Research Article

On the Boundary of Self-Affine Sets

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1. Introduction

Let \((X, \rho)\) be a complete matric space. Recall that a map \(S : X \rightarrow X\) is contractive if there exists a constant \(0 < r < 1\) such that \(\rho(S(x), S(y)) \leq r \rho(x, y)\). We call a finite set of contractive maps \(\{S_j\}_{j=1}^m\) an iterated function system (IFS). It is well known [1] that there exists a unique nonempty compact subset \(K \subset X\) such that \(K = \bigcup_{j=1}^m S_j(K)\). We call \(K\) the invariant set or attractor of the IFS. Moreover, if we associate the IFS with a set of probability weights \(\{p_j > 0 : i = 1, \ldots, m\}\), then there exists a unique probability measure \(\mu\) supported on \(K\) satisfying the equation

\[
\mu(\cdot) = \sum_{j=1}^m p_j \mu(S_j^{-1}(\cdot)).
\]

(1)

We call \(\mu\) the invariant measure.

Let \(A\) be a \(d \times d\) expanding real matrix; that is, all its eigenvalues have modules larger than one. Let \(\lambda\) be the smallest absolute value of \(A\)'s eigenvalues, choose \(c \in (1, \lambda)\), and define \(\|x\|\) for each \(x \in \mathbb{R}^d\) as

\[
\|x\| = \sum_{n=1}^{\infty} c^n |A^{-n}x|,
\]

(2)

where \(|\cdot|\) is the Euclidian norm in \(\mathbb{R}^d\). Then \(\|\cdot\|\) is a norm in \(\mathbb{R}^d\). Let \(\rho(x, y) = \|x - y\|\) be the induced metric. It is easy to check that the map \(S(x) = A^{-1}(x + c)\) with \(x, c \in \mathbb{R}^d\) is contractive under the metric \(\rho\).

Let \(A\) be a \(d \times d\) expanding real matrix and \(D = \{d_1, d_2, \ldots, d_m\} \subset \mathbb{R}^d\). We call the family of maps on \(\mathbb{R}^d\)

\[
S_i(x) = A^{-1}(x + d_i), \quad i = 1, 2, \ldots, m
\]

(3)

a self-affine IFS. The corresponding invariant set \(K\) and invariant measure \(\mu\) are called a self-affine set and a self-affine measure of the IFS, respectively. Furthermore, if the matrix \(A\) in (3) is an orthonormal matrix multiple a constant, then such IFS is called self-similar, and the invariant set and invariant measure are called self-similar set and self-similar measure of the IFS, respectively.

Our main interests in this note are the structures and properties of the boundary \(\partial K\) of a self-affine set \(K\). For self-similar IFS, Lau and Xu [2] showed that \(\dim_H(\partial K) < d\) provided that the self-similar IFS satisfies the open set condition (OSC). He et al. [3] studied the calculation of \(\dim_H(\partial K)\) for integral self-similar IFS. Furthermore, the overlapping cases were considered by Lau and Ngai in [4]. For self-affine sets, however, less is known about \(K\) and \(\partial K\) (see [5–7]). There is no method to compute the Hausdorff dimension and the Lebesgue measure of \(\partial K\) for overlapping self-affine set.

Motivated by these results, we consider the Lebesgue measures of the boundaries of integral self-affine sets. We prove that they have Lebesgue measure zero.
Theorem 1. Let \( \{ \phi_j \}_{j=1}^m \) be a self-affine IFS defined on \( \mathbb{R}^d \). Assume that \( A \) and \( d_j \) are all integral. Let \( K \) be the self-affine set of the IFS; then \( \mathcal{L}(\partial K) = 0 \).

Consider two IFSs \( \{ S_j \}_{j=1}^n \) and \( \{ S_j \}_{j=1}^m \), \( m < n \) (they may not be self-affine). Let \( K_1 \) and \( K_2 \) be the invariant sets, respectively; then \( K_1 \subseteq K_2 \), so \( \dim(K_1) \leq \dim(K_2) \). We think about the natural question: what is the relationship between \( \partial K_1 \) and \( \partial K_2 \)?

We prove that any one case of \( \dim_{H}(\partial K_2) = \dim_{H}(\partial K_1) \), \( \dim_{H}(\partial K_2) < \dim_{H}(\partial K_1) \), and \( \dim_{H}(\partial K_2) > \dim_{H}(\partial K_1) \) may occur.

