Research Article

On Convergence in $L$-Valued Fuzzy Topological Spaces

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We introduce the concept of $L$-fuzzy neighborhood systems using complete $MV$-algebras and present important links with the theory of $L$-fuzzy topological spaces. We investigate the relationships among the degrees of $L$-fuzzy $r$-adherent points ($r$-convergent, $r$-cluster, and $r$-limit, resp.) in an $L$-fuzzy topological spaces. Also, we investigate the concept of $LF$-continuous functions and their properties.

1. Introduction

Šostak [1–3] introduced a new definition of $L$-fuzzy topology as the concept of the degree of the openness of fuzzy set. It is an extension of $I = [0, 1]$-fuzzy topology defined by Chang [4]. It has been developed in many directions [5–11]. The study of neighborhood systems and convergence of nets in Chang fuzzy topology was initiated by Pao-Ming and Ying-Ming [11] and Liu and Luo [12]. In [13] Ying introduced the degree to which a fuzzy point $x_t$ belongs to a fuzzy subset $\lambda$ by $m(x_t, \lambda) = \min(1, 1 - t + \lambda(x))$ and gave the idea of graded neighborhood on fuzzy topological spaces. This plays an important role in the theory of convergence in Chang fuzzy topology see also [14–18]. Following Ying [13], Demirci [5] introduced the idea of graded neighborhood systems in smooth topological spaces [19] (a smooth topology is similar to fuzzy topology as defined by Šostak [1], Hazra and Samanta [6]) in a different approach but restricted himself to the $I$-valued fuzzy sets.

In this paper, we study the concept of $L$-fuzzy neighborhood systems and present important links with the theory of $L$-fuzzy topological spaces and investigate some of their properties. We investigate the relationships among the degrees of $L$-fuzzy $r$-adherent points ($r$-convergent, $r$-cluster, and $r$-limit, resp.) nets in an $L$-fuzzy topological spaces. Also, we give some related examples to illustrate some of the introduced notions. In the end, we characterize $LF$-continuous functions in terms of some of the various notions introduced in this paper.

2. Preliminaries

Throughout the text we consider $(L, \leq, \wedge, \vee, 0, 1)$ as a completely distributive lattice with 0 and 1, respectively, being the universal upper and lower bound and $L_0 = L - \{0\}$. A lattice $L$ is called order dense if for each $a, b \in L$ such that $a < b$, there exist $c \in L$ such that $a < c < b$. If $L$ is a completely distributive lattice and $x \triangleleft \bigvee_{i \in \Gamma} y_i$, then there must be $i_0 \in \Gamma$ such that $x \triangleleft y_{i_0}$, where $x \triangleleft a$ means $K \subseteq L$, $a \leq \bigvee K \Rightarrow \exists y \in K$ such that $x \leq y$. If $a \triangleleft b$ and $c \triangleleft d$, we always assume $a \wedge c \triangleleft b \wedge d$ [20] and some properties of $\triangleleft$ can be found in [12].

A completely distributive lattice $L = (L, \leq, \wedge, \vee, \circ, \rightarrow, 0, 1)$ (or $L$, in short) is called a residuated lattice [9, 21–23] if it satisfies the following conditions: for each $x, y, z \in L$,

(R1) $(L, \circ, 1)$ is a commutative monoid,
(R2) if $x \leq y$, then $x \circ z \leq y \circ z$ ($\circ$ is isotone operation),
(R3) (Galois correspondence) $x \leq y \Rightarrow z \equiv x \circ y \leq z$.

In a residuated lattice $L$, $x' = x \rightarrow 0$ is called complement of $x \in L$. 
A residuated lattice $L$ is called a $BL$-algebra [9, 21, 23] if it satisfies the following conditions: for each $x, y, z \in L$,

(B1) $x \land y = x \circ (x \rightarrow y)$,

(B2) $x \lor y = [(x \rightarrow y) \rightarrow y] \land [(y \rightarrow x) \rightarrow x]$,

(B3) $x \rightarrow y = (x \circ y) \lor y = (x \circ y) \lor (x \rightarrow y) = 1$.

A $BL$-algebra is called an $MV$-algebra if $x = x''$, for each $x \in L$.

**Lemma 1** (see [9, 21, 23]). Let $L$ be a complete $MV$-algebra. For each $x, y, z \in L$, $\{y_i, x_i \mid i \in I\} \subset L$, one has the following properties:

1. $x \circ y \leq x \land y \leq x \lor y$,
2. $x \circ y \leq x, y$,
3. If $y \leq z, (x \circ y) \leq (x \circ z), x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow y \geq x \rightarrow y$,
4. $x \circ y = (x \rightarrow y')'$,
5. $x \leq y$ iff $x' \geq y'$,
6. $x \rightarrow y = y' \rightarrow x'$,
7. $\bigwedge_{i \in I} (x \circ y_i) = x \circ \bigwedge_{i \in I} y_i$,
8. $\bigvee_{i \in I} (x \circ y_i) = \bigvee_{i \in I} (y_i)$,
9. $x \rightarrow 1 = 1, 0 \rightarrow x = 1, x \rightarrow x = 1$,
10. $x \leq y \Leftrightarrow x \rightarrow y = 1$ and $1 \rightarrow x = x$,
11. $x \rightarrow \bigwedge_{i \in I} y_i = \bigwedge_{i \in I} (x \rightarrow y_i)$,
12. $\bigvee_{i \in I} (x \rightarrow y_i) = x \rightarrow \bigvee_{i \in I} y_i$,
13. $x \rightarrow \bigwedge_{i \in I} y_i = \bigwedge_{i \in I} (x \rightarrow y_i)$,
14. $\bigwedge_{i \in I} y_i = x \rightarrow \bigvee_{i \in I} y_i$,
15. $\bigwedge_{i \in I} y_i = (\bigwedge_{i \in I} y_i)'$ and $\bigvee_{i \in I} y_i = (\bigvee_{i \in I} y_i)'$.

In this paper, we always assume that $L$ is a complete $MV$-algebra. Let $X$ be a nonempty set, and the family $L^X$ denotes the set of all $L$-fuzzy subsets of a given set $X$. For $\alpha, \beta \in L^X$, we denote $\alpha \rightarrow \beta$ as $(\alpha \circ \beta) \circ \alpha$ and $\alpha \circ \beta \circ \alpha$ as $\alpha \rightarrow \beta$.

A fuzzy point $x_i$ for $t \in L_0$ is an element of $L^X$ such that

$$x_i(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases} \quad (1)$$

The set of all fuzzy points in $X$ is denoted by $Pt(X)$. For $\lambda \in L^X$ and $x_i \in Pt(X), x_i \in \lambda$ if and only if $t \leq \lambda(x)$.

Given a mapping $\phi : X \rightarrow Y$, we write $\phi-$ for the mapping $L^Y \rightarrow L^X$ defined by $\phi^{-}(\mu) = \mu * \phi$; we write $\phi-$ for the mapping $L^X \rightarrow L^Y$ defined by $\phi^{-}(\mu)(y) = \bigvee \{\mu(x) \mid \phi(x) = y\}$ for all $\mu \in L^X, y \in Y$.

For a given set $X$, define a binary mapping $S(\cdot) : L^X \times L^X \rightarrow L$ as

$$S(\lambda, \mu) = \bigvee \{\lambda(x) \rightarrow \mu(x) \}, \quad \forall (\lambda, \mu) \in L^X \times L^X. \quad (2)$$

For each $\lambda, \mu \in L^X, S(\lambda, \mu)$ can be interpreted as the degree to which $\lambda$ is fuzzy included in $\mu$. It is called the $L$-fuzzy inclusion order [24].

