Research Article

A Smoothing Inexact Newton Method for Nonlinear Complementarity Problems

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Received 15 September 2014; Accepted 15 December 2014

Academic Editor: Neculai Andrei

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A smoothing inexact Newton method is presented for solving nonlinear complementarity problems. Different from the existing exact methods, the associated subproblems are not necessary to be exactly solved to obtain the search directions. Under suitable assumptions, global convergence and superlinear convergence are established for the developed inexact algorithm, which are extensions of the exact case. On the one hand, results of numerical experiments indicate that our algorithm is effective for the benchmark test problems available in the literature. On the other hand, suitable choice of inexact parameters can improve the numerical performance of the developed algorithm.

1. Introduction

In the study of equilibria problems from economics, engineering, and management sciences, a complementarity problem (CP) often appears as the prominent mathematical model of the equilibria problems. Thus, it is the most practical interest to develop a robust and efficient algorithm for solving CP in the past decades (see the very recently published book [1] and the references therein). In this paper, we consider a nonlinear complementarity problem (denoted by $\text{NCP}(F)$, for short): find a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0, \quad (1)$$

where $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable function. Due to the extensive applications, $\text{NCP}(F)$ has attracted great attention of researchers (see, e.g., [2, 3] and the references therein). On the one hand, there have been many theoretical results on the existence of solutions and their structural properties. On the other hand, many attempts have been made to develop implementable algorithms for the solution of $\text{NCP}(F)$.

A popular way to solve the $\text{NCP}(F)$ is to reformulate (1) to a nonsmooth equation via an NCP-function. Function $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be the NCP-function if

$$\phi(a, b) = 0 \iff a \geq 0, \quad b \geq 0, \quad ab = 0. \quad (2)$$

Define $\Phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$, given by

$$\Phi(x) = \begin{pmatrix} \phi(x_1, F_1(x)) \\ \phi(x_2, F_2(x)) \\ \vdots \\ \phi(x_n, F_n(x)) \end{pmatrix}. \quad (3)$$

Then, problem (1) is equivalent to

$$\Phi(x) = 0. \quad (4)$$

Thus, any efficient algorithm for solving (4) can be directly applied to find the solution of problem (1).

Smoothing method is a fundamental approach to solve the nonsmooth equation (4). In this connection, one can see, for example, [4–16]. The basic idea of this method is to construct a smooth function to approximate Φ . Let $\mu > 0$

be a given smoothing parameter. We define a continuously differentiable function $\Phi_\mu : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for any $\mu > 0$ and $x \in \mathbb{R}^n$ there holds

$$\|\Phi(x) - \Phi_\mu(x)\| \rightarrow 0, \quad \text{as } \mu \rightarrow 0. \quad (5)$$

Then, problem (4) is approximated by a smooth equation:

$$\Phi_\mu(x) = 0. \quad (6)$$

Let $\{\mu^k\}$ be a given positive sequence which tends to 0. Then, we can obtain an approximate solution of (4) by solving (6) with $\mu = \mu^k$.

Recently, there are many different smoothing functions being employed to smooth the problem (4). Among them, the Fischer-Burmeister function and the minimum function are two popular ones, which are defined by

$$\phi(a, b) = a + b - \sqrt{a^2 + b^2}, \quad \forall (a, b) \in \mathbb{R}^2, \quad (7)$$

$$\phi(a, b) = \min\{a, b\}, \quad \forall (a, b) \in \mathbb{R}^2, \quad (8)$$

respectively. With the 2-norm of (a, b) in the Fischer-Burmeister function being replaced by a more general p -norm with $p \in (1, \infty)$, Chen and Pan proposed a family of new NCP-function in [6]. By combining the Fischer-Burmeister function and the minimum function, Liu and Wu presented a smoothing function in [11] as follows:

$$\begin{aligned} \phi_\theta(a, b) &= a + b - \sqrt{\theta(a - b)^2 + (1 - \theta)(a^2 + b^2)}, \\ \theta &\in [0, 1] \quad \forall (a, b) \in \mathbb{R}^2. \end{aligned} \quad (9)$$

In [13], a symmetric perturbed Fischer-Burmeister function is constructed:

$$\begin{aligned} \phi(\mu, a, b) &= (1 + \mu)(a + b) \\ &\quad - \sqrt{(a + \mu b)^2 + (b + \mu a)^2 + \mu^2}, \\ &\quad \forall (\mu, a, b) \in \mathbb{R}^3. \end{aligned} \quad (10)$$

Very recently, in [15], a more general smoothing function with the p -norm ($p \in (1, \infty)$) was presented. It is shown that for the nonmonotone smoothing Newton algorithm developed in [14] the numerical performance of algorithm is greatly improved if $p = 1.1$.

In this paper, we first write (8) as

$$\min\{a, b\} = \frac{1}{2}(a + b - |a - b|), \quad \forall (a, b) \in \mathbb{R}^2, \quad (11)$$

then, we intend to investigate a new smoothing method of $|\cdot|$, and in virtue of this new method, we will design a smoothing inexact Newton algorithm to solve the obtained smooth equations. Since an inexact parameter at each iteration is admissible to obtain an inexact Newton search direction, the developed algorithm is more helpful to numerical computation than the similar ones available in the literature. By suitably choosing a sequence of inexact parameters in

advance, numerical performance of the developed algorithm in this paper can be improved. On the other hand, without the assumption of strict complementarity, we can establish the theory of convergence for our algorithm, which is weaker than that in the existing results.

The rest of this paper is organized as follows. In Section 2, we study a smoothing method of the absolute function. Section 3 is devoted to development of a smoothing inexact Newton algorithm to solve the nonlinear complementarity problem. In Section 4, the global convergence and the superlinear convergence are established. Numerical results are reported in Section 5. Some final remarks are made in Section 6.

The following notions will be used throughout this paper. For any vector or matrix A , A^T denotes the transposition of A . $\mathbb{R}^n$ denotes the space of n -dimensional real vectors. $\mathbb{R}_+^n$ and $\mathbb{R}_{++}^n$ denote the nonnegative and the positive subspaces in $\mathbb{R}^n$, respectively. For any vector $v \in \mathbb{R}^n$, $\text{diag}\{v_i : i \in N\}$ denotes a diagonal matrix, whose i th diagonal element is v_i and $\text{vec}\{v_i : i \in N\}$ the vector v , N represents the set $\{1, 2, \dots, n\}$. I represents the identity matrix with a suitable dimension. $\|\cdot\|$ stands for the 2-norm. For any $\alpha, \beta \in \mathbb{R}_{++}$, $\alpha = O(\beta)$ and $\alpha = o(\beta)$ represent that α/β is uniformly bounded and that α/β tends to zero as $\beta \rightarrow 0$, respectively.

2. Smoothing the Absolute Function

In this section, we will study a smoothing method of the absolute function.

We first present a function $\varphi_\mu : \mathbb{R} \rightarrow \mathbb{R}$, given by

$$\varphi_\mu(t) = \frac{2}{\pi} \arctan\left(\frac{t}{\mu}\right). \quad (12)$$

It is clear that

$$\lim_{\mu \rightarrow 0^+} \varphi_\mu(t) = \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \\ -1, & \text{if } t < 0. \end{cases} \quad (13)$$

Note that the generalized derivative of the absolute function $|\cdot|$ is calculated by

$$\text{sign}(t) = \begin{cases} 1, & \text{if } t > 0, \\ [-1, 1], & \text{if } t = 0, \\ -1, & \text{if } t < 0. \end{cases} \quad (14)$$

We can conclude that, except for $t = 0$, $\varphi_\mu(t)$ is a good approximation to the generalized derivative of $|t|$ with a sufficiently small μ . Actually, the following result was proved in [17].

