Research Article

Fixed Point Theorems for Ćirić-Berinde Type Contractive Multivalued Mappings

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We give a Ćirić-Berinde type contractive condition for multivalued mappings and analyze the existence of fixed point for these mappings.

1. Introduction and Preliminaries

In 2012, Samet et al. [1] introduced the notions of α - ψ -contractive mapping and α -admissible mappings in metric spaces and obtained corresponding fixed point results, which are generalizations of ordered fixed point results (see [1]). Since then, by using their idea, some authors investigated fixed point results in the field. Asl et al. [2] extended some of results in [1] to multivalued mappings by introducing the notions of α_* - ψ -contractive mapping and α_* -admissible mapping.

Recently, Salimi et al. [3] modified the notions of α - ψ -contractive mapping and α -admissible mappings by introducing another function η . And then, they gave generalizations of the results of Samet et al. [1] and Karapınar and Samet [4]. Hussain et al. [5] extended these modified notions to multivalued mappings. That is, they introduced the notion of α - η -contractive multifunctions and gave fixed point results for these multifunctions.

Very recently, Ali et al. [6] generalized and extended the notion of α - ψ -contractive mapping by introducing the notion of (α, ψ, ξ) -contractive multivalued mappings and obtained fixed point theorems for these mappings in complete metric spaces.

The purpose of this paper is to introduce the notion of Ćirić-Berinde type contractive multivalued mappings and to generalize and extend the notion of α - η -contractive multifunctions and to establish fixed point theorems for Ćirić-Berinde type contractive multivalued mappings.

Let (X, d) be a metric space. We denote by $CB(X)$ the class of nonempty closed and bounded subsets of X and by $CL(X)$ the class of nonempty closed subsets of X . Let $H(\cdot, \cdot)$ be the generalized Hausdorff distance on $CL(X)$; that is, for all $A, B \in CL(X)$,

$$H(A, B) = \begin{cases} \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise,} \end{cases} \quad (1)$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$ is the distance from point a to subset B .

For $A, B \in CL(X)$, let $D(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y)$. Then, we have $D(A, B) \leq H(A, B)$ for all $A, B \in CL(X)$. From now on, we denote by

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2} \{d(x, Ty) + d(y, Tx)\} \right\} \quad (2)$$

for a multivalued map $T : X \rightarrow CL(X)$ and $x, y \in X$.

We denote by Ξ the class of all functions $\xi : [0, \infty) \rightarrow [0, \infty)$ such that

- (1) ξ is continuous;
- (2) ξ is nondecreasing on $[0, \infty)$;
- (3) $\xi(t) = 0$ if and only if $t = 0$;

(4) ξ is subadditive.

Also, we denote by Ψ the family of all nondecreasing functions $\psi : [0, \infty) \rightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n th iterate of ψ .

Note that if $\psi \in \Psi$, then $\psi(0) = 0$ and $0 < \psi(t) < t$ for all $t > 0$.

Let (X, d) be a metric space, and let $\alpha : X \times X \rightarrow [0, \infty)$ be a function.

We consider the following conditions:

- (1) for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} x_n = x$, we have

$$\alpha(x_n, x) \geq 1 \quad \forall n \in \mathbb{N}; \quad (3)$$

- (2) for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and a cluster point x of $\{x_n\}$, we have

$$\liminf_{n \rightarrow \infty} \alpha(x_n, x) \geq 1; \quad (4)$$

- (3) for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and a cluster point x of $\{x_n\}$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)}, x) \geq 1 \quad \forall k \in \mathbb{N}. \quad (5)$$

Remark 1. (1) implies (2) and (2) implies (3).

Note that if (X, d) is a metric space and $\xi \in \Xi$, then $(X, \xi \circ d)$ is a metric space.