**Lemma 2** (see [24]). For each $\lambda, \mu, \rho, \mu_i \in L^X, i \in \Gamma$ and $e, x_i \in Pt(X)$, the following properties hold:

1. $\lambda \leq \mu \Rightarrow S(\lambda, \mu) = 1$,
2. $\lambda \leq \mu \Rightarrow S(\rho, \lambda) \leq S(\rho, \mu)$ and $S(\lambda, \rho) \leq S(\mu, \rho)$, for all $\rho \in L^X$,
3. $S(\lambda, \lambda) = \lambda(x), \forall x \in L^X$,
4. $S(\lambda, \lambda) = 0$ and only if $t = 1 \land \lambda(x) = 0$,
5. $S(\lambda, \lambda) \land S(\epsilon, \mu) = S(\epsilon, \lambda \land \mu), \forall \epsilon, \lambda, \mu \in L^X$,
6. $S(\lambda, \bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} S(\lambda, \mu_i), \forall \{\mu_i\}_{i \in I} \subset L^X$,
7. $S(\lambda, \bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} S(\lambda, \mu_i), \forall \{\mu_i\}_{i \in I} \subset L^X$.

**Lemma 3** (see [16]). Let $f : X \rightarrow Y$ be a mapping. Then the following statement hold:

1. $S(\lambda, \mu) \leq S(f^{-}(\lambda), f^{-}(\mu)), \forall \lambda, \mu \in L^X$,
2. $S(\rho, \nu) \leq S(f^{-}(\rho), f^{-}(\nu)), \forall \rho, \nu \in L^Y$,

In particular, if the mapping $f : X \rightarrow Y$ is bijective, and then the equalities hold.

**Definition 4** (see [1, 9]). A map $T : L^X \rightarrow L$ is called an $L$-fuzzy topology on $X$ if it satisfies the following conditions:

1. $T(1_X) = T(0_X) = 1$,
2. $T(\mu_1 \land \mu_2) \supseteq T(\mu_1) \land T(\mu_2)$, for all $\mu_1, \mu_2 \in L^X$,
3. $T(\bigvee_{i \in I} \mu_i) \supseteq \bigwedge_{i \in I} T(\mu_i)$, for any $\{\mu_i\}_{i \in I} \subset L^X$.

The pair $(X, T)$ is called an $L$-fuzzy topological space.

Let $T_1$ and $T_2$ be $L$-fuzzy topologies on $X$. We say that $T_1$ is finer than $T_2$ (or $T_2$ is coarser than $T_1$), denoted by $T_2 \leq T_1$, if $T_2(\lambda) \subseteq T_1(\lambda)$ for all $\lambda \in L^X$. Let $(X, T_1)$ and $(Y, T_2)$ be $L$-fuzzy topological spaces. A map $f : (X, T_1) \rightarrow (Y, T_2)$ is $L$-fuzzy continuous ($L$-continuous, for short) if $T_2(\lambda) \subseteq T_1(f^{-}(\lambda)), \forall \lambda \in L^X$.

**Theorem 5** (see [7, 9]). Let $(X, T)$ be an $L$-fuzzy topological space. For each $r \in L_0$ and $\lambda \in L^X$, one defines operators $I_T, C_T : L^X \times L_0 \rightarrow L^X$ as follows:

$$I_T(\lambda, r) = \bigvee \{\rho \in L^X \mid \rho \leq \lambda, T(\rho) \geq r\} \quad (3)$$

$$C_T(\lambda, r) = \bigwedge \{\nu \in L^X \mid \lambda \leq \nu, T(\nu') \geq r\}. \quad (4)$$

For each $\lambda, \mu \in L^X$ and $r, s \in L_0$, one has the following properties:

1. $I_T(1_X, r) = 1_X$,
2. $I_T(\lambda, r) \leq \lambda$,
3. If $\lambda \leq \mu$ and $r \leq s$, then $T_2(\lambda, s) \leq T_2(\mu, r)$,
4. $I_T(\lambda \land \mu, r \land s) \supseteq I_T(\lambda, r) \land I_T(\mu, s)$,
5. $I_T(\lambda, r, r) = I_T(\lambda, r)$,
6. $I_T(\lambda', r) = (C_T(\lambda, r))'$.
Definition 6 (see [12]). Let $D$ be a directed set. A function $T : D \to \text{Pt}(X)$ is called a fuzzy net in $X$. Let $\lambda \in L^X$, and one says that $T$ is a fuzzy net in $\lambda$ if $T(n) \in \lambda$ for every $n \in D$.

Definition 7 (see [12, 25]). Let $T$ be a fuzzy net and $\lambda \in L^X$.

1. $T$ is often in $\lambda$ if for each $n \in D$, there exists $n_0 \in D$ such that $n_0 \geq n$ and $T(n_0) \in \lambda$.
2. $T$ is finally in $\lambda$ if there exists $n_0 \in D$ such that for each $n \in D$ with $n \geq n_0$, one has $T(n) \in \lambda$.

Definition 8 (see [12, 25]). Let $T : D \to \text{Pt}(X)$ and $U : E \to \text{Pt}(X)$ be two fuzzy nets. A fuzzy net $U$ is called a subnet of $T$ if there exists a function $N : E \to D$, called by a cofinal selection on $T$, such that

1. $U = T \circ N$;
2. for every $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$, for $m \geq m_0$.

3. L-Fuzzy Neighborhood Systems

Definition 9. Let $\lambda \in L^X$ and $x_i \in \text{Pt}(X)$. Then the degree to which $x_i$ belongs to $\lambda$ is

$$S(x_i, \lambda) = \bigwedge_{x \in X} (t \rightarrow \lambda(x)) .$$

Definition 10. Let $(X, \mathcal{F})$ be an $L$-fuzzy topological space, $\lambda \in L^X$, $e \in \text{Pt}(X)$, and $r \in L_0$. The degree to which $\lambda$ is a $r$-neighborhood of $e$ is defined by

$$\mathcal{N}^\mathcal{F}_e(\lambda, r) = \bigvee \{S(e, \mu) \mid \mu \leq \lambda, \ r \leq \mathcal{F}(\mu)\} .$$

A mapping $\mathcal{N}^\mathcal{F}_e : L^X \times L_0 \to L$ is called the $L$-fuzzy neighborhood system of $e$.

Theorem 11. Let $(X, \mathcal{F})$ be an $L$-fuzzy topological space and let $(\mathcal{N}^\mathcal{F}_e)$ be the fuzzy neighborhood system of $e$. For all $\lambda, \mu \in L^X$ and $r, s \in L_0$, the following properties hold:

1. $(\mathcal{N}^\mathcal{F}_e)(0_X, r) = S(e, 0_X)$ and $(\mathcal{N}^\mathcal{F}_e)(1_X, r) = 1$,
2. $(\mathcal{N}^\mathcal{F}_e)(\lambda, r) \leq S(e, \lambda)$,
3. $(\mathcal{N}^\mathcal{F}_e)(\lambda, r) \geq (\mathcal{N}^\mathcal{F}_e)(\lambda, s)$, if $r \leq s$,
4. $(\mathcal{N}^\mathcal{F}_e)(\lambda, r) \leq (\mathcal{N}^\mathcal{F}_e)(\mu, r)$, if $\lambda \leq \mu$,
5. $(\mathcal{N}^\mathcal{F}_e)(\lambda_1, r) \wedge (\mathcal{N}^\mathcal{F}_e)(\lambda_2, r) \leq (\mathcal{N}^\mathcal{F}_e)(\lambda_1 \wedge \lambda_2, r)$,
6. $(\mathcal{N}^\mathcal{F}_e)(\lambda, r) \leq \bigvee \{S(e, \mu) \mid \mu \leq \lambda, S(d, \mu) \leq (\mathcal{N}^\mathcal{F}_e)(d, r) \forall d \in \text{Pt}(X)\}$,
7. $(\mathcal{N}^\mathcal{F}_e)(x_i, r) = \bigwedge_{x \in X} (t \rightarrow (\mathcal{N}^\mathcal{F}_e)(\lambda, r))$.

