Research Article

Generalized Solutions for Nonlocal Elliptic Equations and Systems with Nonlinear Singularities

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We use the topological degree method to study the existence of solutions for nonlocal elliptic equations (systems) with a strong singular nonlinearity.

1. Introduction and Main Results

Given \( s \in (0, 1) \), an integer \( n > 2s \), and a bounded open set \( \Omega \) of \( \mathbb{R}^n \) with Lipschitz boundary, let \( K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty) \) be a function satisfying the following properties:

(i) \( \gamma K \in L^1(\mathbb{R}^n) \) with \( \gamma(x) = \min\{|x|^2, 1\} \).

(ii) There exists \( \theta > 0 \) such that \( K(x) \geq \theta|x|^{-(n+2s)} \) for any \( x \in \mathbb{R}^n \setminus \{0\} \).

(iii) \( K(x) = K(-x) \) for any \( x \in \mathbb{R}^n \setminus \{0\} \).

The so-called nonlocal elliptic operator \( \mathcal{L}_K \) is defined by

\[
\mathcal{L}_K u(x) = \frac{1}{2} \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) K(y) \, dy,
\]

\[ x \in \mathbb{R}^n. \]

In particular, when \( K(x) = |x|^{-(n+2s)} \), \( \mathcal{L}_K \) is equal to the fractional Laplace operator \( (-\Delta)^s \) (up to normalization factors).

For a Carathéodory function \( f : \Omega \times \mathbb{R} \to \mathbb{R} \), the following problem

\[
\mathcal{L}_K u + f(x, u) = 0 \quad \text{in } \Omega,
\]

\[ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \]

and its special case

\[
(-\Delta)^s u = f(x, u) \quad \text{in } \Omega,
\]

\[ u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \]

have been widely studied under various contexts; see a recent survey [1] for details.

1.1. Previous Work. Motivated by the work of Caffarelli and Silvestre [2], several authors have considered an equivalent problem of (3) by means of an auxiliary variable; see [2–6]. Precisely, let \( (x, y) \) denote the points in \( C_{\Omega} := \Omega \times (0, \infty) \subset \mathbb{R}^{n+1}_+ \) and \( \partial \mathcal{C}_{\Omega} = \partial \Omega \times (0, \infty) \). Take \( \alpha = 2s \) and \( X^{\alpha}_0(\mathcal{C}_{\Omega}) \) as the completion of \( C^{\infty}_0(\Omega \times (0, \infty)) \) with respect to the norm

\[
\| z \|_{X^{\alpha}_0(\mathcal{C}_{\Omega})} = \left( k \int_{\mathcal{C}_{\Omega}} y^{1-\alpha} \left| \nabla z \right|^2 \, dx \, dy \right)^{1/2},
\]

where \( k \) is a normalization constant. For \( w \in X^{\alpha}_0(\mathcal{C}_{\Omega}) \), let

\[
L_{\alpha} w := - \text{div} \left( y^{1-\alpha} \nabla w \right),
\]

\[
\frac{\partial w}{\partial y^0} := k_{\alpha} \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial w}{\partial y}
\]
and consider the problem
\[
L_\alpha w = 0 \quad \text{in} \:C_\Omega,
\]
\[
\nu = 0 \quad \text{in} \: \partial_L C_\Omega,
\]
\[
\frac{\partial w}{\partial \nu^\alpha} = f (x, w) \quad \text{in} \: \Omega \times \{y = 0\}.
\]

An energy solution to this problem is a function \(w \in X^\alpha_\Omega(\Omega)\) such that
\[
k_\alpha \int_{\Omega} y^{1-\alpha} \langle \nabla w, \nabla \varphi \rangle \, dx \, dy = \int_{\Omega} f (x, w) \varphi \, dx
\]
\[
\forall \varphi \in X^\alpha_\Omega(\Omega).
\]

Such an energy solution \(w\) yields a function \(u = w(\cdot, 0)\) in the sense of traces, which belongs to the space \(H^{\alpha/2}_0(\Omega)\) and is a weak solution of (3). The converse is also true. The reader may refer to [2–6] for dealing with (3) with this method. In particular, Stinga and Torrea [6] generalized the arguments introduced the following Hilbert space results in [2] to the fractional powers \(L^\sigma\), \(0 < \sigma < 1\), of a linear second order partial differential operator \(L\) that is nonnegative, densely defined, and self-adjoint in \(L^2(\Omega, dp)\) with a positive measure \(dp\) on \(\Omega\).