Let (X, d) be a metric space, and let $T : X \rightarrow CL(X)$ be a multivalued mapping. Then, we say that

- (1) T is called α_* -admissible [2] if

$$\alpha(x, y) \geq 1 \quad \text{implies} \quad \alpha_*(Tx, Ty) \geq 1, \quad (6)$$

where $\alpha_*(Tx, Ty) = \inf\{\alpha(a, b) : a \in Tx, b \in Ty\}$;

- (2) T is called α -admissible [7] if, for each $x \in X$ and $y \in Tx$ with $\alpha(x, y) \geq 1$, we have $\alpha(y, z) \geq 1$ for all $z \in Ty$.

Lemma 2. Let (X, d) be a metric space, and let $T : X \rightarrow CL(X)$ be a multivalued mapping. If T is α_* -admissible, then it is α -admissible.

Proof. Suppose that T is an α_* -admissible mapping.

Let $x \in X$ and $y \in Tx$ be such that $\alpha(x, y) \geq 1$.

Let $z \in Ty$ be given.

Since T is α_* -admissible, $\alpha(y, z) \geq \alpha_*(Tx, Ty) \geq 1$. $\square$

Lemma 3. Let (X, d) be a metric space, and let $\xi \in \Xi$ and $B \in CL(X)$.

If $a \in X$ and $\xi(d(a, B)) < c$, then there exists $b \in B$ such that $\xi(d(a, b)) < c$.

Proof. Let $\epsilon = c - \xi(d(a, B))$.

Since $\xi(d(a, B)) < c$ and $\xi \circ d$ is metric on X , there exists $b \in B$ such that $\xi(d(a, b)) < \xi(d(a, B)) + \epsilon$ by definition of infimum. Hence, $\xi(d(a, b)) < c$. $\square$

Let (X, d) be a metric space.

A function $f : X \rightarrow [0, \infty)$ is called *upper semicontinuous* if, for each $x \in X$ and $\{x_n\} \subset X$ with $\lim_{n \rightarrow \infty} x_n = x$, we have $\lim_{n \rightarrow \infty} f(x_n) \leq f(x)$.

A function $f : X \rightarrow [0, \infty)$ is called *lower semicontinuous* if, for each $x \in X$ and $\{x_n\} \subset X$ with $\lim_{n \rightarrow \infty} x_n = x$, we have $f(x) \leq \lim_{n \rightarrow \infty} f(x_n)$.

For a multivalued map $T : X \rightarrow CL(X)$, let $f_T : X \rightarrow [0, \infty)$ be a function defined by $f_T(x) = d(x, Tx)$.

2. Fixed Point Theorems

In this section, we establish fixed point theorems for Ćirić-Berinde type contractive multivalued mappings.

Theorem 4. Let (X, d) be a complete metric space, and let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. Suppose that a multivalued mapping $T : X \rightarrow CL(X)$ is α -admissible.

Assume that, for all $x, y \in X$, $\alpha(x, y) \geq 1$ implies

$$\xi(H(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(d(y, Tx)), \quad (7)$$

where $L \geq 0$, $\xi \in \Xi$, and $\psi \in \Psi$ is strictly increasing.

Also, suppose that the following are satisfied:

- (1) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;

- (2) either T is continuous or f_T is lower semicontinuous.

Then T has a fixed point in X .

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$ be such that $\alpha(x_0, x_1) \geq 1$. Let c be a real number with $\xi(d(x_0, x_1)) < \xi(c)$.

If $x_0 = x_1$, then x_1 is a fixed point.

Let $x_0 \neq x_1$.

If $x_1 \in Tx_1$, then x_1 is a fixed point. Let $x_1 \notin Tx_1$. Then $d(x_1, Tx_1) > 0$.