Proof. (1), (3), and (4) are easily proved. (2) is proved from the following:

$$\mathcal{N}^\mathcal{F}_e(\lambda, r) = \bigvee \{S(e, \mu) \mid \mu \leq \lambda, r \leq \mathcal{F}(\mu)\}$$

$$\leq \bigvee \{S(e, \bigvee \mu) \mid \mu \leq \lambda, r \leq \mathcal{F}(\mu)\}$$

(by Lemma 2 (2))

$$\leq \bigvee \{S(e, \bigvee \mu) \mid \mu \leq \lambda, r \leq \mathcal{F}(\bigvee \mu)\}$$

$$\leq S(e, \lambda) .$$

(6)

In (5) if $a \subset (\mathcal{N}^\mathcal{F}_e)(\lambda_1, r) \wedge (\mathcal{N}^\mathcal{F}_e)(\lambda_2, r)$, then $a \subset (\mathcal{N}^\mathcal{F}_e)(\lambda_1, r)$ and $a \subset (\mathcal{N}^\mathcal{F}_e)(\lambda_2, r)$, and there exists $\rho_1 \in L^X$ with $\rho_1 \leq \lambda_1$ and $r \leq \mathcal{F}(\rho_1)$ such that $a \subset S(e, \rho_1)$. Again, there exists $\rho_1 \in L^X$ with $\rho_2 \leq \lambda_2$ and $r \leq \mathcal{F}(\rho_2)$ such that $a \subset S(e, \rho_2)$. So, $\rho_1 \wedge \rho_2 \leq \lambda_1 \wedge \lambda_2$, $r \leq \mathcal{F}(\rho_1) \wedge \mathcal{F}(\rho_2)$, and $a \subset S(e, \rho_1 \wedge \rho_2) = S(e, \rho_1 \wedge \rho_2) \leq (\mathcal{N}^\mathcal{F}_e)(\lambda_1 \wedge \lambda_2, r \wedge s)$. Hence,

$$(\mathcal{N}^\mathcal{F}_e)(\lambda_1 \wedge \lambda_2, r \wedge s) \leq (\mathcal{N}^\mathcal{F}_e)(\lambda_1, r) \wedge (\mathcal{N}^\mathcal{F}_e)(\lambda_2, r) .$$

(7)

In (6) if $r \leq \mathcal{F}(\mu)$, then $S(d, \mu) \leq (\mathcal{N}^\mathcal{F}_e)(d, r)$, for each $d \in \text{Pt}(X)$. It implies

$$\mathcal{N}^\mathcal{F}_e(\lambda, r) = \bigvee \{S(e, \mu) \mid \mu \leq \lambda, r \leq \mathcal{F}(\mu)\}$$

$$= \bigvee \{\mathcal{N}^\mathcal{F}_e(\mu, r) \mid \mu \leq \lambda\}$$

$$S(d, \mu) = (\mathcal{N}^\mathcal{F}_e)(d, r),$$

$$\forall d \in \text{Pt}(X) .$$

$$\leq \bigvee \{\mathcal{N}^\mathcal{F}_e(\mu, r) \mid \mu \leq \lambda\}$$

$$\leq S(d, \mu) \leq (\mathcal{N}^\mathcal{F}_e)(d, r),$$

$$\forall d \in \text{Pt}(X) .$$

(8)

(7) is proved from

$$(\mathcal{N}^\mathcal{F}_e)(\lambda, r) = \bigvee \{S(x_i, \mu) \mid \mu \leq \lambda, \mathcal{F}(\mu) \geq r\}$$

$$= \bigvee \{\bigwedge_{x \in X} (t \rightarrow \mu(x)) \mid \mu \leq \lambda, \mathcal{F}(\mu) \geq r\}$$
\[
\begin{align*}
&= \bigwedge_{x \in X} \left\{ t \rightarrow \bigvee \{ \mu(x) \mid \mu \leq \lambda, \mathcal{T}(\mu) \geq r \} \right\} \\
&= \bigwedge_{x \in X} \left( t \rightarrow \left( N^\mathcal{S}_{\chi}(\lambda, r) \right) \right).
\end{align*}
\]

(by Lemma 2 (7))

\[
\begin{align*}
&\leq \bigwedge_{x \in X} \left( t \rightarrow \left( \bigvee_{i \in \Gamma} \left( \lambda_i \right) \right) \right). \\
&\leq \bigwedge_{x \in X} \left( t \rightarrow \left( \bigvee_{i \in \Gamma} \left( \lambda_i \right) \right) \right).
\end{align*}
\]

So there exists \( r_i \in L_0 \), with \( S(e, \lambda_i) = N_e(\lambda_i, r_i) \) such that \( a \prec r_i \). Put \( r = \bigwedge_{i \in \Gamma} r_i \), and then \( a \prec r \). By Theorem II, we have

\[
S(e, \lambda_i) \leq N_e(\lambda_i, r_i) \leq N_e(\lambda_i, r) \leq S(e, \lambda_i).
\]

It implies \( S(e, \lambda_i) = N_e(\lambda_i, r) \). Furthermore, by Lemma 2(7), we have

\[
S \left( e, \bigvee_{i \in \Gamma} \lambda_i \right) = \bigvee_{i \in \Gamma} S(e, \lambda_i).
\]

Proof.\( \quad \) (LO1) It is easily proved from Theorem II(1).

(LO2) It is proved from the following:

\[
\mathcal{T}_{\mathcal{S}}(\lambda) = \bigvee_{r \in L_0} S(e, \lambda) = N^\mathcal{N}(\lambda, r), \forall e \in \text{Pt}(X).
\]

Then one has the following:

(a) \( \mathcal{T}_{\mathcal{S}} \) is an \( L \)-fuzzy topology on \( X \);

(b) if \( (N^\mathcal{S})_e \) is the \( L \)-fuzzy neighborhood system of \( e \) induced by \( (X, \mathcal{T}) \), then \( \mathcal{T}_{\mathcal{S}}(\lambda) = \mathcal{T} \);

(c) if \( N^\mathcal{S} \) satisfies the conditions (6) and (7), then

\[
\mathcal{T}_{\mathcal{S}}(\lambda) = \bigvee_{r \in L_0} S(e, \lambda) = N^\mathcal{S}(\lambda, r), \forall x \in X;
\]

(d) \( N^\mathcal{S} \) is \( N^\mathcal{S} \).

Thus, \( \lambda = \bigvee_{i \in \Gamma} \mu_i \). So \( \mathcal{T}(\lambda) \geq r_0 \geq a \). Hence, \( \mathcal{T}(\lambda) \leq \mathcal{T}(\lambda) \). We can easily obtain \( \mathcal{T}(\lambda) \geq \mathcal{T}(\lambda) \).

(c) We only show that \( S(x_i, \lambda) = N_{x_i}(\lambda, r), \forall x_i \in \text{Pt}(X) \) if and only if \( S(x_i, \lambda) = \lambda(x) = N_{x_i}(\lambda, r), \forall x \in X \).

\( \Rightarrow \) It is trivial.

\( \Leftarrow \) From condition (7),

\[
N_{x_i}(\lambda, r) = \bigwedge_{x \in X} \left( t \rightarrow \bigvee_{i \in \Gamma} \left( \lambda_i \right) \right).
\]

Therefore, \( \mathcal{T}(\lambda) \geq \mathcal{T}(\lambda) \).