Proposition 1. *For any given constant $\alpha > 0$, there is a constant $M_\alpha > 0$ independent of μ and t such that*

$$\begin{aligned} 0 &\leq \text{sign}(t) - \varphi_\mu(t) \leq M_\alpha \mu, \quad \forall t : t \geq \alpha, \quad \forall \mu > 0, \\ 0 &\leq \varphi_\mu(t) - \text{sign}(t) \leq M_\alpha \mu, \quad \forall t : t \leq -\alpha, \quad \forall \mu > 0. \end{aligned} \quad (15)$$

By Proposition 1, to obtain an approximation of $|t|$, we calculate the integral of φ_μ :

$$\begin{aligned} \int \varphi_\mu(t) dt &= \frac{2}{\pi} \int \arctan\left(\frac{t}{\mu}\right) dt \\ &= t\varphi_\mu(t) - \frac{1}{\pi} \mu \ln\left(1 + \frac{t^2}{\mu^2}\right) \triangleq \phi_\mu(t). \end{aligned} \quad (16)$$

Then, it is natural that we use $\phi_\mu(t)$ to approximate $|t|$. Actually, we have the following result (see [17]).

Proposition 2. (1) For any $\mu > 0$, it holds that

$$|t| \leq \phi_\mu(t), \quad \forall t \in \mathbb{R}. \quad (17)$$

The above inequality holds strictly for all $t \neq 0$.

(2) For any $t \in \mathbb{R}^n$, $\lim_{\mu \rightarrow 0^+} \phi_\mu(t) = |t|$.

(3) $\lim_{\mu \rightarrow 0^+} \text{dist}(\phi'_\mu(t), \partial h(t)) = 0$, where $h(t) = |t|$ and $\text{dist}(v, S)$ denotes the distance from the point v to the set S .

3. A Smoothing Inexact Newton Algorithm for NCP(F)

In this section, we will develop a smoothing inexact Newton algorithm for solving a smooth equation obtained by reformulating the NCP(F).

Since

$$\begin{aligned} \min\{x_i, F_i(x)\} &= \frac{1}{2}(x_i + F_i(x) - |x_i - F_i(x)|) \triangleq \phi_i(x), \\ &\quad (i \in N), \end{aligned} \quad (18)$$

we construct an approximation of $\phi_i(x)$ by Proposition 2, defined by

$$\begin{aligned} \phi_i(\mu, x) &= \frac{1}{2}(x_i + F_i(x) - \phi_\mu(x_i - F_i(x))) \\ &= \frac{1}{2}\left(x_i + F_i(x) - \frac{2}{\pi}(x_i - F_i(x)) \arctan\left(\frac{x_i - F_i(x)}{\mu}\right) \right. \\ &\quad \left. + \frac{1}{\pi} \mu \ln\left(1 + \frac{(x_i - F_i(x))^2}{\mu^2}\right)\right). \end{aligned} \quad (19)$$

Define $\Phi_\mu : \mathbb{R}^n \rightarrow \mathbb{R}^n$, given by

$$\Phi_\mu(x) = \begin{pmatrix} \phi_1(\mu, x) \\ \vdots \\ \phi_n(\mu, x) \end{pmatrix}. \quad (20)$$

Then, $\Phi(x) = 0$ is approximated by a smooth equation:

$$\Phi_\mu(x) = 0. \quad (21)$$

Remark 3. The above smoothing method has been used to deal with NCP(F) in [17]. Then, by solving a generalized Newton equation:

$$\nabla \Phi_\mu(x^k)^T d + \Phi(x^k) = 0, \quad (22)$$

so as to obtain a search direction d at k -iteration for the developed algorithm in [17]. Different from the standard Newton method, $\Phi(x^k)$ is employed to replace $\Phi_\mu(x^k)$ in (22).

Taking into account the advantage of the standard smoothing Newton method (see, e.g., [12, 15, 16, 18]) in adjusting the value of smoothing parameter automatically, we further transform problem (21) into a smooth optimization problem.

Denote $z = (\mu, x) \in \mathbb{R}_{++} \times \mathbb{R}^n$. We define a function $H : \mathbb{R}_{++} \times \mathbb{R}^n \rightarrow \mathbb{R}_{++} \times \mathbb{R}^n$ by

$$H(z) = \begin{pmatrix} \mu \\ \Phi(z) \end{pmatrix} \quad (23)$$

with $\Phi(z) = (\phi_1(z), \phi_2(z), \dots, \phi_n(z))^T$. Then, corresponding to any solution x^* of (21), $z^* = (0, x^*)$ is an optimal solution of the following minimization problem:

$$\min \Psi(z) \triangleq \|H(z)\|^2 = \mu^2 + \|\Phi(z)\|^2. \quad (24)$$

Conversely, if $z^* = (\mu^*, x^*)$ is an optimal solution of problem (24) with $\Psi(z^*) = 0$, then, x^* solves the system of (21).

Next, we focus on developing an efficient algorithm to solve problem (24). Before presentation of such an algorithm, we further investigate the properties of problem (24). The following definitions are useful to describe the properties of H .

Definition 4. A matrix $M \in \mathbb{R}^{n \times n}$ is said to be a P_0 matrix if all principal minors of M are nonnegative.

Definition 5. A function $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be a P_0 function if for all $x, y \in \mathbb{R}^n$ with $x \neq y$, there holds that

$$\max_{i \in N} (x_i - y_i) [F_i(x) - F_i(y)] \geq 0. \quad (25)$$

Definition 6 (see [19, 20]). Suppose that $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a locally Lipschitz continuous function, which has the generalized Jacobian $\partial \Psi(x)$ in the sense of Clarke [21], it is said to be semismooth (or strongly semismooth) at a point $x \in \mathbb{R}^n$ if and only if for any $V \in \partial \Psi(x + h)$, as $h \rightarrow 0$,

$$\|Vh - \Psi'(x, h)\| = o(\|h\|), \quad (\text{or } O(\|h\|^2)),$$

$$\|\Psi(x + h) - \Psi(x) - \Psi'(x, h)\| = o(\|h\|), \quad (\text{or } O(\|h\|^2)). \quad (26)$$

We now prove the following results.

Lemma 7. Let $z = (\mu, x)$ and H be defined by (23). Then, consider the following:

- (i) H is continuously differentiable at any $z = (\mu, x) \in \mathbb{R}_{++} \times \mathbb{R}^n$ with its Jacobian matrix

$$H'(z) = \begin{pmatrix} 1 & 0 \\ A(z) & B(z) \end{pmatrix}, \quad (27)$$

where

$$\begin{aligned} A(z) &= \frac{1}{2} \text{vec} \left\{ \frac{1}{\pi} \ln \left(1 + \frac{(x_i - F_i(x))^2}{\mu^2} \right) : i \in N \right\}, \\ B(z) &= \frac{1}{2} (I - D(z) + (I + D(z))F'(x)), \\ D(z) &= \frac{2}{\pi} \text{diag} \left\{ \arctan \left(\frac{x_i - F_i(x)}{\mu} \right) : i \in N \right\}. \end{aligned} \quad (28)$$

Furthermore, if F is a P_0 -function, then, H' is nonsingular for any $\mu > 0$.