2. Proofs of Results

For an IFS \( \{ S_j \}_{j=1}^n \) on \( \mathbb{R}^d \), we use the following notations throughout the paper. Let \( \Sigma = \{ 1, \ldots, m \} \) (or \( \Sigma^* \) if there is no confusion), and \( \Sigma^* = \bigcup_{n \leq m} \Sigma^n \). For any \( i_1 i_2 \cdots i_k \in \Sigma^n \) and \( j = j_1 j_2 \cdots j_k \in \Sigma^k \), let \( I_j = i_1 i_2 \cdots i_k j_1 j_2 \cdots j_k \) and

\[
P_I = p_{i_1} p_{i_2} \cdots p_{i_k}, \quad S_I = S_{i_1} \circ S_{i_2} \cdots \circ S_{i_k},
\]

\[
d_i = d_{i_1} + Ad_{i_2} + \cdots + A^{i_k-1} d_{i_k},
\]

\[
\mathcal{D}_n = \mathcal{D} + AD + \cdots + A^{n-1} \mathcal{D}.
\]

Also, we use \( \mathcal{L}(E) \), \( E^0 \), and \( \partial E \) to denote the Lebesgue measure, the interior, and the boundary of a subset \( E \subset \mathbb{R}^d \), respectively.

Theorem 2. Let \( \{ \phi_j \}_{j=1}^m \) and \( \{ \psi_j \}_{j=1}^k \) be two contractive IFSs on \( \mathbb{R}^d \) under some norm \( \| \cdot \| \) with the invariant sets \( K_1 \) and \( K_2 \), respectively. If the invariant set \( K_1 \) contains interior points, then there exist \( a, n \in \mathbb{N} \) and \( \alpha \in \mathbb{Z}^d \) such that the IFS \( \mathcal{F} = \{ \phi_j : 1 \leq i \leq m \} \) and \( \mathcal{F} \cup \mathcal{G} \) generate the same attractor \( aK_1 + \alpha \), where \( \mathcal{G} = \{ \psi_j : 1 \leq j \leq k \} \) and \( \phi_j(x) = a \phi_j(a^{-1}(x - \alpha)) + \alpha \), \( j = 1, \ldots, m \).

Proof. Observe that

\[
\bigcup_{j=1}^m \phi_j(aK_1 + \alpha) = \bigcup_{j=1}^m (a \phi_j(K_1) + \alpha)
\]

\[
= a \left( \bigcup_{j=1}^m \phi_j(K_1) \right) + \alpha = aK_1 + \alpha.
\]

This means that \( aK_1 + \alpha \) is the invariant set of \( \{ \phi_j \}_{j=1}^m \) for any \( a > 0 \) and \( \alpha \in \mathbb{R}^d \). Hence it is also the invariant set of the IFS \( \mathcal{F} \). Now we need only to prove that \( aK_1 + \alpha \) is the invariant set of \( \mathcal{F} \cup \mathcal{G} \) for some \( a, n \in \mathbb{N} \) and \( \alpha \in \mathbb{Z}^d \).

Note that \( K_1 \) contains interior points; we can find a constant \( r > 0 \) and a point \( x_0 \in K_1 \) with rational entries such that \( B_2(x_0) \subset K_1 \). Hence \( B_{2r}(0) \subset aK_1 - ax_0 \) for all positive real number \( a > 0 \). Since \( \{ \psi_j \}_{j=1}^k \) are contractive in the norm \( \| \cdot \| \), we can choose integers \( a, n \in \mathbb{N} \) large enough such that \( K_2 \subset B_{2r}(0) \) and \( \| \psi_j(aK_1 + \alpha) \| < ar \) for all \( j \in \Sigma_k \) with \( |J| \geq n \), where \( |E| \) is the diameter of the set \( E \subset \mathbb{R}^d \) under the norm \( \| \cdot \| \). Also, we can assume that \( \alpha = -ax_0 \in \mathbb{Z}^d \). Noting \( K_2 \subseteq B_{2r}(0) \subseteq B_{2ar}(0) \subseteq aK_1 + \alpha \), \( \| \psi_{j_i-j_n}(aK_1 + \alpha) \| < ar \) and observing

\[
\psi_{j_i-j_n}(aK_1 + \alpha) \cap K_2 \supseteq \psi_{j_i-j_n}(K_2) \cap K_2 \neq \emptyset,
\]

we have

\[
\psi_{j_i-j_n}(aK_1 + \alpha) \subseteq aK_1 + \alpha.
\]