(d) From the proof of Theorem II(6), we easily obtain \( N^\mathcal{S}_{\mathcal{S}} \geq N^\mathcal{S} \).

If \( a \prec (N^\mathcal{S})_e(\lambda, r) = \bigvee_{i \in \Gamma} S(e, \mu_i) \mid \mu \leq \lambda, r \prec \mathcal{T}(\mu_i) \), there exists \( \mu_0 \) with \( \mu_0 \leq \lambda, r \prec \mathcal{T}(\mu_0) \) such that \( a \prec S(e, \mu_0) \). Note that

\[
\mathcal{T}(\mu_0) = \bigvee_{t \in L_0} S(e, \mu_0) = N^\mathcal{N}(\mu_0, t), \forall e \in \text{Pt}(X),
\]
and there exists \( t_0 \in L_0 \) with \( S(\epsilon, \mu_0) = N_{\epsilon}(\mu_0, t_0) \) such that \( r < t_0 \) (thus \( r \leq t_0 \)). So \( a < N_{\epsilon}(\mu_0, t_0) \leq N_{\epsilon}(\mu, r) \). Therefore, \( N_{\epsilon, r} \leq N_{\epsilon} \). □

By Theorem 12, we have the following corollary.

**Corollary 13.** The set of all \( L \)-fuzzy topologies on \( X \) and the set of all \( L \)-fuzzy neighborhood systems on \( X \) are in one to one correspondence.

**Example 14.** Let \( L = [0, 1] \), \( X = \{a, b\} \) be a set, \( x \rightarrow y = \min(1 - x + y, 1) \), and let \( \mu \in L^X \) be defined as follows:

\[
\mu(a) = 0.3, \quad \mu(b) = 0.4.
\]

We define an \( L \)-fuzzy topology on \( X \) as

\[
\mathcal{T}(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0_X \text{ or } 1_X, \\
\frac{1}{2}, & \text{if } \lambda = \mu, \\
0, & \text{otherwise}.
\end{cases}
\]

From Definition 10, \( N_{\epsilon, x, 1}, N_{\epsilon, x, 2} : L^X \times L_0 \rightarrow L \) as follows:

\[
N_{\epsilon, x, 1}(\lambda, r) = \begin{cases} 
1, & \text{if } \lambda = 1_X, \ r \in L_0, \\
0.3, & \text{if } 1_X \neq \lambda \geq \mu, \ 0 < r \leq \frac{1}{2}, \\
0, & \text{otherwise},
\end{cases}
\]

\[
N_{\epsilon, x, 2}(\lambda, r) = \begin{cases} 
1, & \text{if } \lambda = 1_X, \ r \in L_0, \\
0.4, & \text{if } 1_X \neq \lambda \geq \mu, \ 0 < r \leq \frac{1}{2}, \\
0, & \text{otherwise}.
\end{cases}
\]

From Theorem 12(c), we have

\[
\mathcal{T}_{x, x} \mathcal{s}(\lambda) = \begin{cases} 
1, & \text{if } \lambda = 0_X \text{ or } 1_X, \\
\frac{1}{2}, & \text{if } \lambda = \mu, \\
0, & \text{otherwise}.
\end{cases}
\]

4. R-Convergence

**Definition 15.** Let \((X, \mathcal{T})\) be an \( L \)-fuzzy topological space, \( \lambda \in L^X \), \( e \in \text{Pt}(X) \), and \( r \in L_0 \). The degree to which a fuzzy net \( T \) in \( X \) is \( r \)-convergent to \( e \) and \( T \) is \( r \)-cluster to \( e \) are defined, respectively, as follows:

\[
\text{Con}_x(T, r) = \bigwedge \{ N'_{\epsilon}(\lambda, r) \mid T \text{ is often in } \lambda' \},
\]

\[
\text{Cl}_x(T, r) = \bigwedge \{ N'_e(\lambda, r) \mid T \text{ is finally in } \lambda' \}.
\]

**Definition 16.** Let \((X, \mathcal{T})\) be an \( L \)-fuzzy topological space, \( \lambda \in L^X \), \( e \in \text{Pt}(X) \), and \( r \in L_0 \). The degree to which \( e \) is \( r \)-adherent point of \( e \) is defined by

\[
\text{Ad}_e(\lambda, r) = N'_e(\lambda, r).
\]

**Proposition 17.** Let \((X, \mathcal{T})\) be an \( L \)-fuzzy topological space. For each \( \lambda \in L^X \), \( e, x_i \in \text{Pt}(X) \) and \( r \in L_0 \), one has

(1) \( S(e, I_{\mathcal{T}}(\lambda, r)) = N'_e(\lambda, r) \),

(2) \( S(e, C'_{\mathcal{T}}(\lambda, r)) = Ad'_e(\lambda, r) \),

(3) \( Ad_{x_i}(\lambda, r) = \bigvee_{x \in X} (t \otimes Ad_x(\lambda, r)) \).

**Proof.** (1) From Lemma 2(7), we have

\[
S(e, I_{\mathcal{T}}(\lambda, r)) = S(e, \bigvee \{ \mu_i \mid \mu_i \leq \lambda, \mathcal{T}(\mu_i) \geq r \})
\]

\[
= \bigvee \{ S(e, \mu_i) \mid \mu_i \leq \lambda, \mathcal{T}(\mu_i) \geq r \}
\]

\[
= N'_e(\lambda, r).
\]

(2) From Theorem 5, we have

\[
S(e, C'_{\mathcal{T}}(\lambda, r)) = S(e, I_{\mathcal{T}}(\lambda', r))
\]

\[
= N'_e(\lambda', r) \quad \text{(by (1))}
\]

\[
= Ad'_e(\lambda, r).
\]

(3) From Theorem 7(7), we have

\[
Ad_{x_i}(\lambda, r) = N'_{x_i}(\lambda', r)
\]

\[
= \left( \bigwedge_{x \in X} (t \rightarrow N'_{x_i}(\lambda', r)) \right)'
\]

\[
= \bigvee_{x \in X} (t \rightarrow N'_{x_i}(\lambda', r))'
\]

\[
= \bigvee_{x \in X} (t \otimes N'_{x_i}(\lambda', r))
\]

(by Lemma 2(4))

\[
= \bigvee_{x \in X} (t \otimes Ad_x(\lambda, r)).
\]

□

**Theorem 18.** Let \((X, \mathcal{T})\) be an \( L \)-fuzzy topological space. Let \( T : D \rightarrow \text{Pt}(X) \) be fuzzy net and let \( U : E \rightarrow \text{Pt}(X) \) be a subnet of \( S \). For \( r, s \in L_0 \), the following properties hold:

(1) if \( r_1 \leq r_2 \), \( \text{Con}_x(T, r_1) \leq \text{Con}_x(T, r_2), \) and \( \text{Cl}_x(T, r_1) \leq \text{Cl}_x(T, r_2) \),

(2) \( \text{Con}_x(T, r) \leq \text{Cl}_x(T, r) \),

(3) \( \text{Cl}_x(U, r) \leq \text{Cl}_x(T, r) \),

(4) \( \text{Con}_x(T, r) \leq \text{Con}_x(U, r) \),

(5) \( \text{Con}_x(T, r) = \bigvee_{x \in X} (t \otimes \text{Con}_x(T, r)), \) and \( \text{Cl}_x(T, r) = \bigvee_{x \in X} (t \otimes \text{Cl}_x(T, r)) \).