- (ii) H is locally Lipschitz continuous and semismooth on $\mathbb{R}^{n+1}$.

Proof. (i) Since Φ is continuous differentiable at any $z = (\mu, x) \in \mathbb{R}_{++} \times \mathbb{R}^n$, then H is continuous differentiable. For any $\mu > 0$, by straightforward calculation, it yields (27) from the definition of H .

Note that, for all $i \in N$, $-1 < D(z)_{ii} < 1$. It is clear that $I - D(z)$ and $I + D(z)$ are two positive diagonal matrices. Since F is a P_0 -function, F' is also a P_0 -matrix for all $x \in \mathbb{R}^n$. Thus, the principal minors of the matrix $(I + D(z))F'(x)$ are nonnegative. By Definition 4, we know that the matrix $(I + D(z))F'(x)$ is a P_0 -matrix. From Theorem 3.3 in [7], it follows that the matrix $B(z)$ is nonsingular. Then, it is concluded that the matrix $H'(z)$ is nonsingular.

(ii) It is clear that H is locally Lipschitz continuous and semismooth on $\mathbb{R}^{n+1}$. The proof is completed. $\square$

With the properties of H in Lemma 7, we first present an algorithm to solve problem (24) similar to the idea in [18, 22–25].

Algorithm 8 (a smoothing inexact Newton method).

Step 0. Choose constants $\delta, \gamma \in (0, 1)$, $\sigma \in (0, 1/2)$, $\mu_0 > 0$ such that $\mu_0\gamma < 1$. Given an initial point $x^0 \in \mathbb{R}^n$, choose a sequence $\{\theta_k\} \subset \mathbb{R}_{++}$ such that $\theta_k \in (0, 1 - \mu_0\gamma)$. Set $z^0 := (\mu_0, x^0)$ and $k := 0$.

Step 1. If $\|H(z^k)\| = 0$, then the algorithm stops. Otherwise, compute

$$\begin{aligned} \beta(z^k) &:= \gamma \min\{1, \Psi(z^k)\}, \\ h(z^k) &:= (\mu_0\beta(z^k), \theta_k\Phi(z^k)). \end{aligned} \quad (29)$$

Step 2. Compute $\Delta z^k := (\Delta\mu^k, \Delta x^k) \in \mathbb{R} \times \mathbb{R}^n$ by

$$H(z^k) + H'(z^k)\Delta z^k = h(z^k). \quad (30)$$

Step 3. Set $\alpha_k := \delta^{m_k}$, where m_k is the smallest nonnegative integer m such that

$$\Psi(z^k + \delta^m \Delta z^k) \leq [1 - 2\sigma(1 - \mu_0\gamma - \theta_k)\delta^m]\Psi(z^k). \quad (31)$$

Step 4. Set $z^{k+1} := z^k + \alpha_k \Delta z^k$ and $k := k + 1$. Return to Step 1.

Remark 9. Similar to the idea in [26], we develop Algorithm 8 by incorporating an inexact parameter θ_k at each iteration to obtain an inexact Newton direction of search in (30). Generally, we choose a sequence $\{\theta_k\}$ in advance, such that $\lim_{k \rightarrow \infty} \theta_k = 0$. Suitable choice of $\{\theta_k\}$ can be used to improve the numerical performance of Algorithm 8 by generating an inexact Newton direction Δz^k in Step 2 of Algorithm 8. The difference between Algorithm 8 and that developed in [26] lies in the distinct smoothing method. In [26], instead of the smoothing function (19), the Fischer-Burmeister function is adopted.

On the other hand, without the assumption of strict complementarity, we will establish the theory of global and local superlinearly convergences for Algorithm 8 in Section 4 under weaker conditions than the existing results.

If $\theta_k \equiv 0$, then, Algorithm 8 reduces to a smoothing Newton algorithm, which is similar to that developed in [18]. However, the definition of h in this paper is different from that in [18].

Denote

$$\Omega := \{z = (\mu, x) \in \mathbb{R}^{n+1} : \mu \geq \mu_0\beta(z)\}. \quad (32)$$

The following result shows that Algorithm 8 is well-defined.

Theorem 10. Suppose that F is a continuous differentiable P_0 -function.

- (1) For the system of linear equations (30) in the unknown variable Δz^k , there exists a unique solution.
- (2) In finitely many back-tracking steps, α_k in Step 3 of Algorithm 8 is obtained to satisfy (31).
- (3) Let $\{z^k\}$ be the sequence generated by Algorithm 8. Then, for all $k > 0$, $z^k \in \Omega$.

Proof. We prove the first result.

Since F is a continuously differentiable P_0 -function, it follows from Lemma 7 that the matrix H' is nonsingular at z^k as $\mu_k > 0$. It implies that the system of linear equations (30) in the unknown variable Δz^k has a unique solution. Thus, Step 2 of Algorithm 8 is well-defined.

We now prove the second result.

By (30), we have

$$\Delta\mu_k = -\mu_k + \mu_0\beta(z^k). \quad (33)$$

From the definitions of $\Psi(z^k)$ and $\beta(z^k)$, it follows that, for all $k \geq 0$,

$$\begin{aligned}\beta(z^k) &\leq \gamma \Psi(z^k)^{1/2}, \\ \mu_k &\leq \Psi(z^k)^{1/2}.\end{aligned}\quad (34)$$

Thus, for any $\alpha \in (0, 1)$

$$\begin{aligned}(\mu_k + \alpha \Delta \mu_k)^2 &= [(1 - \alpha)\mu_k + \alpha \mu_0 \beta(z^k)]^2 \\ &= (1 - \alpha)^2 \mu_k^2 + 2(1 - \alpha) \alpha \mu_0 \mu_k \beta(z^k) \\ &\quad + \alpha^2 \mu_0^2 \beta(z^k)^2 \\ &\leq (1 - \alpha)^2 \mu_k^2 + 2\alpha \mu_0 \gamma \Psi(z^k) + O(\alpha^2).\end{aligned}\quad (35)$$

Denote

$$\varphi(\alpha) = \Phi(z^k + \alpha \Delta z_k) - \Phi(z^k) - \alpha \Phi'(z^k) \Delta z^k. \quad (36)$$

Since Φ is continuous differentiable at $z \in \mathbb{R}_{++} \times \mathbb{R}^n$, then $\|\varphi(\alpha)\| = o(\alpha)$; we conclude from (36) that

$$\begin{aligned}\|\Phi(z^k + \alpha \Delta z^k)\|^2 &= \|\Phi(z^k) + \alpha \Phi'(z^k) \Delta z^k + \varphi(\alpha)\|^2 \\ &= \|(1 - \alpha + \alpha \theta_k) \Phi(z^k) + \varphi(\alpha)\|^2 \\ &\leq (1 - \alpha + \alpha \theta_k)^2 \|\Phi(z^k)\|^2 + o(\alpha).\end{aligned}\quad (37)$$