From (7) we obtain

$$\begin{aligned} 0 &< \xi(d(x_1, Tx_1)) \\ &\leq \xi(H(Tx_0, Tx_1)) \\ &\leq \psi\left(\xi\left(\max\left\{d(x_0, x_1), d(x_0, Tx_0), d(x_1, Tx_1), \right.\right.\right. \\ &\quad \left.\left.\left.\frac{1}{2}\{d(x_0, Tx_1) + d(x_1, Tx_0)\}\right\}\right)\right) \\ &\quad + L\xi(d(x_1, Tx_0)) \\ &\leq \psi\left(\xi\left(\max\left\{d(x_0, x_1), d(x_0, x_1), d(x_1, Tx_1), \right.\right.\right. \\ &\quad \left.\left.\left.\frac{1}{2}\{d(x_0, Tx_1) + d(x_1, x_1)\}\right\}\right)\right) \\ &\quad + L\xi(d(x_1, x_1)) \end{aligned}$$

$$\begin{aligned} &\leq \psi \left(\xi \left(\max \left\{ d(x_0, x_1), d(x_0, x_1), d(x_1, Tx_1), \right. \right. \right. \\ &\quad \left. \left. \left. \frac{1}{2} \{d(x_0, x_1) + d(x_1, Tx_1)\} \right\} \right) \right) \\ &= \psi \left(\xi \left(\max \{d(x_0, x_1), d(x_1, Tx_1)\} \right) \right). \end{aligned} \quad (8)$$

If $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$, then we have $0 < \xi(d(x_1, Tx_1)) \leq \psi(\xi(d(x_1, Tx_1))) < \xi(d(x_1, Tx_1))$, which is a contradiction.

Thus, $\max\{d(x_0, x_1), d(x_1, Tx_1)\} = d(x_0, x_1)$, and hence we have

$$0 < \xi(d(x_1, Tx_1)) \leq \psi(\xi(d(x_0, x_1))) < \psi(\xi(c)). \quad (9)$$

Hence, there exists $x_2 \in Tx_1$ such that

$$\xi(d(x_1, x_2)) < \psi(\xi(c)). \quad (10)$$

Since T is α -admissible, from condition (1) and $x_2 \in Tx_1$, we have

$$\alpha(x_1, x_2) \geq 1. \quad (11)$$

If $x_2 \in Tx_2$, then x_2 is a fixed point. Let $x_2 \notin Tx_2$. Then $d(x_2, Tx_2) > 0$, and so $\xi(d(x_2, Tx_2)) > 0$. From (7) we obtain

$$\begin{aligned} 0 &< \xi(d(x_2, Tx_2)) \\ &\leq \xi(H(Tx_1, Tx_2)) \\ &\leq \psi \left(\xi \left(\max \left\{ d(x_1, x_2), d(x_1, Tx_1), d(x_2, Tx_2), \right. \right. \right. \\ &\quad \left. \left. \left. \frac{1}{2} \{d(x_1, Tx_2) + d(x_2, Tx_1)\} \right\} \right) \right) \\ &\quad + L\xi(d(x_2, Tx_1)) \\ &\leq \psi \left(\xi \left(\max \left\{ d(x_1, x_2), d(x_1, x_2), d(x_2, Tx_2), \right. \right. \right. \\ &\quad \left. \left. \left. \frac{1}{2} \{d(x_1, Tx_2) + d(x_2, Tx_2)\} \right\} \right) \right) \\ &\quad + L\xi(d(x_2, x_2)) \\ &\leq \psi \left(\xi \left(\max \left\{ d(x_1, x_2), d(x_1, x_2), d(x_2, Tx_2), \right. \right. \right. \\ &\quad \left. \left. \left. \frac{1}{2} \{d(x_1, x_2) + d(x_2, Tx_2)\} \right\} \right) \right) \\ &= \psi(\xi(\max\{d(x_1, x_2), d(x_2, Tx_2)\})). \end{aligned} \quad (12)$$

If $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_2, Tx_2)$, then we have $\xi(d(x_2, Tx_2)) \leq \psi(\xi(d(x_2, Tx_2))) < \xi(d(x_2, Tx_2))$, which is a contradiction.