**Proof.** (1) is easily proved.

In (2) if \( T \) is finally in \( \lambda', T \) is often in \( \lambda' \). Hence

\[
\text{Con}_x(T, r) = \bigwedge \{ N'_{x}(\lambda, r) \mid T \text{ is often in } \lambda' \}
\]

\[
\leq \bigwedge \{ N'_{x}(\lambda, r) \mid T \text{ is finally in } \lambda' \}
\]

\[
= \text{Cl}_x(T, r).
\]
In (3) if $T$ is finally in $\lambda'$, $U$ is finally in $\lambda'$. Hence

$$\text{Cl}_e(U, r) = \bigwedge \left\{ \mathcal{A}_{e}^d(\lambda, r) \mid U \text{ is finally in } \lambda' \right\}$$

$$\leq \bigwedge \left\{ \mathcal{A}_{e}^d(\lambda, r) \mid T \text{ is finally in } \lambda' \right\}$$

$$= \text{Cl}_e(T, r).$$

In (4) let $U$ be often in $\lambda'$. We will show that $T$ is often in $\lambda'$. Let $n \in D$. Since $U : E \to \text{Pt}(X)$ is a subnet of $T$, there exists a cofinal selection $N : E \to D$. For each $n \in D$, there exists $m \in E$ such that $N(k) \geq n$ for $k \geq m$. Since $U$ is often in $\lambda'$, for $m \in E$, there exists $m_0 \in E$ such that $m_0 \geq m$ for $U(m_0) \in \lambda'$. Put $n_0 = N(m_0)$. Then $n_0 \geq n$ and $T(n_0) = \lambda(\lambda(n_0)) = (\lambda(n_0))^+ \in \lambda'$. Thus, $U$ is often in $\lambda'$. Hence

$$\text{Con}_e(T, r) = \bigwedge \left\{ \mathcal{A}_{e}^d(\lambda, r) \mid T \text{ is often in } \lambda' \right\}$$

$$\leq \bigwedge \left\{ \mathcal{A}_{e}^d(\lambda, r) \mid U \text{ is often in } \lambda' \right\}$$

$$= \text{Con}_e(U, r).$$

In (5) one has

$$\text{Con}_{x_1}(T, r) = \bigwedge \left\{ \mathcal{A}_{x_1}(\lambda, r) \mid T \text{ is often in } \lambda' \right\}$$

$$= \bigwedge \left\{ \left( \bigwedge_{x \in X} \left( t \to \mathcal{A}_{x_1}(\lambda, r) \right)^{r} \right) \mid T \text{ is finally in } \lambda' \right\}$$

(by Theorem 11 (7))

$$= \bigvee_{x \in X} \bigwedge \left\{ \left( t \to \mathcal{A}_{x_1}(\lambda, r) \right)^{r} \mid T \text{ is finally in } \lambda' \right\}$$

$$= \bigvee_{x \in X} \left\{ \left( t \to \mathcal{A}_{x_1}(\lambda, r) \right)^{r} \mid T \text{ is finally in } \lambda' \right\}$$

(by Theorem 11 (7))

$$= \bigvee_{x \in X} \left\{ \left( t \to \mathcal{A}_{x_1}(\lambda, r) \right)^{r} \mid T \text{ is finally in } \lambda' \right\}$$

(by Lemma 1 (4))

$$= \bigvee_{x \in X} \left\{ \left( t \to \mathcal{A}_{x_1}(\lambda, r) \right)^{r} \mid T \text{ is finally in } \lambda' \right\}$$

$$= \bigvee_{x \in X} \left\{ \left( t \to \text{Con}_x(T, r) \right)^{r} \right\}.$$

The other case is the same.

**Proposition 19.** Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space, let $T$ be a fuzzy net, $e \in \text{Pt}(X)$, and $r \in L_0$. Then one has

$$\text{Ad}_e(\lambda, r) = \bigvee \left\{ \text{Con}_e(T, r) \mid T \text{ is a fuzzy net in } \lambda \right\}$$

$$= \bigwedge \left\{ \text{Cl}_e(T, r) \mid T \text{ is a fuzzy net in } \lambda \right\}.$$

**Proof.** Since $T$ is finally in $\lambda$, $T$ is often in $\lambda$. We easily show that

$$\text{Ad}_e(\lambda, r) = \mathcal{A}_{e}^d(\lambda', r)$$

$$\geq \bigvee \left\{ \text{Cl}_e(T, r) \mid T \text{ is a fuzzy net in } \lambda \right\}$$

$$= \bigvee \left\{ \text{Con}_e(T, r) \mid T \text{ is a fuzzy net in } \lambda \right\}.$$

We only show that

$$\text{Ad}_e(\lambda, r) \leq \bigvee \left\{ \text{Con}_e(T, r) \mid T \text{ is a fuzzy net in } \lambda \right\}.$$  

Let $\text{Ad}_e(\lambda, r) = t$. If $t > 0$, then $\mathcal{A}_{e}^d(\lambda', r) = t$. Put $D = \{ \mu \in L^X \mid \mathcal{A}_{e}^d(\mu, r) > t' \}$. Define a relation on $D$ by

$$\mu_1 \leq \mu_2 \iff \mu_1 \geq \mu_2, \forall \mu_1, \mu_2 \in D.$$

For each $\mu_1, \mu_2 \in D$, since by Theorem II(5),

$$\mathcal{A}_{e}^d(\mu_1 \land \mu_2, r) \geq \mathcal{A}_{e}^d(\mu_1, r) \land \mathcal{A}_{e}^d(\mu_2, r) > t'.$$

Hence, $\mu_1 \land \mu_2 \in D$ and $\mu_1, \mu_2 \leq \mu_1 \land \mu_2$. Thus, $(D, \leq)$ is a directed set. For each $\mu \in D$, that is, $\mathcal{A}_{e}^d(\mu, r) > t'$, we have $\mu \notin \lambda'$; that is, there exists $x \in X$ such that $\lambda(\lambda(x)) > \mu(\lambda(x))$. Thus, we can define a fuzzy net $T_0 : D \to \text{Pt}(X)$ by $T_0(\mu) = x_{\lambda(x)}$ where $T_0(\mu) \in \lambda$ and $\lambda(x) = T_0(\mu(x)) > \mu(\lambda(x))$. We will show that if $\mu \in D$, then $T_0$ is not often in $\mu'$. Suppose that $T_0$ is often in $\mu'$. For $\mu \in D$, there exists $\rho \in D$ such that $\mu \leq \rho$ such that

$$T_0(\rho) = \lambda_{y} \in \mu'$$

and $\lambda(y) = T_0(\rho)(y) > \rho(y)$. Since $\mu \leq \rho$ implies $\mu \geq \rho$, it implies

$$\lambda(y) \leq \mu(\lambda(x)) \leq \rho(\lambda(x)).$$

It is contradiction for the definition of $T_0$. Thus, if $T_0$ is often in $\mu'$, then $\mu \notin D$, that is, $\mathcal{A}_{e}^d(\mu, r) > t'$. Therefore,

$$\bigvee \left\{ \text{Con}_e(T, r) \mid T \text{ is a fuzzy net in } \lambda \right\}$$

$$\geq \text{Con}_e(T, r)$$

$$= \bigwedge \left\{ \mathcal{A}_{e}^d(\mu, r) \mid T \text{ is a fuzzy net in } \lambda \right\}$$

$$= \bigwedge \left\{ \mathcal{A}_{e}^d(\lambda', r) \mid T_0 \text{ is often in } \mu' \right\}$$

$$= \bigwedge \left\{ \mathcal{A}_{e}^d(\lambda, r) \right\}.$$

\[\square\]