Servadei and Valdinoci developed a variational framework to study the problem (2) in a series of papers [7–11, 13–15]. They introduced the following Hilbert space \((X_0(\Omega, K), \langle \cdot, \cdot \rangle_{0, \Omega, K})\) in [7, 8]. Let \(Q = \mathbb{R}^n \setminus (\overline{\Omega} \times \overline{\Omega})\), where \(\overline{\Omega} = \Omega^c \setminus \Omega\), and let \(X(\Omega, K)\) be the space of all Lebesgue measurable functions \(u : \mathbb{R}^n \to \mathbb{R}\) such that \(u|_{\Omega} \in L^2(\Omega)\) and that the map
\[
Q \ni (x, y) \quad \mapsto (u(x) - u(y)) \sqrt{K(x - y)} \quad \text{is in} \: L^2(Q).
\]

\(X(\Omega, K)\) is a Banach space endowed with the so-called Gagliardo norm
\[
\|u\|_{0, \Omega, K} = \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{1/2}
\]
\[
+ \int_Q |u(x) - u(y)|^2 K(x - y) \, dx \, dy \right)^{1/2}
\]
and is contained in \(H^\sigma(\mathbb{R}^n)\). Consider the subspace of \(X(\Omega, K)\):
\[
X_0(\Omega, K) = \{ u \in X(\Omega, K) \mid u = 0 \; \text{a.e. in} \; \mathbb{R}^n \setminus \Omega \}.
\]

It was proved in [12, Theorem 6] that this space is the closure of \(C^\infty_0(\Omega)\) in \(X(\Omega, K)\). Clearly, the space \(X_0(\Omega, K)\) depends on \(K\). In fact, when \(K(x) = |x|^{-n+2\alpha}\), \(X_0(\Omega, K) = \{ u \in H^\sigma(\mathbb{R}^n) \mid u = 0 \; \text{a.e. in} \; \mathbb{R}^n \setminus \Omega \} ([11, Lemma 7-b]). \(X_0(\Omega, K)\) can be endowed with a Hilbert space structure given by the inner product
\[
\langle u, v \rangle_{0, \Omega, K} = \int_Q (u(x) - u(y))(v(x) - v(y)) K(x - y) \, dx \, dy
\]
Then, problem (2) has at least one generalized solution \( u \) in \( X_0(\Omega, K) \); that is, it satisfies
\[
\int_{\mathbb{R}^n} \left( u(x) - u(y) \right) (\phi(x) - \phi(y)) K(x - y) \, dx \, dy \\
= \int_{\Omega} f(x, u(x)) \phi(x) \, dx \quad \forall \phi \in C_0^\infty(\Omega).
\] (18)
In particular, it must have a nontrivial generalized solution if \( b \) is not identically zero.

**Corollary 2.** Under the assumptions of Theorem 1, let \( 1 < \nu < 2, l > 2n/(2n - (n - 2)s) \nu \) and let \( G \in \mathcal{L}_{1,\text{loc}}(\Omega) \). If either \( \lambda G \leq 0 \) or \( (\lambda G)^+ := \max\{0, \lambda G\} \) belongs to \( L^\nu(\Omega) \) with \( \kappa > 2/(2 - \nu) \), then
\[
\mathcal{Q}_K u + \lambda G(x)|u|^{r-2} u + f(x, u) = 0 \quad \text{in} \quad \Omega,
\]
(19)
\[
u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega,
\]
has at least one nontrivial solution in \( X_0(\Omega, K) \) provided \( b \) is not identically zero.

**Corollary 3.** Under the assumptions of Theorem 1, let \( 2 \leq \nu < 2^* \) and \( G \in \mathcal{L}_{1,\text{loc}}(\Omega) \) with \( l > r \) and \( r > (n + 2s)/(n + 2s - (n - 1)(n - 2s)) \). If \( \lambda \) is such that \( \lambda G \leq 0 \), then
\[
\mathcal{Q}_K u + \lambda G(x)|u|^{r-2} u + f(x, u) = 0 \quad \text{in} \quad \Omega,
\]
(20)
\[
u = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega,
\]
has at least one nontrivial solution in \( X_0(\Omega, K) \) provided \( b \) is not identically zero.

In particular, this corollary includes the example at the end of Section 1.2. See Example 1 for more general cases.

Our methods can also be used to study the case of nonlocal elliptic operator systems. Let \( K_1, K_2 : \mathbb{R}^n \to (0, +\infty) \) be functions satisfying conditions (i)–(iii). Given two Carathéodory functions \( f_j : \Omega \times \mathbb{R} \to \mathbb{R}, i = 1, 2 \), consider the following problem:
\[
\mathcal{Q}_{K_i} u + f_j(x, v(x)) \frac{\partial v}{\partial x_j}(x) + f_0(x, v(x)) + a_1(x) = 0,
\]
(26)
\[
\Delta v(x) = \sum_{j=1}^{n} g_j(x, u(x)) \frac{\partial u}{\partial x_j}(x) + g_0(x, u(x)) + a_2(x) = 0
\]
(27)
or
\[
\Delta u(x) = \sum_{j=1}^{n} f_j(x, v(x)) \frac{\partial v}{\partial x_j}(x) + f_0(x, v(x)) + a_1(x) = 0,
\]
\[
\Delta v(x) = \sum_{j=1}^{n} g_j(x, u(x)) \frac{\partial u}{\partial x_j}(x) + g_0(x, u(x)) + a_2(x) = 0
\]