Thus, $\max\{d(x_1, x_2), d(x_2, Tx_2)\} = d(x_1, x_2)$, and hence we have

$$\xi(d(x_2, Tx_2)) \leq \psi(\xi(d(x_1, x_2))) < \psi^2(\xi(c)). \quad (13)$$

Hence, there exists $x_3 \in Tx_2$ such that

$$\xi(d(x_2, x_3)) < \psi^2(\xi(c)). \quad (14)$$

Since T is α -admissible, from $x_2 \in Tx_1$ and $\alpha(x_1, x_2) \geq 1$, we have

$$\alpha(x_2, x_3) \geq 1. \quad (15)$$

By induction, we obtain a sequence $\{x_n\} \subset X$ such that, for all $n \in \mathbb{N} \cup \{0\}$,

$$\alpha(x_n, x_{n+1}) \geq 1,$$

$$x_{n+1} \in Tx_n, \quad x_n \neq x_{n+1}, \quad \xi(d(x_n, x_{n+1})) < \psi^n(\xi(c)). \quad (16)$$

Let $\epsilon > 0$ be given.

Since $\sum_{n=0}^{\infty} \psi^n(\xi(c)) < \infty$, there exists $N \in \mathbb{N}$ such that

$$\sum_{n \geq N} \psi^n(\xi(c)) < \xi(\epsilon). \quad (17)$$

For all $m > n \geq N$, we have

$$\begin{aligned} \xi(d(x_n, x_m)) &\leq \sum_{k=n}^{m-1} \psi^k(\xi(c)) \\ &< \sum_{n \geq N} \psi^n(\xi(c)) < \xi(\epsilon) \end{aligned} \quad (18)$$

which implies $d(x_n, x_m) < \epsilon$ for all $m > n \geq N$. Hence, $\{x_n\}$ is a Cauchy sequence in X .

It follows from the completeness of X that there exists

$$x_* = \lim_{n \rightarrow \infty} x_n \in X. \quad (19)$$

Suppose that T is continuous.

We have

$$\begin{aligned} d(x_*, Tx_*) &\leq d(x_*, x_{n+1}) + d(x_{n+1}, Tx_*) \\ &\leq d(x_*, x_{n+1}) + H(x_n, Tx_*). \end{aligned} \quad (20)$$

By letting $n \rightarrow \infty$ in the above inequality, we obtain $d(x_*, Tx_*) = 0$, and so $x_* \in Tx_*$.

Assume that f_T is lower semicontinuous.

Then, $f_T(x_*) \leq \lim_{n \rightarrow \infty} f_T(x_n)$. Hence, $d(x_*, Tx_*) \leq \lim_{n \rightarrow \infty} d(x_n, Tx_n) \leq \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. Thus, $x_* \in Tx_*$. $\square$

Corollary 5. Let (X, d) be a complete metric space, and let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. Suppose that $T : X \rightarrow CL(X)$ is an α -admissible mapping.

Assume that, for all $x, y \in X$,

$$\xi(\alpha(x, y)H(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(d(y, Tx)), \quad (21)$$

where $L \geq 0$, $\xi \in \Xi$, and $\psi \in \Psi$ is strictly increasing.

Also, suppose that conditions (1) and (2) of Theorem 4 are satisfied.

Then T has a fixed point in X .

Remark 6. If we have $\xi(t) = t$ for all $t \geq 0$, $L = 0$, and T is continuous, then Corollary 5 reduces to Theorem 3.4 of [7].

Let $(X, \leq)$ be an ordered set and $A, B \subset X$. We say that $A \leq B$ whenever, for each $a \in A$, there exists $b \in B$ such that $a \leq b$.

Corollary 7. Let $(X, \leq, d)$ be a complete ordered metric space. Suppose that a multivalued mapping $T : X \rightarrow CL(X)$ satisfies

$$\xi(H(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(d(y, Tx)) \quad (22)$$

for all $x, y \in X$ with $Tx \leq Ty$ (resp., $Ty \leq Tx$), where $L \geq 0$, $\xi \in \Xi$, and $\psi \in \Psi$ is strictly increasing.