**Theorem 20.** Let $(X, \mathcal{T})$ be an $L$-fuzzy topological space and let $T, U : D \to \text{Pt}(X)$ be fuzzy nets such that $T(n) \lor U(n)$ be fuzzy nets for each $n \in D$. Define fuzzy nets $T \lor U, T \land U : D \to \text{Pt}(X)$ by, for each $n \in D$,

$$(T \lor U)(n) = T(n) \lor U(n),$$

$$(T \land U)(n) = T(n) \land U(n).$$
For each \( r \in L_0 \), the following properties hold:

(1) if \( T(n) \leq U(n) \) for all \( n \in D \), then
\[
Cl_e(T, r) \leq Cl_e(U, r), \quad \text{Con}_e(T, r) \leq \text{Con}_e(U, r), \quad (42)
\]

(2) \( Cl_e(T \land U, r) \leq Cl_e(T, r) \land Cl_e(U, r) \),
(3) \( \text{Con}_e(T \lor U, r) \geq \text{Con}_e(T, r) \lor \text{Con}_e(U, r) \),
(4) \( \text{Con}_e(T \land U, r) \leq \text{Con}_e(T, r) \land \text{Con}_e(U, r) \),

(5) if \( T \) is order dense, then \( Cl_e(T \lor U, r) = Cl_e(T, r) \lor Cl_e(U, r) \).

Proof. In (1) let \( U \) be finally (often) in \( \lambda \). Then let \( T \) be finally (often) in \( \lambda \), respectively. Thus it is trivial. (2), (3), and (4) are easily proved.

In (5) since \( T \leq T \lor U \) and \( U \leq T \lor U \), by (1), we have
\[
Cl_e(T \lor U, r) \geq Cl_e(T, r) \lor Cl_e(U, r). \quad (43)
\]

Suppose that \( Cl_e(T \lor U, r) \neq Cl_e(T, r) \lor Cl_e(U, r) \). Since \( L \) is order dense, then there exist \( t \in L_0 \) and a fuzzy point \( e \in Pt(\lambda) \) such that
\[
Cl_e(T \lor U, r) > Cl_e(T, r) \lor Cl_e(U, r). \quad (44)
\]

Since \( Cl_e(T, r) < t \) and \( Cl_e(U, r) < t \), by the definition \( Cl_e \), there exist \( \lambda, \mu \in L^X \) such that \( T \) and \( U \) are finally in \( \lambda^t \) and \( \mu^t \), respectively, with
\[
Cl_e(T, r) \lor Cl_e(U, r) \leq A_e^t(\lambda, r) \lor A_e^t(\mu, r) < t. \quad (45)
\]

Since \( T \) is finally in \( \lambda^t \), there exists \( n_1 \in D \) such that \( T(n) \in \lambda^t \) for every \( n \in D \) with \( n \geq n_1 \). Since \( U \) is finally in \( \mu^t \), there exists \( n_2 \in D \) such that \( U(n) \in \mu^t \) for every \( n \in D \) with \( n \geq n_2 \). Let \( n_3 \in D \) such that \( n_3 \geq n_1 \) and \( n_3 \geq n_2 \). For \( n \geq n_3 \), we have
\[
(T \lor U)(n) \leq \lambda^t \lor \mu^t = (\lambda \lor \mu)^t. \quad (46)
\]

Thus, \( T \lor U \) is finally in \( (\lambda \lor \mu)^t \). It implies
\[
Cl_e(T \lor U, r) \leq A_e^t(\lambda \lor \mu, r)
\leq A_e^t(\lambda, r) \lor A_e^t(\mu, r) < t. \quad (47)
\]

It is a contradiction. Hence, we have
\[
Cl_e(T \lor U, r) \leq Cl_e(T, r) \lor Cl_e(U, r). \quad (48)
\]

Example 21. Let \( \langle L, \rightarrow \rangle \) be defined as Example 14. Let \( X = [a, b] \) be a set and \( \mu \in I^X \) as follows:
\[
\mu(x) = 0.3, \quad \mu(y) = 0.4. \quad (49)
\]

We define \( L \)-fuzzy topology \( \mathcal{T} : I^X \rightarrow I \) as follows:
\[
\mathcal{T}(\lambda) = \begin{cases} 1, & \text{if } \lambda = 0_X \text{ or } 1_X, \\ \lambda, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases} \quad (50)
\]

(1) In general, \( Cl_e(T \land U, r) \neq Cl_e(T, r) \land Cl_e(U, r) \).
Let \( N \) be a natural numbers. Define fuzzy nets \( T, U : N \rightarrow Pt(\lambda) \) by
\[
T(n) = x_n, \quad a_n = 0.8 + (-1)^n 0.2.
U(n) = x_n, \quad b_n = 0.8 + (-1)^n 0.2. \quad (51)
\]

From Theorem 20, \( (T \land U)(n) = x_{0,1} \) is a fuzzy net. Let \( e = x_{0,3} \). From Definition 15, we have for \( 0 < r \leq 1/2 \),
\[
Cl_e(x_{0,6}, r) = 1 - \lambda_e(r) = 1 - m(x_{0,3}, \mu) = 0. \quad (52)
\]

Since \( T \) or \( U \) is finally in \( 1_X \),
\[
Cl_e(T, r) = 1 - \lambda_e(0_X, r) = 1 - m(x_{0,3}, 0_X) = 0.3. \quad (53)
\]

Similarly, \( Cl_e(U, r) = 0.3 \). For \( 0 < r \leq 1/2 \),
\[
0 = Cl_e(T \land U, r) \neq Cl_e(T, r) \land Cl_e(U, r) = 0.3. \quad (54)
\]

(2) In general, \( \text{Con}_e(T \lor U, r) \neq \text{Con}_e(T, r) \lor \text{Con}_e(U, r) \).
Define fuzzy nets \( T, U : N \rightarrow Pt(\lambda) \) by
\[
T(n) = x_n, \quad a_n = 0.6 + (-1)^n 0.2.
U(n) = x_n, \quad b_n = 0.6 + (-1)^n 0.2. \quad (55)
\]

From Theorem 20, \( (T \lor U)(n) = x_{0,1} \) is a fuzzy net. Let \( e = x_{0,3} \). For all \( r \in L_0 \),
\[
A_e(x_{0,6}, r) = 1 - \lambda_e(0_X, r) = 1 - m(x_{0,3}, 0_X) = 0.3. \quad (56)
\]

Since \( T \) or \( U \) is often in \( \mu \), for \( 0 < r \leq 1/2 \),
\[
Cl_e(T, r) = 1 - \lambda_e(0_X, r) = 1 - m(x_{0,3}, \mu) = 0. \quad (57)
\]

Similarly, \( Cl_e(U, r) = 0 \). For \( 0 < r \leq 1/2 \)
\[
0.3 = Cl_e(T \lor U, r) < (\text{Con}_e(T, r) \lor \text{Con}_e(U, r)) = 0. \quad (58)
\]

5. Fuzzy \( r \)-Limit Nets and \( LF \)-Continuous Mappings

Definition 22. Let \( (X, \mathcal{T}) \) be an \( L \)-fuzzy topological space. Let \( T : D \rightarrow Pt(X) \) be fuzzy net in \( X \) and \( r \in L_0 \). Then the degree to which \( T \) is \( r \)-limit to \( e \) is defined, denoted by \( \lim_e(T, r) = t \), if \( Cl_e(T, r) = \text{Con}_e(T, r) \).