Assume
\[
(A) \quad f_j, g_i : \Omega \times \mathbb{R}^n \to \mathbb{R} \text{ are Carathéodory functions,}
\]
\[
i = 0, \ldots, n;
\]
\[
(B) \text{ there exist constants}
\]
\[
r_0 \in \left( \frac{2n}{n+2}, \infty \right),
\]
\[
r_i \in (n, \infty), \quad \frac{n-2}{2n} r_i < \frac{1}{s_i} < \infty, \quad i = 1, \ldots, n.
\] (28)
and measurable functions $b_i, d_i \in L^p_{\text{loc}}(\Omega)$, $i = 0, 1, \ldots, n$, $a_1, a_2 \in L^r_0(\Omega)$, such that

$$f_0(x, 0) = g_0(x, 0) = 0, \quad \forall x \in \Omega,$$

$$|f_i(x, t)| \leq b_i(x) + k_i |t|^r, \quad \forall (x, t) \in \Omega \times \mathbb{R}, \quad i = 0, 1, \ldots, n, \quad (29)$$

$$|g_i(x, t)| \leq d_i(x) + l_i |t|^r, \quad \forall (x, t) \in \Omega \times \mathbb{R}, \quad i = 0, 1, \ldots, n. \quad (29)$$

(C) there exist measurable functions $c(x), \tilde{c}(x) \in L^1(\Omega)$ and constants $1 < q_1, q_2, \tilde{q}_1, \tilde{q}_2 < 2$ such that, for any $(x, t_1, t_2, z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$

$$-\frac{1}{2} |z|^2 - k |t_1|^{q_1/2} |t_2|^{\tilde{q}_2/2} c(x) \leq \sum_{i=1}^n f_i(x, t_2) z_i + f_0(x, t_2) + a_1(x) t_1,$$

$$-\frac{1}{2} |z|^2 - \tilde{k} |t_1|^{\tilde{q}_1/2} |t_2|^{q_2/2} \tilde{c}(x) \leq \sum_{i=1}^n g_i(x, t_2) z_i + g_0(x, t_2) + a_2(x) t_1. \quad (30)$$

Combing the proof of [17] and that of Theorem 4, we can prove the following.

**Theorem 5.** Under the conditions (A), (B), and (C), if $a_1$ or $a_2$ is not zero, then the equations systems (26) and (27) have at least a nontrivial generalized solution $(u, v) \in W^{1,2}_0(\Omega) \times W^{0,2}_0(\Omega)$.

Finally, let us point out that the corresponding results of Theorems 1 and 4 can be also proved if the operator $\mathcal{B}_K$ is replaced by $L^p$ in [6, (10)]. They will be given in other places.

The arrangements of this paper are as follows. In Section 2, we give some necessary preliminaries. The proof of Theorem 1 will be completed in Section 3. In Section 4, we will prove Corollaries 2 and 3 and give an example. Theorem 4 will be proved in Section 5.

## 2. Preliminaries

Firstly, we review the topological degree theory for mappings of class $(\mathcal{B}_1)$ developed in [17]. Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $[E_n]_n$ be a strictly increasing sequence of finite dimensional subspaces of $H$ such that $E = \bigcup_{n=1}^{\infty} E_n$ is dense in $H$. Denote by $P_n$ the orthogonal projection from $H$ onto $E_n$ for each integer $n \in \mathbb{N}$. Let $G$ be an open bounded set in $E$ and let $g$ be a mapping from $G$, the closure of $G$ in $E$, into $H$. Put

$$G_n = G \cap E_n \quad \forall n \in \mathbb{N},$$

$$g_n(x) = P_n(g(x)) \quad \forall n \in \mathbb{N}, \quad x \in G_n. \quad (31)$$

Since $H$ and $E$ induce equivalent topologies on all finite dimensional spaces $E_n$ the subset $G_n \subset E_n$ has the same closure in $E_n, E$, and $H$, denoted by $G_n$.

**Definition 6.** Under the above assumptions, $g$ is said to be of class $(\mathcal{B}_1)$ on $E$ if and only if the following conditions are satisfied:

(a) $g_n : \overline{G_n} \to E_n$ is a continuous mapping for each $n \in \mathbb{N}$.

(b) There is not any sequence $\{x_{n_k}\}_k$ in $E$ such that the sequence $\{x_{n_k}\}_k$ is weakly convergent in $H, x_{n_k} \in \partial_{E_n} G_{n_k}$, and $(g(x_{n_k})), x_{n_k}) \leq 0$ and $(g(x_{n_k})), v) = 0$ for all $k \in \mathbb{N}$ and $v \in E_{n_k}$.

**Lemma 7** (see [17, Lemma 2.3]). Let $H, [E_n]_n, E, G, g$ and $\{g_n\}_n$ be as in Definition 6. Assume that $g$ is of class $(\mathcal{B}_1)$ on $E$.

Then, there exists an integer $n_0$ such that the Leray-Schauder degree $\deg(g_n, G_n, 0)$ is defined and

$$\deg(g_n, G_n, 0) = \deg(g_{n_0}, G_{n_0}, 0) \quad \forall n \geq n_0. \quad (32)$$

It follows that

$$\deg(g, G, 0) := \lim_{n \to \infty} \deg(g_n, G_n, 0) \quad (33)$$

is defined. It was the topological degree of $g$ on $G$ at 0 in [17]. The corresponding versions with usual properties of the Leray-Schauder degree were given in [17, Theorem 2.1]. In particular, the identity map $\text{id}$ is of class $(\mathcal{B}_1)$, and $\deg(\text{id}, G, 0) = 1$ if $0 \in G$. Moreover, the following proposition is key for the proof of our main results.