Assume that, for each $x \in X$ and $y \in Tx$ with $Tx \leq Ty$ (resp., $Ty \leq Tx$), we have $Ty \leq Tz$ (resp., $Tz \leq Ty$) for all $z \in Ty$.

Also, suppose that the following are satisfied:

- (1) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $Tx_0 \leq Tx_1$ (resp., $Tx_1 \leq Tx_0$);
- (2) either T is continuous or f_T is lower semicontinuous.

Then T has a fixed point in X .

Remark 8. If we have $\xi(t) = t$ for all $t \geq 0$, $L = 0$, and T is continuous, then Corollary 7 reduces to Corollary 3.6 of [7].

From Theorem 4 we obtain the following result.

Corollary 9. Let (X, d) be a complete metric space, and let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. Suppose that $T : X \rightarrow CL(X)$ is an α_* -admissible mapping.

Assume that, for all $x, y \in X$, $\alpha(x, y) \geq 1$ implies

$$\xi(H(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(d(y, Tx)), \quad (23)$$

where $L \geq 0$, $\xi \in \Xi$, and $\psi \in \Psi$ is strictly increasing.

Also, suppose that conditions (1) and (2) of Theorem 4 are satisfied.

Then T has a fixed point in X .

Remark 10. If we have $L = 0$ in Corollary 9, then Corollary 9 reduces to Theorem 2.5 of [6].

Corollary 11. Let (X, d) be a complete metric space, and let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. Suppose that $T : X \rightarrow CL(X)$ is an α_* -admissible mapping.

Assume that, for all $x, y \in X$,

$$\xi(\alpha(x, y)H(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(d(y, Tx)), \quad (24)$$

where $L \geq 0$, $\xi \in \Xi$, and $\psi \in \Psi$ is strictly increasing.

Also, suppose that conditions (1) and (2) of Theorem 4 are satisfied.

Then T has a fixed point in X .

Remark 12. In Corollary 11, let $\xi(t) = t$ for all $t \geq 0$ and $\alpha(x, y) = 1$ for all $x, y \in X$ and $\psi(t) = kt$ for all $t \geq 0$, where $k \in [0, 1)$. If T is single valued map, then Corollary 11 reduces to Theorem 2.2 of [8].

Theorem 13. Let (X, d) be a complete metric space, and let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. Suppose that a multivalued mapping $T : X \rightarrow CL(X)$ is α -admissible.

Assume that, for all $x, y \in X$, $\alpha(x, y) \geq 1$ implies

$$\xi(H(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(d(y, Tx)), \quad (25)$$

where $L \geq 0$, $\xi \in \Xi$, and $\psi \in \Psi$ is strictly increasing and upper semicontinuous function. Also, suppose that the following are satisfied:

- (1) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$;
- (2) for a sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and a cluster point x of $\{x_n\}$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N} \cup \{0\}$,

$$\alpha(x_{n(k)}, x) \geq 1. \quad (26)$$

Then T has a fixed point in X .

Proof. Following the proof of Theorem 4, we obtain a sequence $\{x_n\} \subset X$ with $\lim_{n \rightarrow \infty} x_n = x_* \in X$ such that, for all $n \in \mathbb{N} \cup \{0\}$,

$$x_{n+1} \in Tx_n, \quad x_n \neq x_{n+1}, \quad \alpha(x_n, x_{n+1}) \geq 1. \quad (27)$$

From (2) there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\alpha(x_{n(k)}, x_*) \geq 1. \quad (28)$$

Thus, we have

$$\begin{aligned} \xi(d(x_{n(k)+1}, Tx_*)) &= \xi(H(Tx_{n(k)}, Tx_*)) \\ &\leq \psi(\xi(M(x_{n(k)}, x_*))) \\ &\quad + L\xi(d(x_*, x_{n(k)+1})), \end{aligned} \quad (29)$$

where

$$\begin{aligned} M(x_{n(k)}, x_*) &= \max \left\{ d(x_{n(k)}, x_*), d(x_{n(k)}, x_{n(k)+1}), d(x_*, Tx_*), \right. \\ &\quad \left. \frac{1}{2} \{d(x_{n(k)}, Tx_*) + d(x_*, x_{n(k)+1})\} \right\}. \end{aligned} \quad (30)$$