Theorem 23. Let \( (X, \mathcal{T}) \) be an \( L \)-fuzzy topological space and let \( T, U : D \rightarrow Pt(X) \) be fuzzy nets such that \( T(n) \lor U(n) \in Pt(X) \) for each \( n \in D \). If \( L \) is order dense, \( Cl_e(T, r) = \text{Con}_e(T, r) \), and \( Cl_e(U, r) = \text{Con}_e(U, r) \), then
\[
\lim_e(T \lor U, r) = \lim_e(T, r) \lor \lim_e(U, r). \quad (59)
\]
Proof. From Theorem 20, \( T \lor U \) is a fuzzy net. We easily proved it from the following:

\[
\begin{align*}
\text{Cl}_x(T \lor U, r) &= \text{Cl}_x(T, r) \lor \text{Cl}_x(U, r) \\
&= \text{Cl}_x(T, r) \lor \text{Cl}_x(U, r) \quad \text{(by Theorem 20 (2))}
\end{align*}
\]

(since \( \text{Cl}_x(T, r) = \text{Con}_x(T, r) \), \( \text{Cl}_x(U, r) = \text{Con}_x(U, r) \))

\[
\begin{align*}
&= \text{Con}_x(T \lor U, r) \quad \text{(by Theorem 20 (4))} \\
&\leq \text{Cl}_x(T \lor U, r) \quad \text{(by Theorem 20 (2))}.
\end{align*}
\]

(60)

\[ \square \]

Theorem 24. Let \((X, \mathcal{T})\) be L-fuzzy topological space. Let \(T\) be a fuzzy net and \(\mathcal{H} = \{U \mid U \text{ is a subnet of } T\}\). Then, if \(L\) is an order dense, the following statements hold:

1. \( \text{Con}_x(T, r) = \bigwedge_{t \in \mathcal{H}} \text{Cl}_x(U, r) \);
2. \( \text{Cl}_x(T, r) = \bigvee_{t \in \mathcal{H}} \text{Con}_x(U, r) \).

Proof. (1) For each \(U \in \mathcal{H}\), by Theorem 18, we have

\[
\text{Con}_x(T, r) \leq \text{Con}_x(U, r) \leq \text{Cl}_x(U, r) \leq \text{Cl}_x(T, r) .
\]

Hence

\[
\text{Con}_x(T, r) \leq \bigwedge_{U \in \mathcal{H}} \text{Cl}_x(U, r) .
\]

(61)

Suppose

\[
\text{Con}_x(T, r) \nleq \bigwedge_{U \in \mathcal{H}} \text{Cl}_x(U, r) .
\]

(62)

Then there exist \(x_p \in \text{Pt}(X)\) and \(t \in L_0\) such that

\[
\text{Con}_{x_p}(T, r) < t < \bigwedge_{U \in \mathcal{H}} \text{Cl}_{x_p}(U, r) .
\]

(63)

Since \(\text{Con}_{x_p}(T, r) < t\), there exists \(\mu \in L^X\) with \(T\) is often in \(\mu\) such that

\[
\text{Con}_{x_p}(T, r) \leq \mathcal{N}_{x_p}(\mu, r) \leq \bigwedge_{U \in \mathcal{H}} \text{Cl}_{x_p}(U, r) .
\]

(64)

Since \(T\) is often in \(\mu\), for each \(n \in D\) there exists \(N(n) \in D\) with \(N(n) \geq n\) and \(T(N(n)) \in \mu\). Hence there exists a cofinal selection \(N : E \rightarrow D\) such that \(U = T \circ N\). Thus \(U\) is a subnet of \(T\) and \(U\) is finally in \(\mu\). It is a contradiction.

(2) From (1), we have

\[
\bigvee_{U \in \mathcal{H}} \text{Con}_x(U, r) \leq \text{Cl}_x(T, r) .
\]

(66)

Conversely, let \(\text{Cl}_x(T, r) = t > 0\). Then \(\mathcal{N}_x(\lambda, r) \leq t\), for \(T\) is finally in \(\lambda\). Let \(F = \{\mu \mid \mathcal{N}_x(\mu, r) > t\}\). Define a relation on \(E = D \times F\) by

\[
(m, \mu_1) \leq (n, \mu_2) \iff m \leq n, \mu_1 \geq \mu_2 .
\]

(67)

Then \((E, \leq)\) is a directed set. If \(\mu \in F\), then \(T\) is not finally in \(\mu\).

For each \((n, \mu) \in E\), there exists \(N(n, \mu) \in D\) with \(N(n, \mu) \geq n\) such that \(T(N(n, \mu)) \notin \lambda\). So, we can define \(N : E \rightarrow D\).

Then \(E = D \times F\) with \(\mathcal{N}_x(\mu, r) > t\) for \(\mu \in F\). Hence for every \((n, \mu) \geq (n_0, \mu_0)\), since \(n \geq n_0\), we have \(N(n, \mu) \geq n \geq n_0\). Therefore \(N\) is a cofinal selection on \(T\). So \(U = T \circ N\) is a fuzzy subnet of \(T\) and \(U\) is finally to every member of \(F\). If \(U\) is often in \(\lambda\), then \(U\) is not finally of \(\lambda\); that is, \(\lambda \notin F\). Thus

\[
\bigvee_{U \in \mathcal{H}} \text{Con}_x(U, r) = \bigwedge_{\lambda} \{\mathcal{N}_x(\lambda, r) \mid U \text{ is often in } \lambda\} \geq t .
\]

(68)

Since \(t\) is arbitrary, we complete the proof. \[ \square \]

Theorem 25. Let \(L\) be an order dense, let \((X, \mathcal{T})\) be L-fuzzy topological space, and let \(T\) be a fuzzy net. If every subnet \(U\) of \(T\) has a subnet \(K\) of \(U\) such that \(\lim_k(K, r) = t\), then \(\lim_t(T, r) = t\).

Proof. Let \(\mathcal{H} = \{U \mid U \text{ is a subnet of } T\}\). For each \(U \in \mathcal{H}\), since \(U\) has a subnet \(K\) with \(\lim_k(K, r) = t\), by Theorem 18(4), we have

\[
\text{Con}_x(U, r) \leq \text{Con}_x(K, r) = \text{Cl}_x(K, r) = t .
\]

(69)

Hence, by Theorem 24(2),

\[
\text{Cl}_x(T, r) = \bigvee_{U \in \mathcal{H}} \text{Con}_x(U, r) \leq t .
\]

(70)

Conversely, by Theorem 18(2),

\[
t = \text{Con}_x(K, r) = \text{Cl}_x(K, r) \leq \text{Cl}_x(U, r) .
\]

(71)

Hence, by Theorem 24(1),

\[
t \leq \bigwedge_{U \in \mathcal{H}} \text{Cl}_x(U, r) = \text{Con}_x(T, r) .
\]

(72)

Hence, \(\text{Cl}_x(T, r) \leq \text{Con}_x(T, r)\). Since \(\text{Con}_x(T, r) \leq \text{Cl}_x(T, r)\) from Theorem 18(2), \(\text{Cl}_x(T, r) = \text{Con}_x(T, r)\); that is, \(\lim_t(T, r) = t\). \[ \square \]

Example 26. Let \((L = [0, 1], \rightarrow)\) be defined as in Example 21. Let \(N\) be a natural number set. Define a fuzzy net \(T : N \rightarrow \text{Pt}(X)\) by

\[
T(n) = x_{a_n}, \quad a_n = 0.6 + (-1)^n 0.2 .
\]

(73)

Let \(e = x_{0.3}\). Since \(T\) is often in \(\mu\), for \(0 < r \leq 1/2\),

\[
\text{Con}_x(T, r) = 1 - \mathcal{N}_x(\mu, r) = 1 - m(x_{0.3}, 0) = 0 .
\]

(74)

Since \(T\) is finally in \(1_X\), for each \(r \in L_0\),

\[
\text{Cl}_x(T, r) = 1 - \mathcal{N}_x(0, r) = 1 - m(x_{0.3}, 0) = 0 .
\]

(75)

Thus, since \(\text{Con}_x(T, r) \neq \text{Cl}_x(T, r)\) for \(0 < r \leq 1/2\), \(\lim_t(T, r)\) does not exists.