It yields

$$\begin{aligned}\Psi(z^k + \alpha \Delta z_k) &= \mu_{k+1}^2 + \|\Phi(z^k + \alpha \Delta z_k)\|^2 \\ &\leq (1 - \alpha)^2 \mu_k^2 + 2\alpha \mu_0 \gamma \Psi(z^k) \\ &\quad + (1 - \alpha + \alpha \theta_k)^2 \|\Phi(z^k)\|^2 + o(\alpha) + O(\alpha^2) \\ &\leq (1 - \alpha + \alpha \theta_k)^2 \Psi(z^k) + 2\alpha \mu_0 \gamma \Psi(z^k) + o(\alpha) \\ &\leq \Psi(z^k) - 2\alpha(1 - \theta_k) \Psi(z^k) \\ &\quad + \alpha^2(1 - \theta_k)^2 \Psi(z^k) + 2\alpha \mu_0 \gamma \Psi(z^k) + o(\alpha) \\ &\leq [1 - 2(1 - \mu_0 \gamma - \theta_k)\alpha] \Psi(z^k) + o(\alpha).\end{aligned}\quad (38)$$

Since $\theta_k < 1 - \mu_0 \gamma$, there exists a constant $\bar{\alpha} \in (0, 1)$ such that, for any $\alpha \in (0, \bar{\alpha})$ and $\sigma \in (0, 1)$, there holds that

$$\Psi(z^k + \alpha \Delta z^k) \leq [1 - 2\sigma(1 - \mu_0 \gamma - \theta_k)\alpha] \Psi(z^k). \quad (39)$$

This demonstrates that Step 3 of Algorithm 8 is well-defined at each iteration.

Finally, we prove $z^k \in \Omega$ for all $k > 0$.

It is clear that $\mu_0 \beta(z^0) \leq \mu_0 \gamma \leq \mu_0$. In other words, $z^0 \in \Omega$. Suppose that $z^k \in \Omega$ as $k \geq 1$. Then, $\mu_k \geq \mu_0 \beta(z^k)$. By (31), we get $\Psi(z^k) \geq \Psi(z^{k+1})$; then $\beta(z^k) \geq \beta(z^{k+1})$. By (33), we have

$$\begin{aligned}\mu_{k+1} &= (1 - \alpha) \mu_k + \alpha \mu_0 \beta(z^k) \\ &\geq (1 - \alpha) \mu_0 \beta(z^k) + \alpha \mu_0 \beta(z^k) \\ &\geq \mu_0 \beta(z^k) \geq \mu_0 \beta(z^{k+1}).\end{aligned}\quad (40)$$

The last inequality implies that the desired result holds for $k + 1$. By mathematical induction method, we concluded that $z^k \in \Omega$ for all $k > 0$.

We have completed the proof of Theorem 10. $\square$

Remark 11. By Theorem 10, we know that Algorithm 8 is well-defined, and either it stops in finitely many steps or generates an infinite sequence $\{z^k = (\mu_k, x^k)\}$ with $\mu \in \mathbb{R}_{++}$ and $z^k \in \Omega$ for all $k \geq 0$. In the subsequent section, we will analyze the convergence of this sequence.

4. Convergence

In this section, we will establish the global convergence and the superlinear convergence for Algorithm 8.

We first prove the following result.

Lemma 12. Let Φ_μ be defined by (20). If F is a P_0 -function, then, for any $\mu > 0$, Φ_μ is coercive in x . That is,

$$\lim_{\|x\| \rightarrow \infty} \|\Phi_\mu(x)\| \rightarrow +\infty. \quad (41)$$

Proof. As $\|x\| \rightarrow \infty$, there exists a vector sequence $\{x^k\}$ which is unbounded. Then, there is a component $i_0 \in \{1, 2, \dots, n\}$ such that $\{x_{i_0}^k\}$ is unbounded.

Define an index set $J = \{i \in N : \{x_i^k\} \text{ is unbounded}\}$. Then, J is a nonempty set. Without loss of generality, we assume that $\{|x_j^k|\} \rightarrow +\infty$, for all $j \in J$.

Let the sequence $\{\hat{x}^k\}$ be defined by

$$\hat{x}_i^k = \begin{cases} 0, & \text{if } i \in J, \\ x_i^k, & \text{if } i \notin J, \end{cases} \quad i \in N. \quad (42)$$

Then, it is clear that $\{\hat{x}^k\}$ is bounded. Since F is a P_0 -function, by Definition 5, we have

$$\begin{aligned}0 &\leq \max_{i \in N} \{(x_i^k - \hat{x}_i^k) [F_i(x^k) - F_i(\hat{x}^k)]\} \\ &= \max_{i \in J} \{x_i^k [F_i(x^k) - F_i(\hat{x}^k)]\} \\ &= x_j^k [F_j(x^k) - F_j(\hat{x}^k)],\end{aligned}\quad (43)$$

where j is one of the indices at which the max is attained. Since $j \in J$, and j can be supposed to be independent of k , we know $|x_j^k| \rightarrow +\infty$ as $k \rightarrow +\infty$.

Next, we continue the proof in the following six directions.

Case 1 ($x_j^k \rightarrow +\infty$ and $x_j^k - F_j(x^k) \rightarrow +\infty$). Since $\{F_j(\hat{x}^k), k \in N\}$ is bounded by the continuity of F_j and the definition of $\hat{x}^k$, we know that $F_j(x^k) \rightarrow -\infty$ from (43). Thus,

$$\ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) \rightarrow +\infty. \quad (44)$$

It yields

$$\begin{aligned} |\phi_j(\mu, x^k)| &= \frac{1}{2} \left| x_j^k + F_j(x^k) \right. \\ &\quad - \frac{2}{\pi} (x_j^k - F_j(x^k)) \arctan\left(\frac{x_j^k - F_j(x^k)}{\mu}\right) \\ &\quad \left. + \frac{1}{\pi} \mu \ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) \right| \rightarrow +\infty. \end{aligned} \quad (45)$$

Case 2 ($x_j^k \rightarrow -\infty$ and $x_j^k - F_j(x^k) \rightarrow +\infty$). It is clear that

$$\begin{aligned} F_j(x^k) - x_j^k &\rightarrow -\infty, \\ \ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) &\rightarrow +\infty. \end{aligned} \quad (46)$$

In virtue of

$$\lim_{u \rightarrow -\infty} \frac{\ln(1 + u^2)}{u} = 0, \quad (47)$$

we obtain

$$\begin{aligned} |\phi_j(\mu, x^k)| &= \frac{1}{2} \left| x_j^k + F_j(x^k) \right. \\ &\quad - \frac{2}{\pi} (x_j^k - F_j(x^k)) \arctan\left(\frac{x_j^k - F_j(x^k)}{\mu}\right) \\ &\quad \left. + \frac{1}{\pi} \mu \ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) \right| \\ &= \left| (F_j(x^k) - x_j^k) \right. \\ &\quad \left. + \frac{\mu}{2\pi} \ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) + x_j^k \right| \rightarrow +\infty. \end{aligned} \quad (48)$$

Case 3 ($x_j^k \rightarrow +\infty$ and $x_j^k - F_j(x^k) \rightarrow -\infty$). In the same reason as in Case 1, $F_j(x^k) \rightarrow -\infty$. Thus,

$$\ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) \rightarrow +\infty. \quad (49)$$