Therefore

\[
|\psi_{j_i-j_n}(K_2) \cap K_2| \geq \psi_{j_i-j_n}(K_2) \cap K_2 \neq \emptyset,
\]

we have

\[
\psi_{j_i-j_n}(aK_1 + \alpha) \subseteq aK_1 + \alpha.
\]

Therefore

\[
aK_1 + \alpha = \bigcup_{f \in \mathcal{F} \cup \mathcal{G}} f(aK_1 + \alpha) \subseteq \bigcup_{f \in \mathcal{F} \cup \mathcal{G}} f(aK_1 + \alpha) \subseteq aK_1 + \alpha.
\]

We see that \( aK_1 + \alpha \) is the invariant set of \( \mathcal{F} \cup \mathcal{G} \). This completes the proof.

In Theorem 2, IFS \( \mathcal{F} \) is a subset of IFS \( \mathcal{F} \cup \mathcal{G} \) and they have the same invariant set \( aK_1 + \alpha \). So do the same boundary of the invariant set. On the other hand, the invariant set of \( \mathcal{G} \) is \( K_2 \). Obviously, either \( \dim_{H}(\partial(aK_1 + \alpha)) < \dim_{H}(\partial K_2) \) or \( \dim_{H}(\partial(aK_1 + \alpha)) > \dim_{H}(\partial K_2) \) may occur.

In the following, we consider the Lebesgue measure of \( \partial K \) for the self-affine IFS (3). We will prove Theorem 1; that is, \( \mathcal{L}(\partial K) = 0 \) if \( A \) and \( d_j \) are all integral. For this, we first prove some lemmas.

Lemma 3. Let the IFS in (3) be integral; that is, all entries of \( A \) and \( d_j \) are integers. Assume that the self-affine set \( K \) has positive Lebesgue measure; then \( K^0 \neq \emptyset \).

Proof. Note that the fact that \( A \) and \( d_j \) are all integral implies that the IFS is uniformly discrete, and the assertion follows from [7, Theorem 3.1].

Lemma 4. Let the IFS in (3) be integral. Suppose that \( \{ d_j \}_{j=1}^m \) contains a complete set of residues (mod \( AZ^d \)). Then the self-affine measure \( \mu \) in (1) is absolutely continuous with respect to the Lebesgue measure provided that

\[
\sum_{j \in \{ d_i \}_{i=1}^m} p_j = \frac{1}{|\det(A)|}, \quad i = 1, \ldots, m.
\]

Proof. Without loss of generality, assume that \( \overline{A} = \{ d_i \}_{i=1}^m \) is a complete set of residues (mod \( AZ^d \)) with \( |\det(A)| = \ell \). Then \( \overline{A} := A \overline{A} + \cdots + A^{\ell-1} \overline{A} \) is a complete set of residues (mod \( AZ^d \)).

For each \( i \in \{ 1, \ldots, \ell \} \), let \( I_i = \{ j : 1 \leq j \leq m, (d_j - d_i) \in AZ^d \} \) and \( p_i = 1/\ell \# I_i \) if \( j \in I_i \); then we have

\[
\sum_{j \in I_i} p_j = \frac{1}{|\det(A)|}, \quad i = 1, \ldots, \ell.
\]

Hence such probability weights \( \{ p_j \}_{j=1}^m \) satisfying (9) always exist.
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To prove the absolute continuity of \( \mu \), by making use of [8, Theorem 3.5], we need only to show that

\[
\sum_{j \in \Sigma, d_j = z} p_j \leq |\det(A)|^{-n}, \quad \forall n > 0, \ z \in \mathbb{Z}^d.
\]