Since \(\text{Con}_x(T, r) = \text{Cl}_x(T, r) = 0.3\) for \(1/2 < r \leq 1\), \(\lim_t(T, r) = 0.3\).
Theorem 27. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be $L$-fuzzy topological spaces. For every fuzzy net $T$ in $X$, $x_i \in Pt(X)$, $r \in L_0$, and $\lambda \in L^X$, the following statements are equivalent:

1. $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is $L$-fuzzy continuous;
2. $\mathcal{N}_f^{-\infty}(\mu, r) \subseteq \bigvee \{ \mathcal{N}_e(\lambda, r) \mid f^{-\infty}(\lambda) \leq \mu \}$;
3. $\text{Cl}_L(T, r) \leq \text{Cl}_L(f \circ T, r)$;
4. $\text{Con}_L(T, r) \leq \text{Con}_L(f \circ T, r)$;
5. $f^{-\infty}(C_{\mathcal{T}_2}(\lambda, r)) \leq C_{\mathcal{T}_2}(f^{-\infty}(\lambda), r)$;
6. $C_{\mathcal{T}_1}(f^{-\infty}(\mu), r) \leq f^{-\infty}(C_{\mathcal{T}_2}(\mu), r)$;
7. $f^{-\infty}(I_{\mathcal{T}_2}(\mu, r)) \leq I_{\mathcal{T}_1}(f^{-\infty}(\mu), r)$.

Proof. (1) $\Rightarrow$ (2) For any $\rho \in L^Y$ such that $\mathcal{T}_2(\rho) \geq r$ and $\rho \leq \mu$. Since $f$ is $L$-continuous, then $\mathcal{T}_1(f^{-\infty}(\rho)) \geq \mathcal{T}_2(\rho) \geq r$, and we have by Lemma 3(2)

$$S(f^{-\infty}(\epsilon), \rho) \leq S(f, f^{-\infty}(\rho)) \leq \bigvee \{ \mathcal{N}_e(\lambda, r) \mid f^{-\infty}(\lambda) \leq \mu \} \leq \bigvee \{ \mathcal{N}_e(\lambda, r) \mid f^{-\infty}(\lambda) \leq \rho \leq \mu \}.$$ (76)

Thus, $\mathcal{N}_f^{-\infty}(\mu, r) \subseteq \bigvee \{ \mathcal{N}_e(\lambda, r) \mid f^{-\infty}(\lambda) \leq \mu \}$.

(2) $\Rightarrow$ (3) If $f^{-\infty}(\lambda) \leq \mu$ and $f \circ T$ is finally in $\mu'$, there exists $n_0 \in D$ such that, for all $n \geq n_0$, $f(T(n)) \in \mu'$. Let $T(n) = x_i$. Then

$$t \leq \mu' \supseteq \bigvee \{ \mathcal{N}_e(\lambda, r) \mid f^{-\infty}(\lambda) \leq \mu \}.$$ (77)

It implies $T(n) \in \lambda'$. Therefore, $T$ is finally in $\lambda'$. One has

$$\text{Cl}_L(T, r) = \bigvee \{ \mathcal{N}_e(\lambda, r) \mid T \text{ is finally in } \lambda' \} \leq \bigvee \{ \mathcal{N}_e(\lambda, r) \mid f^{-\infty}(\lambda) \leq \mu, f \circ T \text{ is finally in } \mu' \} \leq \bigvee \{ \mathcal{N}_e(\lambda, r) \mid f^{-\infty}(\lambda) \leq \mu' \} = \text{Cl}_L(f \circ T, r).$$ (78)

(3) $\Rightarrow$ (4) Every subnet $U : E \to \text{Pt}(Y)$ of $f(T)$, and there exists a cofinal selection $N : E \to D$ such that $U = f(T) \circ N = f \circ (T \circ N)$. Put $K = T \circ N$. Then $K$ is a subnet of $T$. We can prove it from the following:

$$\text{Con}_L(T, r) \leq \text{Con}_L(K, r) \leq \text{Cl}_L(K, r) \leq \text{Cl}_L(f \circ K, r) \leq \text{Cl}_L(f \circ (T \circ N), r) = \text{Cl}_L(f \circ (U, r))$$ (79)

From Theorem 18(2), we have $\text{Con}_L(T, r) \leq \text{Con}_L(f \circ T, r)$.

(4) $\Rightarrow$ (5) From Theorem 5 and Proposition 17(2),

$$S(x_1, C_{\mathcal{T}_1}(\lambda, r)) = C_{\mathcal{T}_1}(\lambda, r)(x) = \text{Ad}_L(\lambda, r).$$ (80)

It implies

$$C_{\mathcal{T}_1}(\lambda, r)(x) = \text{Ad}_L(\lambda, r).$$ (81)

Thus, we have

$$f^{-\infty}(C_{\mathcal{T}_1}(\lambda, r))(y) = \bigvee \{ C_{\mathcal{T}_1}(\lambda, r)(x) \mid f(x) = y \} = \bigvee \{ \text{Ad}_L(\lambda, r) \mid f(x) = y \} \leq \bigvee \{ \text{Con}_L(f \circ T, r) \mid T \text{ is fuzzy net in } \lambda \} \leq \bigvee \{ \text{Con}_L(f \circ T, r) \mid T \text{ is fuzzy net in } \lambda \} = \text{Cl}_L(f \circ (U, r)).$$ (82)

(5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7) are easily proved.

(7) $\Rightarrow$ (1) We will show that $\mathcal{T}_1(f^{-\infty}(\mu)) \geq \mathcal{T}_2(\mu)$, for all $\mu \in L^Y$.

Let $\mathcal{T}_2(\mu) = 0$. It is trivial.

Let $\mathcal{T}_2(\mu) = r > 0$. Since $\mathcal{T}_N = \mathcal{T}_2$ from Theorem 12(b), we have for all $y \in Y$,

$$S(y, \mu) \geq \mathcal{N}_y(\mu, r).$$ (83)

It implies, for all $x \in X$,

$$S(f(x), \mu) = S(x, f^{-\infty}(\mu)) = \mathcal{N}_{f(x)}(\mu, r).$$ (84)
Since $f^{-}(I_{\mathcal{T}}_{z}(\mu, r)) = f^{-}(\mu)$,
\[ S\left(x, f^{-1}(\mu)\right) \]
\[ = S\left(x, f^{-}(I_{\mathcal{T}}_{z}(\mu, r))\right) \]
\[ = S\left(x, f^{-1}(I_{\mathcal{T}}_{z}(\mu, r))\right) = S\left(x, f^{-}(I_{\mathcal{T}}_{z}(\mu, r))\right) \quad \text{(since } f^{-}(I_{\mathcal{T}}_{z}(\mu, r)) \leq I_{\mathcal{T}}_{z}(f^{-}(\mu), r)) \]
\[ \leq S\left(x, I_{\mathcal{T}}_{z}(f^{-}(\mu), r)\right) \]
\[ = N_{x}(f^{-}(\mu), r) \quad \text{(by Proposition 17 (1))}. \]

Thus, by Theorem II(2), we have
\[ S\left(x, f^{-}(\mu)\right) = N_{x}(f^{-}(\mu), r). \quad \text{(86)} \]

Hence, $\mathcal{T}_{z}(f^{-}(\mu)) \geq r$. \hfill \Box

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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**References**


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