It yields

$$\begin{aligned} |\phi_j(\mu, x^k)| &= \frac{1}{2} \left| x_j^k + F_j(x^k) \right. \\ &\quad - \frac{2}{\pi} (x_j^k - F_j(x^k)) \arctan\left(\frac{x_j^k - F_j(x^k)}{\mu}\right) \\ &\quad \left. + \frac{1}{\pi} \mu \ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) \right| \\ &= \left| x_j^k + \frac{\mu}{2\pi} \ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) \right| \rightarrow +\infty. \end{aligned} \quad (50)$$

Case 4 ($x_j^k \rightarrow -\infty$ and $x_j^k - F_j(x^k) \rightarrow -\infty$). Similar to Case 2, we can obtain

$$\begin{aligned} |\phi_j(\mu, x^k)| &= \frac{1}{2} \left| x_j^k + F_j(x^k) \right. \\ &\quad - \frac{2}{\pi} (x_j^k - F_j(x^k)) \arctan\left(\frac{x_j^k - F_j(x^k)}{\mu}\right) \\ &\quad \left. + \frac{1}{\pi} \mu \ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) \right| \\ &= \left| x_j^k + \frac{\mu}{2\pi} \ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) \right| \\ &= \left| (x_j^k - F_j(x^k)) + \frac{\mu}{2\pi} \ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) \right. \\ &\quad \left. + F_j(x^k) \right| \rightarrow +\infty. \end{aligned} \quad (51)$$

Case 5 ($x_j^k \rightarrow +\infty$ and $x_j^k - F_j(x^k)$ is bounded). On the one hand, it is clear that

$$w = -\frac{2}{\pi} (x_j^k - F_j(x^k)) \arctan\left(\frac{x_j^k - F_j(x^k)}{\mu}\right) + \frac{1}{\pi} \mu \ln\left(1 + \frac{(x_j^k - F_j(x^k))^2}{\mu^2}\right) \quad (52)$$

is bounded. On the other hand, $x_j^k \rightarrow +\infty$ and $x_j^k - F_j(x^k)$ is bounded; we know $F_j(x^k) \rightarrow +\infty$. Thus, $x_j^k + F_j(x^k) \rightarrow +\infty$. It yields

$$|\phi_j(\mu, x^k)| = \frac{1}{2} |x_j^k + F_j(x^k) + w| \rightarrow +\infty. \quad (53)$$

Case 6 ($x_j^k \rightarrow -\infty$ and $x_j^k - F_j(x^k)$ is bounded). Similar to Case 5, it is easy to prove that $|\phi_j(\mu, x^k)| \rightarrow +\infty$.

The proof is completed. $\square$

Remark 13. By Lemma 12, we can remove the assumption that the level set of the merit function is bounded. In addition, different from [13, 22, 27], the result of Lemma 12 is obtained in this paper for the nonsymmetric smoothing function.

Before statement of main results, we need the following assumption.

Assumption 14. The solution set S of NCP(F) (1) is nonempty and bounded.

Remark 15. Assumption 14 is a relatively weak condition to ensure the convergence of Algorithm 8. For example, in [26], it is assumed that the level sets

$$L(z^0) = \{z \in R^{n+1}, \Phi(z) \leq \Phi(z_0)\} \quad (54)$$

are bounded to prove the convergence of algorithm. Up to our knowledge, for the Fischer-Burmeister smoothing function, (54) is proved to be true under the condition that F in NCP(F) (1) is a uniform P -function.

However, with our smoothing method, we can prove that (54) holds. Since the proof is only involved with the condition that F is a P_0 -function, Assumption 14 is weaker than that in [26].

With Lemma 12 and Assumption 14, we are now in a position to establish the convergence theory for Algorithm 8.

Theorem 16. Let $\{z^k = (\mu_k, x^k)\}$ be the iteration sequence generated by Algorithm 8. Under Assumption 14, the following statements are true.

- (i) $\{\Psi(z^k)\}$ and $\{\mu_k\}$ generated by Algorithm 8 are two monotonically decreasing and bounded sequences, whose limits are 0.

- (ii) Any accumulation point of $\{z^k\}$ is a solution of (24).

- (iii) Under Assumption 14, $\{z^k\}$ has at least one accumulation point $z^* = (\mu_*, x^*)$ with $H(z^*) = 0$ and $x^* \in S$.

Proof. (i) From Steps 2 and 3 of Algorithm 8, it is clear that $\{\Psi(z^k)\}$, $\{\beta(z^k)\}$, and $\{\mu_k\}$ are three monotonically decreasing and bounded sequences.

(ii) By Lemma 12, we conclude that the sequence $\{z^k\}$ is bounded. Then, without loss of generality, we suppose that as $k \rightarrow \infty$, there exists z^* such that

$$z^k \rightarrow z^*, \quad \beta(z^k) \rightarrow \beta_*, \quad \Psi(z^k) \rightarrow \Psi_*, \quad \mu^k \rightarrow \mu_*. \quad (55)$$

If $\Psi_* > 0$, then, by the definition of $\beta(z^k)$, $\beta_* > 0$ and $\mu_* > 0$. From Lemma 7, it follows that $H'(z^*)$ is nonsingular. Thus, there exist a closed neighborhood $N(z^*)$ and a constant $\bar{\alpha} \in (0, 1]$, such that, for any $z \in N(z^*)$ and nonnegative integer m satisfying $\delta^m \in (0, \bar{\alpha}]$, the following inequality holds true:

$$\Psi(z^k + \delta^m \Delta z^k) \leq [1 - 2\sigma(1 - \mu_0\gamma - \theta_k)\delta^m] \Psi(z^k). \quad (56)$$

If k is large enough such that $m^k \leq m$ and $\delta^{m^k} \geq \delta^m$, then,

$$\begin{aligned} \Psi(z^{k+1}) &\leq [1 - 2\sigma(1 - \mu_0\gamma - \theta_k)\delta^{m^k}] \Psi(z^k) \\ &\leq [1 - 2\sigma(1 - \mu_0\gamma - \theta_k)\delta^m] \Psi(z^k). \end{aligned} \quad (57)$$

Therefore, as $k \rightarrow \infty$, it follows from $\Psi_* > 0$ that

$$2\sigma(1 - \mu_0\gamma - \theta_k) \leq 0. \quad (58)$$

It contradicts $(1 - \mu_0\gamma - \theta_k) > 0$. We conclude that $\Psi(z^k) \rightarrow 0$ and $\mu_k \rightarrow 0$.

(iii) By Assumption 14, we know that $\Phi^{-1}(0)$ is nonempty and bounded. Thus, $\{z^k\}$ has at least one accumulation point $z^* = (\mu_*, x^*)$ with $H(z^*) = 0$ and $x^* \in S$. $\square$

Theorem 17. Suppose that Assumption 14 is satisfied and $z^* = (\mu_*, x^*)$ is an accumulation point of the sequence $\{z^k\}$ generated by Algorithm 8. If all $V \in \partial H(z^*)$ are nonsingular, then, $\{z^k\}$ converges to z^* superlinearly; that is, $\|z^{k+1} - z^*\| = o(\|z^k - z^*\|)$. Moreover, $\mu_{k+1} = o(\mu_k)$.