**Proposition 8** (see [17, Corollary 2.1]). Let $H, [E_n]_n, E$ and $G$ be as in Definition 6. Let $g$ be a mapping from $G$ into $H$ such that $g_m$ is continuous on $G_m$ for any $m \in \mathbb{N}$. Suppose that $G$ contains 0 and

$$\langle g(x), x \rangle > 0, \quad \forall x \in \partial_{E} G. \quad (34)$$

Then there is a weakly Cauchy sequence $\{x_{n}\}_n$ in $G$ such that

$$\lim_{n \to \infty} \langle g(x_n), v \rangle = 0, \quad \forall v \in E. \quad (35)$$

Next, we need the following results on the space $X_0(\Omega, K)$.

**Lemma 9.** (a) $X_0(\Omega, K)$ and $X(\Omega, K)$ are continuously embedded in $H^1(\mathbb{R}^n)$ and $H^1(\Omega)$, respectively ([8, Lemma 5]).

(b) If $\Omega \subset \mathbb{R}^n$ is a bounded open set with continuous boundary, the embedding $X_0(\Omega, K) \hookrightarrow L^p(\mathbb{R}^n)$ is compact for any $p \in [1, 2^*_n]$ ([8, Lemma 8] and [11, Lemma 9-1]).

(c) The embedding $X_0(\Omega, K) \hookrightarrow L^p(\mathbb{R}^n)$ is continuous for $p = 2^*_n$ ([3, Lemma 9-1]).

(d) The embedding $H^1(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is continuous for any $p \in [1, 2^*_n]$ ([18, Theorem 6.5]).

(e) If $\Omega$ is an open set in $\mathbb{R}^n$ of class $C^0, 1$ with bounded boundary, then there exist continuous embeddings $W^{1,2}_0(\Omega) \hookrightarrow W^{1,2}_0(\Omega)$ and $W^{1,2}_0(\Omega) \hookrightarrow W^{1,2}_0(\Omega)$ for any $p \in [1, \infty)$ and $s \in (0, 1)$ ([18, Proposition 2.2]).
**Lemma 10.** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with boundary of class $C^{1,1}$. Then, the space $X_0(\Omega, K)$ is separable. Furthermore, there exists a sequence $\{v_m\}_m$ in $C_0^\infty(\Omega)$ such that $\{v_m\}_m$ is a maximal orthogonal set of $X_0(\Omega, K)$. 

**Proof.** By Proposition 9(11) of [9], there exists a Hilbert basis $\{e_k\}_{k \geq 1}$ of $X_0(\Omega, K)$, which implies separability of $X_0(\Omega, K)$. So $\{e_k\}_{k=1}^\infty$ is a dense countable subset of $X_0(\Omega, K)$. Let $\{f_m\}_{m=1}^\infty$ denote this countable set. Since $C_0^\infty(\Omega)$ is dense in $X_0(\Omega, K)$ by [12], $\{f_m\}_{m=1}^\infty$ is also dense in $X_0(\Omega, K)$. Then, $\{f_m\}_{m=1}^\infty$ is dense in $X_0(\Omega, K)$. Making the Hilbert-Schmidt orthogonalization procedure for $\{f_m\}_{m=1}^\infty$, we obtain an orthogonal set $\{e_m\}_{m=1}^\infty$, which is also a maximal in $X_0(\Omega, K)$. 

**3. Proof of Theorem 1**

Take an increasing sequence of open subsets of $\Omega$, $\Omega_k$, such that each of them has $C^{1,1}$-boundary and that:

$$\Omega_k \subset \Omega_{k+1} \quad \forall k \in \mathbb{N},$$

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k.$$  

By Lemma 10, we may choose a sequence $\{v_{1,m}\}_m$ in $C_0^\infty(\Omega_1)$ such that $\{v_{1,m}\}_m$ is a maximal orthogonal set of $X_0(\Omega_1, K)$. Then, we can find a sequence $\{v_{2,m}\}_m$ in $C_0^\infty(\Omega_2) \setminus C_0^\infty(\Omega_1, K)$ such that $\{v_{2,m}\}_m$ is a maximal orthogonal set of $X_0(\Omega_2, K)$. The mathematical induction, it is easy to find the set $\{v_{k,m}\}_m$ is a maximal orthogonal set of $X_0(\Omega_k, K)$ for every $k \in \mathbb{N}$. Let $E_m$, be the vector subspace of $X_0(\Omega, K)$ spanned by $\{e_1, e_2, \ldots, e_m\}$, and $E = \bigcup_{m=1}^{\infty} E_m$. For conveniences we set $H = X_0(\Omega, K)$ and denote by $P_m$ the orthogonal projection from $H$ onto $E_m$.

