuzzy linear spaces need not be an interval-valued fuzzy linear space.

Now define two fuzzy sets η_1 and η_2 in V by

$$\begin{aligned} &\eta_1(e) = 0.65, \\ &\eta_1(x) = 0.76 = \eta_1(y), \\ &\eta_1(z) = 0.7, \\ &\eta_2(e) = 0.68, \\ &\eta_2(x) = 0.72, \\ &\eta_2(y) = \eta_2(z) = 0.75. \end{aligned} \tag{34}$$

We observe that η_1 and η_2 are fuzzy linear spaces over V.

Define $(\eta_1 \cap \eta_2)(x) = \min\{\eta_1(x), \eta_2(y)\}$.

Then $(\eta_1 \cap \eta_2)(e) = 0.65$, $(\eta_1 \cap \eta_2)(x) = 0.72$, $(\eta_1 \cap \eta_2)(y) = 0.75$, and $(\eta_1 \cap \eta_2)(z) = 0.7$.

So $(\eta_1 \cap \eta_2)$ is fuzzy subset of V.

Here we note that $(\overline{\mu}_1)^-(\alpha x * \beta y) \leq \eta_1(\alpha x * \beta y) \leq (\overline{\mu}_1)^+(\alpha x * \beta y)$ for all $x, y \in V$ and

$$\begin{aligned} &(\overline{\mu}_2)^-(\alpha x * \beta y) \\ &\quad \leq \eta_2(\alpha x * \beta y) \leq (\overline{\mu}_2)^+(\alpha x * \beta y) \quad \forall x, y \in V, \\ &(\eta_1 \cap \eta_2)(x * z) \\ &\quad \leq \max^i\{(\eta_1 \cap \eta_2)(x), (\eta_1 \cap \eta_2)(z)\} \\ &\quad \Longrightarrow (\eta_1 \cap \eta_2)(y) \leq \max\{0.72, 0.7\} = 0.72. \end{aligned} \tag{35}$$

But $(\eta_1 \cap \eta_y)(y) = 0.75 \leq 0.7$, which is absurd.

So, the intersection of two ICLSs need not be an ICLS.

That is, the P union of two ICLSs, $(\mathscr{A}_1 \cup \mathscr{A}_2)^I{}_P = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cup \eta_2)$ need not be an ICLS.

(iv) Let $V = \{e, x, y, z\}$ be the Klein 4-group as in Example 14.

Define two interval-valued fuzzy sets $\overline{\mu}_1$ and $\overline{\mu}_2$ as follows:

$$\begin{aligned} &\overline{\mu}_1(e) = [0.9, 1], \\ &\overline{\mu}_1(x) = [0.5, 0.6] = \overline{\mu}_1(y), \\ &\overline{\mu}_1(z) = [0.6, 0.7], \\ &\overline{\mu}_2(e) = [0.8, 0.9], \\ &\overline{\mu}_2(x) = [0.7, 0.8], \\ &\overline{\mu}_2(y) = [0.3, 0.5] = \overline{\mu}_2(z). \end{aligned} \tag{36}$$

Observe that $\overline{\mu}_1$ and $\overline{\mu}_2$ are interval-valued fuzzy linear space of V.

Define $(\overline{\mu}_1 \cup \overline{\mu}_2)(x) = \max\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}$ for all $x, y \in V$.

Therefore, $(\overline{\mu}_1 \cup \overline{\mu}_2)(e) = [0.9, 1]$, $(\overline{\mu}_1 \cup \overline{\mu}_2)(x) = [0.7, 0.8]$, $(\overline{\mu}_1 \cup \overline{\mu}_2)(y) = [0.5, 0.6]$, $(\overline{\mu}_1 \cup \overline{\mu}_2)(z) = [0.6, 0.7]$.

Thus $\overline{\mu}_1 \cup \overline{\mu}_2$ is an i-v fuzzy subset of V

$$\begin{aligned} &(\overline{\mu}_1 \cup \overline{\mu}_2)(x * z) \\ &\quad \geq \min\{(\overline{\mu}_1 \cup \overline{\mu}_2)(x), (\overline{\mu}_1 \cup \overline{\mu}_2)(z)\} \\ &\quad \Longrightarrow (\overline{\mu}_1 \cup \overline{\mu}_2)(y) \geq \min\{[0.7, 0.8], [0.6, 0.7]\} \\ &\quad = [0.6, 0.7]. \end{aligned} \tag{37}$$

But $(\overline{\mu}_1 \cup \overline{\mu}_2)(y) = [0.5, 0.6] \geq [0.6, 0.7]$, which is absurd.