Proof. By Theorem 16, we have $H(z^*) = 0$ and $x^* \in S$. Because all $V \in \partial H(z^*)$ are nonsingular, it follows that for all z^k sufficiently close to z^* ,

$$\|H'(z^k)^{-1}\| = O(1). \quad (59)$$

From Lemma 7, it follows that $H(\cdot)$ is semismooth at z^* . Hence, for all z^k sufficiently close to z^* , we have

$$\|H(z^k) - H(z^*) - H'(z^k)(z^k - z^*)\| = o(\|z^k - z^*\|). \quad (60)$$

On the other hand, Lemma 7 implies that $H(\cdot)$ is locally Lipschitz continuous near z^* . Therefore, for all z^k sufficiently close to z^* , we have

$$\|H(z^k)\| = O(\|z^k - z^*\|). \quad (61)$$

Since $\lim_{k \rightarrow \infty} \theta_k = 0$, it is concluded that $\theta_k \|H(z^k)\| = o\|z^k - z^*\|$. Thus, by the definitions of $h(z)$ and $\beta(z)$, we have

$$\begin{aligned} \|h(z^k)\| &\leq |\mu_0 \beta(z^k)| + \|\theta_k \Phi(z^k)\| \\ &\leq \mu_0 \gamma \Psi(z^k) + \theta_k \|H(z^k)\| \\ &= o(\|z^k - z^*\|). \end{aligned} \quad (62)$$

Then, in view of (59), (60), and (62), it is obtained that

$$\begin{aligned} &\|z^k + \Delta z^k - z^*\| \\ &= \|z^k + H'(z^k)^{-1} [-H(z^k) + h(z^k)] - z^*\| \\ &\leq \|H'(z^k)^{-1}\| \\ &\quad \cdot (\|H(z^k) - H(z^*) - H'(z^k)(z^k - z^*)\| + \|h(z^k)\|) \\ &= o(\|z^k - z^*\|). \end{aligned} \quad (63)$$

On the other hand, from (61), it follows that

$$\begin{aligned} &\Psi(z^k + \Delta z^k) \\ &= \|H(z^k + \Delta z^k)\|^2 \\ &= O(\|z^k + \Delta z^k - z^*\|^2) = o(\|z^k - z^*\|^2) \\ &= o(\|H(z^k)\|^2) = o(\Psi(z^k)). \end{aligned} \quad (64)$$

Thus, as z^k sufficiently close to z^* , we have $z^{k+1} = z^k + \Delta z^k$. It yields

$$\mu_{k+1} = \mu_k + \Delta \mu_k = \mu_0 \gamma \|H(z^k)\|^2. \quad (65)$$

In virtue of (65), we obtain

$$\frac{\mu_{k+1}}{\mu_k} = \frac{\|H(z^k)\|^2}{\|H(z^{k-1})\|^2} = \frac{o(\Psi(z^{k-1}))}{\Psi(z^{k-1})}. \quad (66)$$

As z^k sufficiently close to z^* , we know $\mu_{k+1} = o(\mu_k)$. The proof has been completed. $\square$

5. Numerical Experiments

In this section, we test the numerical performance of Algorithm 8 for solving benchmark test problems in NCP.

Algorithm 8 is implemented in MATLAB2008a on a PC 2.00 GHZ CPU with 2.00 GB RAM with the operation system

of Windows 7. Throughout the experiments, the parameters in Algorithm 8 are chosen as follows:

$$\mu_0 = 0.01, \quad \sigma = 0.25, \quad \delta = 0.85, \quad \gamma = 0.2, \quad \theta_k = \frac{1}{4^{k+1}}. \quad (67)$$

We use $\|H(z^k)\| < 10^{-8}$ as the termination criterion.

Numerical results are reported in Tables 1–9, where we use the following denotations for conciseness:

IT: the number of iterations,

ST: the initial point x^0 ,

CT: the CPU time depleted by the algorithm,

x^* : a solution of the NCP,

x^k : the final value of x ,

o : zero vector with n dimension,

e : unit vector with n dimension,

F: The algorithm fails to get a solution.

The test problems are from the literature (see, e.g., [22, 27, 28]).

Problem 1. In problem (1), $x \in \mathbb{R}^3$ and $F(x) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is given by

$$F(x) = \begin{pmatrix} x_2 \\ x_3 \\ -x_2 + x_3 + 1 \end{pmatrix}. \quad (68)$$

This problem has infinitely many solutions $(0, \lambda, 0)$, where $\lambda \in [0, 1]$. The test results are listed in Table 1 by using different initial points.

Problem 2 (modified Mathiesen problem). In problem (1), $x \in \mathbb{R}^4$ and $F(x) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is given by

$$F(x) = \begin{pmatrix} -x_2 + x_3 + x_4 \\ x_1 - \frac{(4.5x_3 + 2.7x_4)}{(x_2 + 1)} \\ 5 - x_1 - \frac{(0.5x_3 + 0.3x_4)}{(x_3 + 1)} \\ 3 - x_1 \end{pmatrix}. \quad (69)$$

This problem has infinitely many solutions $(\lambda, 0, 0, 0)$, where $\lambda \in [0, 3]$. The solutions are degenerate for $\lambda = 0$ or $\lambda = 3$ and nondegenerate for $\lambda \in (0, 3)$. With different starting points, we report results in Table 2.

Problem 3 (Kojima-Shindo problem). In problem (1), $x \in \mathbb{R}^4$ and $F(x) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is given by

$$F(x) = \begin{pmatrix} 3x_1^2 + 2x_1x_2 + 2x_2^2 + x_3 + 3x_4 - 6 \\ 2x_1^2 + x_1 + x_2^2 + 10x_3 + 2x_4 - 2 \\ 3x_1^2 + x_1x_2 + 2x_2^2 + 2x_3 + 9x_4 - 9 \\ x_1^2 + x_2^2 + 2x_3 + 3x_4 - 3 \end{pmatrix}. \quad (70)$$

TABLE 1: Numerical results of Problem 1.

ST	IT	x^k	$ x^T y $	$\ H^k\ $	CPU
o	4	(0, 0.0000, 0)	$4.83E - 09$	$8.38E - 09$	0.085
e	5	(0, 0.4998, 0)	$3.44E - 11$	$3.97E - 11$	0.103
$10e$	6	(0, 1.0000, 0)	$1.05E - 11$	$9.09E - 12$	0.124
$-e$	5	(0, 0.0023, 0)	$2.09E - 09$	$3.61E - 09$	0.105
$-10e$	6	(0, 0.0071, 0)	$2.49E - 11$	$4.28E - 11$	0.124
$-100e$	6	(0, 0.0399, 0)	$1.50E - 09$	$2.50E - 09$	0.125

TABLE 2: Numerical results of Problem 2.

ST	IT	x^k	$ x^T y $	$\ H^k\ $	CPU
e	5	(0.37, 0, 3, 0)	$1.27E - 08$	$3.91E - 09$	0.304
o	5	(0, 0, 0, 0)	$6.59E - 11$	$3.81E - 11$	0.292
$-2e$	6	(0.002, 0, 0, 0)	$1.32E - 08$	$3.31E - 09$	0.356
$-11e$	6	(1.85, 0, 0, 0)	$1.06E - 08$	$2.64E - 09$	0.416
$-20e$	6	(0.05, 0, 0, 0)	$1.38E - 08$	$3.46E - 09$	0.366

TABLE 3: Numerical results of Problem 3.