**Lemma 11.** (a) $E$ is dense in $X_0(\Omega, K)$. 

(b) For each $m \in C_0^\infty(\Omega)$, there are $k \in \mathbb{N}$ and a sequence $\{u_k\}_m$ in $E$ such that the supports of all $u_k$ are contained in $\Omega_k$ and that $u_k$ converges to $u$ in $X_0(\Omega, K)$ as $m \to \infty$.

(c) For every $m \in \mathbb{N}$ and for every given $u \in X_0(\Omega, K)$, there exists a unique $T_m(u)$ in $X_0(\Omega, K)$ such that:

$$\langle T_m(u), v \rangle_{0,\Omega,K} = -\int_{\Omega} f(x, u(x)) v(x) \, dx$$

$$\forall v \in X_0(\Omega, K).$$

(37)

Moreover, if $v \in C_0^\infty(\Omega, K)$, then:

$$\langle T_m(u), v \rangle_{0,\Omega,K} = \langle T_k(u), v \rangle_{0,\Omega,K} \quad \forall m \geq k.$$  

(38)

(d) Suppose that a sequence $\{u_k\}_k \subset X_0(\Omega, K)$ weakly converges to $u$ in $X_0(\Omega, K)$. Then, $\{T_m(u_k)\}_k$ weakly converges to $T_m(u)$ in $X_0(\Omega, K)$ for $m = 1, 2, \ldots$.

(e) For every given $u \in E$ (the support of $u$ must be contained in some $\Omega_m$, by the construction of $E$), there exists a unique $T(u)$ in $X_0(\Omega, K)$ such that:

$$\langle T(u), v \rangle_{0,\Omega,K} = -\int_{\Omega} f(x, u(x)) v(x) \, dx$$

$$\forall v \in X_0(\Omega, K).$$

(39)

**Proof.** (a) Since $\{e_k\}_k$ is a maximal orthogonal set of $X_0(\Omega, K)$, $E$ is dense in $X_0(\Omega, K)$. 

(b) For a given $u \in C_0^\infty(\Omega)$, by the choices of $\{\Omega_m\}_m$, there exists $k \in \mathbb{N}$ such that the support of $u$ is contained in $\Omega_k$. Let $\{v_{1,m}\}_k \subset \mathbb{N}$ be the maximal orthogonal set of $X_0(\Omega_k, K)$ as constructed above. Then, $\langle T_{1,m}(u), v \rangle_{0,\Omega,K} \in X_0(\Omega, K)$. Hence, we can find a sequence $\{v_{2,m}\}_k$ is a maximal orthogonal set in $\mathbb{N}$ such that $\|v_{2,m} - u\|_{0,\Omega,K} \to 0$ as $m \to \infty$.

(c) By (16), we can write $f(x, t) = b(x) + f_0(x, t)$ for $t \in \Omega \times \mathbb{R}$. Then, $f_0(x, 0) = 0 \forall x \in \Omega$, and (15) implies:

$$|f_0(x, t)| \leq a(x) + b(x) + |t|^q-1 \quad \forall (x, t) \in \Omega \times \mathbb{R}.$$  

(41)

Note that $2^* = \min\{2 \Omega (m-n) = \min\{2 \mathbb{R}, 2 \mathbb{N} \} = 2 \mathbb{N} \} = 2 \mathbb{N}$.

$$\frac{2n}{n + 2s} < r < s \quad \forall x \in \Omega_m.$$  

(42)

For given $m \in \mathbb{N}$, by [19, Theorem 3.2.4] (see also [20, page 30]) we have a continuous mapping $\mathcal{S}_m$ from $L^{(q-1)}(\Omega)$ into $L'(\Omega_m)$, where:

$$\mathcal{S}_m(u)(x) = f_0(x, u(x)) \quad \forall x \in \Omega_m.$$  

(43)

For $u, v \in X_0(\Omega, K)$, we have $u \in L^{(q-1)}(\Omega)$ and $v \in L^2(\Omega)$ by Lemma 9(b) and (c). Thus, $\mathcal{S}_m(u) \in L'(\Omega_m)$ and:

$$\left| \int_{\Omega} f_0(x, u(x)) v(x) \, dx \right| \leq \left\| \mathcal{S}_m(u) \right\|_{L'(\Omega_m)} \| v \|_{L^2}.$$  

(44)
Using Lemma 9(c) again, there exists a constant $C > 0$ such that $\|v\|_{L^2} \leq C\|v\|_{0,\Omega,K}$ for all $v \in X_0(\Omega,K)$. It follows that
\[
\left| - \int_{\Omega_m} f_0(x, u(x)) v(x) \, dx - \int_{\Omega} b(x) v(x) \, dx \right| \leq C \left( \|\mathfrak{F}_m(u)\|_{L^2(\Omega)} + \|b\|_{L^2(\Omega)} \right) \|v\|_{0,\Omega,K}.
\] (45)

Hence, Riesz representative theorem yields a unique $T_m(u) \in X_0(\Omega,K)$ such that
\[
\langle T_m(u), v \rangle_{0,\Omega,K} = \int_{\Omega_m} \mathfrak{F}_m(u)(x) v(x) \, dx - \int_{\Omega} b(x) v(x) \, dx \quad \forall v \in X_0(\Omega,K).
\] (46)

Since $\{u_k\} \subset G$ converges to $u \in G$, Lemma 11(f) from Lemma 9(b) and (c) deduce that
\[
\mathcal{R}(s) \rightarrow \mathcal{R}(s)^*.
\]

Clearly, $(T(u), v)_{0,\Omega,K} = (T_m(u), v)_{0,\Omega,K} \forall v \in X_0$ for all $m \geq m_0$.