We have

$$\lim_{k \rightarrow \infty} M(x_{n(k)}, x_*) = d(x_*, Tx_*), \quad (31)$$

and so

$$\lim_{k \rightarrow \infty} \xi(M(x_{n(k)}, x_*)) = \xi(d(x_*, Tx_*)). \quad (32)$$

Suppose that $d(x_*, Tx_*) \neq 0$.

Since ψ is upper semicontinuous,

$$\lim_{k \rightarrow \infty} \psi(\xi(M(x_{n(k)}, x_*))) \leq \psi(\xi(d(x_*, Tx_*)). \quad (33)$$

Letting $k \rightarrow \infty$ in inequality (29) and using continuity of ξ , we obtain

$$\begin{aligned} 0 &< \xi(d(x_*, Tx_*)) \\ &\leq \lim_{k \rightarrow \infty} \psi(\xi(M(x_{n(k)}, x_*))) + \lim_{k \rightarrow \infty} L\xi(d(x_*, x_{n(k)+1})) \\ &\leq \psi(\xi(d(x_*, Tx_*))) \\ &< \xi(d(x_*, Tx_*)) \end{aligned} \quad (34)$$

which is a contradiction. Hence, $d(x_*, Tx_*) = 0$, and hence x_* is a fixed point of T . $\square$

The following example shows that upper semicontinuity of ψ cannot be dropped in Theorem 13.

Example 14. Let $X = [0, \infty)$, and let $d(x, y) = |x - y|$ for all $x, y \geq 0$.

Define a mapping $T : X \rightarrow CL(X)$ by

$$Tx = \begin{cases} \left\{\frac{1}{2}, 1\right\} & (x = 0), \\ \left\{\frac{3}{4}x\right\} & (0 < x \leq 1), \\ \{2x\} & (x > 1). \end{cases} \quad (35)$$

Let $\xi(t) = t$ for all $t \geq 0$, and let

$$\psi(t) = \begin{cases} \frac{4}{5}t & (t \geq 1), \\ \frac{3}{4}t & (0 \leq t < 1). \end{cases} \quad (36)$$

Then, $\xi \in \Xi$, and $\psi \in \Psi$ and ψ is a strictly increasing function.

Let $\alpha, \eta : X \times X \rightarrow [0, \infty)$ be defined by

$$\alpha(x, y) = \begin{cases} 4 & (0 \leq x, y \leq 1), \\ 0 & \text{otherwise.} \end{cases} \quad (37)$$

Obviously, condition (2) of Theorem 13 is satisfied. Condition (1) of Theorem 13 is satisfied with $x_0 = 1/4$.

We show that (7) is satisfied.

Let $x, y \in X$ be such that $\alpha(x, y) \geq 1$.

Then, $0 \leq x, y \leq 1$.

If $x = y$, then obviously (7) is satisfied.

Let $x \neq y$.

If $x = 0$ and $0 < y \leq 1$, then we obtain

$$\begin{aligned} \xi(H(Tx, Ty)) &= H\left(\left\{\frac{1}{2}, 1\right\}, \frac{3}{4}y\right) \\ &\leq \frac{1}{4} \leq \psi(d(x, Tx)) \leq \psi(\xi(M(x, y))). \end{aligned} \quad (38)$$

Let $0 < x \leq 1$ and $0 < y \leq 1$.

Then, we have

$$\begin{aligned} \xi(H(Tx, Ty)) &= d(Tx, Ty) = d\left(\frac{3}{4}x, \frac{3}{4}y\right) \\ &= \frac{3}{4}|x - y| = \psi(d(x, y)) \\ &\leq \psi(\xi(M(x, y))). \end{aligned} \quad (39)$$

Thus, (7) is satisfied.