ST	IT	x^k	$ x^T y $	$\ H^k\ $	CPU
e	6	x_2	$2.99E - 08$	$1.42E - 09$	0.679
$10e$	7	x_2	$3.54E - 08$	$1.77E - 09$	0.729
$10^2 e$	8	x_2	$3.02E - 08$	$1.54E - 09$	0.877
$10^3 e$	9	x_1	$1.07E - 08$	$9.99E - 09$	0.965

TABLE 4: Solution of Problem 4 with random initial points.

ST	IT	$ x^T y $	$\ H^k\ $	CPU
e	18	$8.52E - 10$	$1.49E - 10$	5.539
o	22	$2.50E - 10$	$4.38E - 11$	6.538
$-e$	38	$3.41E - 09$	$5.96E - 10$	11.306
rand(5, 1)	16	$4.20E - 10$	$7.35E - 11$	4.796

TABLE 5: Numerical results of Problem 5.

ST	IT	$ x^T y $	$\ H^k\ $	CPU
o	6	$1.38E - 08$	$4.97E - 09$	0.858
e	7	$5.21E - 09$	$1.89E - 09$	1.109
$-e$	7	$1.32E - 08$	$4.77E - 09$	1.003
$-10e$	7	$1.32E - 08$	$4.77E - 09$	0.990
$-10^2 e$	8	$1.32E - 08$	$4.76E - 09$	1.142
$-10^3 e$	8	$1.32E - 08$	$4.76E - 09$	1.079
$-10^4 e$	9	$1.40E - 08$	$2.47E - 09$	1.268

TABLE 6: Numerical results of Problem 6.

ST	IT	$ x^T y $	$\ H^k\ $	CPU
e	7	$4.01E - 11$	$1.39E - 11$	0.248
$10e$	9	$8.61E - 09$	$2.97E - 09$	0.358
$10^2 e$	10	$1.13E - 09$	$3.91E - 10$	0.316
$10^3 e$	9	$2.45E - 09$	$8.48E - 10$	0.359
$-e$	8	$2.46E - 11$	$8.51E - 12$	0.294
$-10e$	15	$9.46E - 12$	$3.28E - 12$	0.504
$-10^3 e$	24	$8.96E - 10$	$3.10E - 10$	0.759

TABLE 7: Numerical results of Problem 7.

n	IT	$ x^T y $	$\ H^k\ $	CPU
128	8	$1.30E - 10$	$1.48E - 09$	0.040
256	8	$1.15E - 10$	$1.90E - 09$	0.155
512	9	$1.55E - 11$	$3.50E - 10$	1.010
800	10	$1.99E - 11$	$6.44E - 10$	2.701
1000	10	$1.58E - 11$	$4.99E - 10$	4.738

TABLE 8: Effect of inexact parameter in Problem 3.

ST	θ_k	IT	$ x^T y $	$\ H^k\ $	CPU
e	0	6	$7.95E - 09$	$3.89E - 10$	0.70
	0.2	6	$1.46E - 08$	$7.27E - 10$	0.66
	0.4	7	$1.30E - 09$	$8.41E - 11$	0.73
	0.6	9	$4.77E - 09$	$4.60E - 10$	0.91
	0.8	13	$7.14E - 08$	$4.05E - 09$	1.33
$10^2 e$	0	F	F	F	F
	0.2	8	$2.97E - 08$	$1.52E - 09$	0.92
	0.4	10	$4.25E - 12$	$1.71E - 12$	1.12
	0.6	10	$8.98E - 09$	$1.43E - 09$	1.01
	0.8	15	$3.66E - 08$	$6.99E - 09$	1.57

This problem has one degenerate solution $x_1 = (\sqrt{6}/2, 0, 0, 1/2)^T$ and one nondegenerate solution $x_2 = (1, 0, 3, 0)^T$. We use different initial points and the test results are listed in Table 3.

Problem 4. In problem (1), $x \in \mathbb{R}^5$ and $F(x) = (F_1(x), \dots, F_5(x))^T$ where

$$F_i(x) = 2 \exp\left(\sum_{i=1}^5 (x_i - i + 2)^2\right) (x_i - i + 2), \quad i = 1, \dots, 5. \quad (71)$$

This problem has one solution $(0, 0, 1, 2, 3)$. We use different starting points and the last initial point x^0 is randomly generated whose elements are in the interval $(0, 1)$. The test results are listed in Table 4.

Problem 5. In problem (1), $x \in \mathbb{R}^7$ and $F(x) : \mathbb{R}^7 \rightarrow \mathbb{R}^7$ is given by

$$F(x) = \begin{pmatrix} 2x_1 - x_3 + x_5 + 3x_6 - 1 \\ x_2 + 2x_5 + x_6 - x_7 - 3 \\ -x_1 + 2x_3 + x_4 + x_5 + 2x_6 - 4x_7 + 1 \\ x_3 + x_4 + x_5 - x_6 - 1 \\ -x_1 - 2x_2 - x_3 - x_4 + 5 \\ -3x_1 - x_2 - 2x_3 + x_4 + 4 \\ x_2 + 4x_3 - 1.5 \end{pmatrix}. \quad (72)$$

The test results are listed in Table 5 by using different initial points.

Problem 6. In problem (1), $x \in \mathbb{R}^4$ and $F(x) : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is given by

$$F(x) = \begin{pmatrix} x_1^3 - 8 \\ x_2 - x_3 + x_2^3 + 3 \\ x_2 + x_3 + 2x_3^3 - 3 \\ x_4 + 2x_4^3 \end{pmatrix}. \quad (73)$$

In this problem, $F(x)$ is a P_0 -function. It has only one solution $(2, 0, 1, 0)$. With different initial points, the results are listed in Table 6.

Problem 7. In problem (1), $x \in \mathbb{R}^n$ and $F(x) = Mx + q$ with

$$[M]_{ii} = 4(i - 1) + 1, \quad i = 1, 2, \dots, n,$$

$$[M]_{ij} = [M]_{ii} + 1, \quad i = 1, 2, \dots, n-1, \quad j = i+1, \dots, n,$$

$$[M]_{ji} = [M]_{jj} + 1, \quad j = 1, 2, \dots, n-1, \quad i = j+1, \dots, n,$$

$$q = (-1, -1, \dots, -1)^T. \quad (74)$$

This problem has only solutions $x^* = (1, 0, \dots, 0)^T$. From the initial point $x^0 = (1, 1, \dots, 1)^T$, we solve this problem with different dimensions. The test results are listed in Table 7.

In the end of this section, we intend to test the effect of the inexact parameter θ_k on the efficiency of Algorithm 8.

In Tables 8 and 9, for Problems 3 (not a P_0 -function) and 6, we take different values of θ_k , $\theta_k = 0, 0.2, 0.4, 0.6, 0.8$, and implement Algorithm 8 to find the solutions of Problems 3 and 6, respectively.

TABLE 9: Effect of inexact parameter in Problem 6.

ST	θ_0	IT	$ x^T y $	$\ H^k\ $	CPU
e	0	6	$2.01E - 09$	$6.78E - 10$	0.22
	0.2	6	$1.18E - 08$	$4.06E - 09$	0.22
	0.4	7	$7.50E - 11$	$4.66E - 11$	0.27
	0.6	9	$1.32E - 11$	$1.99E - 10$	0.27
	0.8	13	$9.29E - 09$	$7.16E - 09$	0.39
$10e$	0	F	F	F	F
	0.2	8	$9.13E - 09$	$3.15E - 09$	0.28
	0.4	10	$9.03E - 11$	$3.11E - 11$	0.34
	0.6	11	$7.19E - 12$	$4.06E - 11$	0.36
	0.8	14	$8.25E - 09$	$9.51E - 09$	0.44

From Tables 8 and 9, it is revealed that, for $\theta_k = 0$ (which corresponds to the smoothing exact Newton method), Algorithm 8 may fail for some initial points. On the other hand, a suitable value of inexact parameter may greatly improve the efficiency of Algorithm 8.