(f) By the construction of $E_m$, we have an integer $m_0 \in \mathbb{N}$ such that each $u \in E_m$ has a support contained in $\Omega_{m_0}$. Let $\{u_k\} \subset E_m$ converge to $u \in E_m$. By (d), $(T_n(u_k))_k$ weakly converges to $T_n(u)$ in $X_0(\Omega, K)$ for every $n \in \mathbb{N}$. Then, (e) implies that $(T(u_k), v)_{0,\Omega,K} \rightarrow (T(u), v)_{0,\Omega,K} \forall v \in X_0(\Omega,K)$ as $k \to \infty$. In particular, since $v \in E_m$ satisfies $P_m v = v$, we have
\[
\langle P_m \circ T(u_k), v \rangle_{0,\Omega,K} = \langle T(u), P_m v \rangle_{0,\Omega,K} \quad \forall v \in X_0(\Omega,K).
\] (52)

This shows that $\{P_m \circ T(u_k)\}_k$ weakly converges to $P_m T(u)$ in $E_m$. However, the strong convergence and the weak ones on finitely dimensional space $E_m$ are equivalent. Hence, $P_m \circ T(u_k) \rightarrow P_m \circ T(u)$ as $m \to \infty$.

(g) Since $1 \leq p < 2 < 2^*$, by Lemma 9(b) and (c) there is a constant $C > 0$ such that $\|u\|_{L^p} \leq C\|u\|_{0,\Omega,K} \forall u \in X_0(\Omega,K)$. It follows from this and (17) that
\[
\langle u + T(u) \rangle_{0,\Omega,K} = \|u\|_{0,\Omega,K}^2 - \int_{\Omega} f(x, u(x)) u(x) \, dx \geq \|u\|_{0,\Omega,K}^2 - \int_{\Omega} \left( \beta |u|^p + |c| \right) \, dx \geq \|u\|_{0,\Omega,K}^2 - \int_{\Omega} |u|^p \, dx - \|c\|_{L^1} \geq \|u\|_{0,\Omega,K}^2 - \beta C p \|u\|^p_{0,\Omega,K} - \|c\|_{L^1}.
\]

This leads to (g).

**Proof of Theorem 1.** Let $\beta, C$, and $c$ be as above. Since $1 \leq p < 2$, we have $R > 0$ such that
\[
1 - \beta C p R^{p-2} - \|c\|_{L^1} \cdot R^{p-2} > \frac{1}{4}.
\] (54)

Let $G = \{u \in E : \|u\|_{0,\Omega,K} < R\}$. Define $g : G^E \rightarrow X_0(\Omega, K)$ by
\[
g(u) = u + T(u) \quad \forall u \in G^E.
\] (55)

Let us prove that $g$ is of class $(B_1)$ on $G^E$. Note that $G_n = G_n \cap E_n$ has the same closure in $E_n$, $E$, and $H$, denoted by $G_n$. Let $g_n : G_n \rightarrow E_n$ be defined by $g_n(u) = P_n(g(u)) = P_n u + P_{n m} \circ T(u) = u + P_{n m} \circ T(u)$ for each $n \in \mathbb{N}$. Suppose that a sequence $\{u_k\} \subset G_n$ converges to $u \in G_n$. Lemma 11(f)
implies $P_n \circ T(u_k) \to P_n \circ T(u)$ in $E_n$. Hence, $g_n$ is continuous. By the proof of Lemma II(g) and (54) we deduce that

$$
\langle g(u), u \rangle_{0, \Omega, K} = \langle u + T(u), u \rangle_{0, \Omega, K} \geq \frac{R^2}{4}
$$

$$
\forall u \in \partial_G G = \{ u \in E : \|u\|_{0, \Omega, K} = R \}.
$$

This implies that Definition 6(b) is satisfied. Hence, $g$ is of class $(B_i)$ on $G^e$. Moreover, it also shows that $g$ satisfies the conditions of Proposition 8. Thus, we have a weakly Cauchy sequence $\{u_n\}_n \subset G$ such that

$$
\lim_{n \to \infty} \langle g(u_n), v \rangle_{0, \Omega, K} = 0 \quad \forall v \in E.
$$