We now show that T is α -admissible.

Let $x \in X$ be given, and let $y \in Tx$ be such that $\alpha(x, y) \geq 1$.

Then, $0 \leq x, y \leq 1$.

Obviously, $\alpha(y, z) \geq 1$ for all $z \in Ty$ whenever $0 < y \leq 1$.

If $y = 0$, then $Ty = \{1/2, 1\}$. Hence, for all $z \in Ty$, $\alpha(y, z) \geq 1$.

Hence, T is α -admissible. Thus, all hypotheses of Theorem 13 are satisfied. However, T has no fixed points.

Note that ψ is not upper semicontinuous.

Corollary 15. Let (X, d) be a complete metric space, and let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. Suppose that $T : X \rightarrow CL(X)$ is an α -admissible mapping.

Assume that, for all $x, y \in X$,

$$\xi(\alpha(x, y)H(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(d(y, Tx)), \quad (40)$$

where $L \geq 0$, $\xi \in \Xi$, and $\psi \in \Psi$ is strictly increasing and upper semicontinuous function. Also, suppose that conditions (1) and (2) of Theorem 13 are satisfied.

Then T has a fixed point in X .

Corollary 16. Let $(X, \leq, d)$ be a complete ordered metric space. Suppose that a multivalued mapping $T : X \rightarrow CL(X)$ satisfies

$$\xi(H(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(d(y, Tx)) \quad (41)$$

for all $x, y \in X$ with $Tx \leq Ty$ (resp., $Ty \leq Tx$), where $L \geq 0$, $\xi \in \Xi$, and $\psi \in \Psi$ is strictly increasing and upper semicontinuous function.

Assume that, for each $x \in X$ and $y \in Tx$ with $Tx \leq Ty$ (resp., $Ty \leq Tx$), we have $Ty \leq Tz$ (resp., $Tz \leq Ty$) for all $z \in Ty$.

Also, suppose that the following are satisfied:

- (1) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $Tx_0 \leq Tx_1$ (resp., $Tx_1 \leq Tx_0$);
- (2) for a sequence $\{x_n\}$ in X with $x_n \leq x_{n+1}$ (resp., $x_{n+1} \leq x_n$) for all $n \in \mathbb{N} \cup \{0\}$ and a cluster point x of $\{x_n\}$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that, for all $k \in \mathbb{N} \cup \{0\}$,

$$x_{n(k)} \leq x \quad (\text{resp., } x \leq x_{n(k)}). \quad (42)$$

Then T has a fixed point in X .

Remark 17. Corollary 16 is a generalization and extension of the result of [9] to multivalued mappings.

Corollary 18. Let (X, d) be a complete metric space, and let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. Suppose that a multivalued mapping $T : X \rightarrow CL(X)$ is α_* -admissible.

Assume that, for all $x, y \in X$, $\alpha(x, y) \geq 1$ implies

$$\xi(H(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(d(y, Tx)), \quad (43)$$

where $L \geq 0$, $\xi \in \Xi$, and $\psi \in \Psi$ is strictly increasing and upper semicontinuous function.

Also, suppose that conditions (1) and (2) of Theorem 13 are satisfied.

Then T has a fixed point in X .

Remark 19. By taking $L = 0$ in Corollary 18 and by applying Remark 1, Corollary 18 reduces to Theorem 2.6 of [6].

Corollary 20. Let (X, d) be a complete metric space, and let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. Suppose that $T : X \rightarrow CL(X)$ is an α_* -admissible mapping.

Assume that, for all $x, y \in X$,

$$\xi(\alpha(x, y)H(Tx, Ty)) \leq \psi(\xi(M(x, y))) + L\xi(d(y, Tx)), \quad (44)$$

where $L \geq 0$, $\xi \in \Xi$, and $\psi \in \Psi$ is strictly increasing and upper semicontinuous function.

Also, suppose that conditions (1) and (2) of Theorem 13 are satisfied.

Then T has a fixed point in X .

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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