From the numerical results, we conclude as follows:

- (1) In Tables 1–7, the choice of initial point only incurs weak impact on the CPU time and the iteration number of Algorithm 8. It indicates that the developed algorithm in this paper is robust even if for the randomly generated initial point.
- (2) From the results in Tables 8 and 9, the inexact parameter θ_k may play critical role in improve the numerical performance of Algorithm 8.

6. Final Remarks

In this paper, a smoothing inexact Newton method has been proposed for solving nonlinear complementarity problems based on a new smoothing function. Then, an implementable algorithm was developed. Under a suitable assumption, the global convergence and the superlinear convergence have been established for the algorithm. Results of numerical experiments indicate that our algorithm is effective for the benchmark test problems available in the literature.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This research is supported by the Major Program of the National Social Science Foundation of China (14ZDB136), the National Natural Science Foundation of China (Grant nos. 71210003, 11461015), and the Natural Science Foundation of Hunan Province (13JJ3002).

References

- [1] S. A. Gabriel, A. J. Conejo, J. D. Fuller, B. F. Hobbs, and C. Ruiz, *Complementarity Modeling in Energy Markets*, vol. 180 of *International Series in Operations Research and Management Science*, Springer, New York, NY, USA, 2013.
- [2] P. T. Harker and J.-S. Pang, "Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications," *Mathematical Programming*, vol. 48, no. 1–3, pp. 161–220, 1990.
- [3] M. C. Ferris and J. S. Pang, "Engineering and economic applications of complementarity problems," *SIAM Review*, vol. 39, no. 4, pp. 669–713, 1997.
- [4] J. Zhang and K.-C. Zhang, "A variant smoothing Newton method for P_0 -NCP based on a new smoothing function," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 1–8, 2009.
- [5] X. Chen, "Superlinear convergence of smoothing quasi-Newton methods for nonsmooth equations," *Journal of Computational and Applied Mathematics*, vol. 80, no. 1, pp. 105–126, 1997.
- [6] J. S. Chen and S. H. Pan, "A family of NCP functions and a descent method for the nonlinear complementarity problem," *Computational Optimization and Applications*, vol. 40, no. 3, pp. 389–404, 2008.
- [7] B. Chen and P. T. Harker, "Smooth approximations to nonlinear complementarity problems," *SIAM Journal on Optimization*, vol. 7, no. 2, pp. 403–420, 1997.
- [8] X. H. Chen and C. F. Ma, "A regularization smoothing Newton method for solving nonlinear complementarity problem," *Nonlinear Analysis: Real World Applications*, vol. 10, no. 3, pp. 1702–1711, 2009.
- [9] X. Chen, "Smoothing methods for complementarity problems and their applications: a survey," *Journal of the Operations Research Society of Japan*, vol. 43, no. 1, pp. 32–47, 2000.
- [10] X. Chen, L. Qi, and D. Sun, "Global and superlinear convergence of the smoothing Newton method and its application to general box constrained variational inequalities," *Mathematics of Computation*, vol. 67, no. 222, pp. 519–540, 1998.
- [11] X.-H. Liu and W. Wu, "Coerciveness of some merit functions over symmetric cones," *Journal of Industrial and Management Optimization*, vol. 5, no. 3, pp. 603–613, 2009.

- [12] Z. H. Huang, J. Y. Han, and Z. W. Chen, "Predictor-corrector smoothing Newton method, based on a new smoothing function, for solving the nonlinear complementarity problem with a P_0 function," *Journal of Optimization Theory and Applications*, vol. 117, no. 1, pp. 39–68, 2003.
- [13] Z. Huang, J. Han, D. Xu, and L. Zhang, "The non-interior continuation methods for solving the P_0 function nonlinear complementarity problem," *Science in China Series A: Mathematics*, vol. 44, no. 9, pp. 1107–1114, 2001.
- [14] J. G. Zhu, H. W. Liu, and C. H. Liu, "A family of new smoothing functions and a nonmonotone smoothing Newton method for the nonlinear complementarity problems," *Journal of Applied Mathematics and Computing*, vol. 37, no. 1-2, pp. 647–662, 2011.
- [15] J. Zhu and B. Hao, "A new class of smoothing functions and a smoothing Newton method for complementarity problems," *Optimization Letters*, vol. 7, no. 3, pp. 481–497, 2013.
- [16] J. Tang, L. Dong, J. Zhou, and L. Fang, "A smoothing Newton method for nonlinear complementarity problems," *Computational and Applied Mathematics*, vol. 32, no. 1, pp. 107–118, 2013.
- [17] Q. Li and D. H. Li, "A smoothing Newton method for nonlinear complementarity problems," *Advanced Modeling and Optimization*, vol. 13, no. 2, pp. 141–152, 2011.
- [18] L. P. Zhang, J. Y. Han, and Z. H. Huang, "Superlinear/quadratic one-step smoothing Newton method for P_0 -NCP," *Acta Mathematica Sinica*, vol. 21, no. 1, pp. 117–128, 2005.
- [19] R. Mifflin, "Semi-smooth and semi-convex function in constrained optimization," *Mathematical Programming*, vol. 15, pp. 957–972, 1997.
- [20] L. Q. Qi and J. Sun, "A nonsmooth version of Newton's method," *Mathematical Programming*, vol. 58, no. 3, pp. 353–367, 1993.
- [21] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, NY, USA, 1983.
- [22] L. Fang, "A new one-step smoothing Newton method for nonlinear complementarity problem with P_0 -function," *Applied Mathematics and Computation*, vol. 216, no. 4, pp. 1087–1095, 2010.
- [23] Z.-H. Huang, L. Qi, and D. Sun, "Sub-quadratic convergence of a smoothing Newton algorithm for the P_0 - and monotone LCP," *Mathematical Programming*, vol. 99, pp. 423–441, 2004.
- [24] S. Liu, J. Tang, and C. Ma, "A new modified one-step smoothing Newton method for solving the general mixed complementarity problem," *Applied Mathematics and Computation*, vol. 216, no. 4, pp. 1140–1149, 2010.
- [25] L. Qi, "Convergence analysis of some algorithms for solving nonsmooth equations," *SIAM Journal on Control and Optimization*, vol. 18, pp. 227–244, 1993.
- [26] S.-P. Rui and C.-X. Xu, "A smoothing inexact Newton method for nonlinear complementarity problems," *Journal of Computational and Applied Mathematics*, vol. 233, no. 9, pp. 2332–2338, 2010.
- [27] J. Tang, S. Liu, and C. Ma, "One-step smoothing Newton method for solving the mixed complementarity problem with a P_0 function," *Applied Mathematics and Computation*, vol. 215, pp. 2326–2336, 2009.
- [28] L. Qi and D. Sun, "Smoothing functions and smoothing Newton method for complementarity and variational inequality problems," *Journal of Optimization Theory and Applications*, vol. 113, no. 1, pp. 121–147, 2002.