We will prove this by induction on \( n \). By (9), the inequality (11) holds for \( n = 1 \). Assume that (11) holds for \( n = k \). Let \( z = d_1 + A z_1 \) with \( d_1 \in \mathcal{D} \) and \( z_1 \in \mathbb{Z}^d \). If \( j \in \Sigma \), \( j \in \Sigma \), and \( d_{ij} = z_1 \), then \( d_j + Ad_j = d_1 + A z_1 \), so \( (d_j - d_i) \in \mathbb{A}^d \), and let \( d_j = d_i + A e_j \) with \( e_j \in \mathbb{Z}^d \); we have \( e_j + d_i = z_1 \). Therefore

\[
\sum_{j \in \Sigma, d_j = z} p_j \leq \sum_{j \in \Sigma, d_j = z} p_j \sum_{j \in \Sigma, d_j = z} p_j \leq |\det(A)|^{-k} \sum_{j \in \Sigma, d_j = z} p_j \leq |\det(A)|^{-k+1}.
\]

Hence (11) is also true for \( n = k + 1 \). This completes the proof. \( \square \)

**Remark.** Lemma 4 gives a sufficient condition for the existence of \( L^1 \)-solutions of integral refinement equations:

\[
f(x) = |\det(A)| \sum_{j=1}^{m} p_j f(Ax - d_j)
\]

provided that \( \{d_1, \ldots, d_m\} \subset \mathbb{Z}^d \) contains a complete set of residues (mod \( A \mathbb{Z}^d \)). Condition (9) ensures that the refinement equation has a unique (up to a scalar multiple) bounded \( L^1 \)-solution with compact support if \( p_j \)'s satisfy (9). Condition (9) is an extension of the “sum role.”

**Lemma 5.** Let the IFS in (3) be integral. Suppose \( \{d_j\}_{j=1}^{m} \subset \mathbb{Z}^d \) contains a complete set of residues (mod \( A \mathbb{Z}^d \)); \( K \) is the corresponding self-affine set. Then \( \mathcal{L}(\partial K) = 0 \).

**Proof.** Lemma 4 implies that there exist probability weights \( \{p_j\}_{j=1}^{m} \) such that the corresponding self-affine measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure and so \( \mathcal{L}(K) > 0 \).

Lemma 3 implies that \( K^o \neq \emptyset \), so \( K^o \) is a nonempty invariant open set (i.e., \( \bigcup_{n=1}^{m} S_n(K^o) \subset K^o \) and \( \mu(K^o) > 0 \)). Then [8, Theorem 4.13] implies that \( \mu(\partial K) = 0 \). On the other hand, [8, Theorem 3.12] implies that the Lebesgue measure restricted on \( K \) is also absolutely continuous with respect to \( \mu \). Hence \( \mathcal{L}(\partial K) = 0 \). \( \square \)

Now we can prove the main theorem of the paper.

**Proof of Theorem 1.** If \( K^o = \emptyset \), then \( \partial K = K \) and Lemma 3 implies that \( \mathcal{L}(\partial K) = 0 \).

Now we consider the case \( K^o \neq \emptyset \). Let \( \phi_i(x) = A^{-1}(x + d_i) \), \( \psi_j(x) = a \phi_i(a^{-1}(x - \alpha)) + \alpha = A^{-1}(x - \alpha + ad_j + A \alpha) \), \( j = 1, \ldots, m \), and \( \psi_i(x) = A^{-1}(x + z_i), i = 1, \ldots, k \), where \( \mathcal{L} = \{z_1 = 0, \ldots, z_k\} \) is a complete set of residues (mod \( A \mathbb{Z}^d \)). Making use of Theorem 2 and the notations there, exist \( a, n \in \mathbb{N} \) and \( \alpha \in \mathbb{Z}^d \) such that the IFSs \( \mathcal{T} \) and \( \mathcal{T} \cup \mathcal{G} \) have the same attractor \( aK + \alpha \). Let \( \mathcal{D} = a \mathcal{D} - \alpha + A \alpha \). Then \( \mathcal{T} \cup \mathcal{G} = \{A^{-n}(x + d) : d \in \mathcal{D} \cup \mathcal{L} \} \). Note that \( \mathcal{D} \cup \mathcal{L} \) contains a complete set \( \mathcal{L}_n \) of residues (mod \( A \mathbb{Z}^d \)).

Lemma 5 implies that \( \mathcal{L}(\partial K) = a^{-d} \mathcal{L}(\partial(aK + \alpha)) = 0 \). We complete the proof. \( \square \)

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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**References**


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