Now define two fuzzy sets η_1 and η_2 in V by

$$\begin{aligned} &\eta_1(e) = 0.3, \\ &\eta_1(x) = 0.4, \\ &\eta_1(y) = \eta_1(z) = 0.9, \\ &\eta_2(e) = 0.4, \\ &\eta_2(x) = \eta_2(y) = 0.85, \\ &\eta_2(z) = 0.6. \end{aligned} \tag{38}$$

We observe that η_1 and η_2 are fuzzy linear spaces over V.

Here we note that $\eta_1(\alpha x * \beta y) \notin ((\overline{\mu}_1)^-(\alpha x * \beta y), (\overline{\mu}_1)^+(\alpha x * \beta y))$ for all $x, y \in V$ and

$$\eta_2(\alpha x * \beta y) \notin \left((\overline{\mu}_2)^-(\alpha x * \beta y), (\overline{\mu}_2)^+(\alpha x * \beta y)\right) \quad \forall x, y \in V. \tag{39}$$

Now define $(\eta_1 \cup \eta_2)(x) = \max\{\eta_1(x), \eta_2(y)\}$

Then, $(\eta_1 \cup \eta_2)(e) = 0.4$, $(\eta_1 \cup \eta_2)(x) = 0.85$, $(\eta_1 \cap \eta_2)(y) = (\eta_1 \cup \eta_2)(z) = 0.9$.

So $(\eta_1 \cup \eta_2)$ is fuzzy subset of V

$$\begin{aligned} &(\eta_1 \cup \eta_2)(x * y) \\ &\quad \leq \max\{(\eta_1 \cup \eta_2)(x), (\eta_1 \cup \eta_2)(y)\}. \end{aligned} \tag{40}$$

By routine calculations it can be seen that the union of two fuzzy linear spaces is also a fuzzy linear space.

That is, P-union of two ECLSs $(\mathscr{A}_1 \cup \mathscr{A}_2)^E{}_P = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cup \eta_2)$ need not be an ECLS.

(v) From (i) and (ii), R-union of two ICLSs $(\mathscr{A}_1 \cup \mathscr{A}_2)^I{}_R = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be an ICLS.

(vi) From (ii) and (iv), R-union of two ECLSs $(\mathscr{A}_1 \cup \mathscr{A}_2)^E{}_R = (\overline{\mu}_1 \cup \overline{\mu}_2, \eta_1 \cap \eta_2)$ need not be an ECLS. □

Theorem 30. *Let $\mathscr{A} = (\overline{\mu}, \eta)$ be a cubic linear space of V. If $\mathscr{A}$ is an ICLS (resp., ECLS), then $\mathscr{A}^c$ is an ICLS (resp., ECLS).*

Proof. Since $\mathscr{A} = (\overline{\mu}, \eta)$ is an ICLS (resp., ECLS) in V, we have $(\overline{\mu})^-(\alpha x * \beta y) \leq \eta(\alpha x * \beta y) \leq (\overline{\mu})^+(\alpha x * \beta y)$ (resp.$\eta(\alpha x * \beta y) \notin ((\overline{\mu})^-(\alpha x * \beta y), (\overline{\mu})^+(\alpha x * \beta y))$) for all $x, y \in V$. This implies that

$$\begin{aligned} &1 - (\overline{\mu})^-(\alpha x * \beta y) \leq 1 - \eta(\alpha x * \beta y) \leq 1 - (\overline{\mu})^+(\alpha x * \beta y) \\ &\text{resp., } 1 - \eta(\alpha x * \beta y) \notin \left(1 - (\overline{\mu})^-(\alpha x * \beta y), 1 - (\overline{\mu})^+(\alpha x * \beta y)\right). \end{aligned} \tag{41}$$

Hence $\mathscr{A}^c = \{\langle (\overline{\mu})^c(\alpha x * \beta y), 1 - \eta(\alpha x * \beta y)\rangle \mid x, y \in V\}$ is an ICLS (resp., ECLS) in V. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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