Let $u$ be the weak limit of $\{u_n\}_n$ in $X_0(\Omega, K)$. For a given $v \in E$, the support of it is contained in some $\Omega_k$, and thus

$$
\langle g(u_n), v \rangle_{0, \Omega, K} = \langle u_n + T(u_n), v \rangle_{0, \Omega, K} = \langle u_n + T_k(u_n), v \rangle_{0, \Omega, K}
$$

$$
\forall m \geq k, \quad \forall n \in \mathbb{N}
$$

by Lemma II(c). For each fixed $n \in \mathbb{N}$, there exists $\bar{n} \in \mathbb{N}$ such that $u_n \in C^0_0(\Omega_{\bar{n}})$. Hence, Lemma II(c) yields

$$
\langle g(u_n), v \rangle_{0, \Omega, K} = \langle u_n + T(u_n), v \rangle_{0, \Omega, K} = \langle u_n + T_k(u_n), v \rangle_{0, \Omega, K}
$$

$$
\forall m \geq \max \{k, \bar{n}\}.
$$

Taking $n \to \infty$ in both sides of $\langle g(u_n), v \rangle_{0, \Omega, K} = \langle u_n + T_k(u_n), v \rangle_{0, \Omega, K}$ and using (57) and Lemma II(d), we deduce

$$
\langle u + T_m(u), v \rangle_{0, \Omega, K} = 0, \quad \forall m \geq k.
$$

For any given $v \in C^0_0(\Omega)$, by Lemma II(b), we have an integer $k$ and a sequence $\{\bar{v}_i\}_i$ $E$ such that $\text{Supp}(\bar{v}_i) \subset \Omega_{\bar{n}}$ for any $\bar{n} \in \mathbb{N}$ and that $\bar{v}_i \to v$ in $X_0(\Omega, K)$. So (60) leads to

$$
\langle u + T_m(u), v \rangle_{0, \Omega, K} = \lim_{i \to \infty} \langle u + T_m(u), \bar{v}_i \rangle_{0, \Omega, K} = 0
$$

$$
\forall m \geq k;
$$

that is, $\langle u, v \rangle_{0, \Omega, K} - \int_\Omega f(x, u(x))v(x)dx = 0 \forall m \geq k$. Letting $m \to \infty$, we get

$$
\langle u, v \rangle_{0, \Omega, K} - \int_\Omega f(x, u(x))v(x)dx = 0,
$$

which shows that $u$ is a generalized solution. Note that $u$ might be zero! But $u = 0$ is not a solution if $b = f(\cdot, 0)$ takes nonzero values on a nonzero measure set. The proof is completed.

4. Proofs of Corollaries and Examples

**Proof of Corollary 2.** Let $\tilde{f}(x, t) = \lambda G(x)|t|^{\gamma-2} + f(x, t)$. It suffices to check that $\tilde{f}$ satisfies (15)–(17). Clearly, we can assume $\lambda \neq 0$. Let $\rho = l/r$. Since $1 < \nu < 2$, we have $(n-2)s > n-2s$ and thus

$$
l > \frac{2n}{2n - (n - 2)s} > \frac{2n}{n + 2s} = r.
$$

Then, $\rho > 1$ and

$$
1 < (\nu - 1)\rho' = (\nu - 1)\frac{\rho}{\rho - 1} = \frac{l}{l - r} < \frac{n + 2s}{n - 2s} = 2s - 1.
$$

By Young’s inequality, we obtain

$$
|\lambda G(x)||t|^{\gamma-2} = |\lambda G(x)||t|^{\gamma-1} \leq \frac{|\lambda|}{\rho} |G(x)|^\rho + \frac{|\lambda|}{\rho'} |t|^{(\nu-1)\rho'}.
$$

Note that $G \in L^1_{\text{loc}}(\Omega)$ and $r\rho = l$ imply $|G|^\rho \in L^1_{\text{loc}}(\Omega)$. Let $\bar{q} - 1 = \max\{q - 1, (\nu - 1)\rho'\}$, which sits in $(1, 2s - 1)$. From these and (15), it follows that

$$
|\tilde{f}(x, t)| \leq a(x) + \frac{|\lambda|}{\rho} |G(x)|^\rho + \frac{|\lambda|}{\rho'} |G(x)|^{\rho'} + a|t|^{\rho'-1} \leq \left( a(x) + \frac{|\lambda|}{\rho} |G(x)|^\rho + C \right) + \bar{a}|t|^{\rho'-1}
$$

a.e. $x \in \Omega$, $\forall t \in \mathbb{R}$ for some constants $C > 0$ and $\bar{a} > 0$, where Young’s inequality is used again. So $\tilde{f}$ satisfies (15). Moreover, $\tilde{b}(x) := \tilde{f}(x, 0) = f(x, 0) = b(x)$; that is, $\tilde{f}$ satisfies (16).

Finally, let us check that (17) holds for $\tilde{f}$. If $\lambda G \leq 0$, then

$$
-\tilde{f}(x, t) = -\lambda G(x)|t|^{\gamma-2} - f(x, t) t \geq -\beta |t|^{\rho} - c(x).
$$

For another case, observe that

$$
-\tilde{f}(x, t) = -\lambda G(x)|t|^{\gamma-2} - f(x, t) t \geq -\beta |t|^{\rho} - c(x) - (\lambda G)_+ |t|^{\gamma-2}.
$$

Since $\kappa > 2/(2 - \nu) > 1$, we may choose a real number $\sigma$ in $(2/(2 - \nu), \kappa)$. Let $\sigma' = \sigma/(\sigma - 1)$. By Young’s inequality, we have

$$
(\lambda G)_+(x)|t|^{\gamma-2} \leq \frac{1}{\sigma'} |(\lambda G)_+(x)|^\sigma + \frac{1}{\sigma'} |t|^{\sigma'}.
$$
Note that \(|(\lambda G)^{\alpha}t| \in L^1(\Omega)\) since \(\sigma < \kappa\). Moreover, \(1 < \alpha\)' and
\[
\frac{2}{2 - \gamma} < \sigma \iff \frac{2 - \gamma}{2} > \frac{1}{\sigma} \iff \frac{2}{\sigma - 1} = \sigma' \\
\iff \alpha' < 2.
\] (70)

Let \(\overline{\lambda} = \max\{\rho, \alpha'\}\), which belongs to \([1, 2]\). Using Young's inequality, we can derive
\[
-\beta|t|^\overline{\lambda} - \frac{1}{\sigma}|t|^\alpha' \geq -\overline{\lambda}|t|^{\overline{\lambda}} - C \quad \forall t \in \mathbb{R}
\] (71)
for some constants \(\overline{\lambda} > 0\) and \(C > 0\). Hence, for a.e. \(x \in \Omega\) and all \(t \in \mathbb{R}\), we have
\[
-\int_\Omega \tilde{f}(x, t) = -\lambda G(x)|t|^\gamma - f(x, t) t
\] (72)
\[
\geq -\overline{\lambda}|t|^\overline{\lambda} - \tilde{c}(x),
\]
where \(\tilde{c}(x) = c(x) + (1/\sigma)(\lambda G(x)^{\alpha})^{\alpha'} + C\) belongs to \(L^1(\Omega)\) as above. This shows that (77) is true for \(\tilde{f}\). The desired conclusion follows from Theorem 1 immediately.

**Proof of Corollary 3.** Let \(\tilde{f}(x, t) = \lambda G(x)|t|^\gamma - t f(x, t)\). Since \(\lambda G \leq 0\), we see that \(\tilde{f}\) satisfies (17) from the above proof. It remains to prove that \(\tilde{f}\) satisfies (15)-(16). Let \(\delta' = \delta/(\delta - 1)\). Note that
\[
\delta > \frac{n + 2s}{n + 2s - (\delta - 1)(n - 2s)}
\] (73)
\[
\iff \frac{n + 2s - (\delta - 1)(n - 2s)}{n + 2s} = 1 - \frac{(\delta - 1)(n - 2s)}{n + 2s}
\] (74)
\[
< \frac{\delta - 1}{\delta} \iff 1 - \frac{\delta}{\delta} < \frac{(\delta - 1)(n - 2s)}{n + 2s}
\] (75)
\[
\iff \frac{\delta - 1}{\delta} < \frac{n + 2s}{n - 2s}.
\]
By Young's inequality, we have
\[
|G(x)| |t|^\gamma \leq \frac{|G(x)|^\delta}{\delta} + \frac{|t|^{(\gamma - 1)\delta'}}{\delta'}. \tag{76}
\]
Now, \(l \geq r\delta\) implies \(l/\delta \geq r\) and \(|G|^\delta \in L^1(\Omega)\) because \(G \in L^1_{\text{loc}}(\Omega)\). We obtain \(|G|^{\delta'} \in L^1_{\text{loc}}(\Omega)\). As in the proof of Corollary 2, using Young's inequality, we may derive from this and (73)-(74) that \(\tilde{f}\) satisfies (15) and (16).

**Example 1.** Let \(n \geq 2\) and \(\Omega \subset \mathbb{R}^n\) be as above. Consider
\[
\mathcal{B}_K u + G(x) u + h(u) = 0 \quad \text{in} \ \Omega,
\]
\[
u(x, u) = 0 \quad \text{in} \ \mathbb{R}^n \setminus \Omega,
\] (75)
where \(G \leq 0\) belongs to \(L^1_{\text{loc}}(\Omega)\) with \(l \in ((n + 2s)/4s, n/2s)\), \(h \in C(\mathbb{R})\) is absolutely continuous, \(h(0) \neq 0\), \(\sup_{t \in \mathbb{R}} |h(t)| < \infty\), and \(|h'(t)| \leq e_1 + e_2 |t|^\rho, \text{a.e.}, t \in \mathbb{R}, 0 \leq \rho < 4s/(n - 2s)\). Then, (75) has a nontrivial generalized solution.

In fact, taking \(v = 2\) in Corollary 3, we should require \(\theta > (n + 2s)/4s\). Since \(e \in ((n + 2s)/4s, n/2s)\), there is sufficiently small \(e > 0\) such that
\[
\frac{n + 2s}{4s} + e < \frac{n}{2s} < \frac{n + 2s}{4s} + e\]
(76)
This means that we can take \(\theta = (n + 2s)/4s + e\). Moreover, \(1 \leq \rho + 1 < 1 + 4s/(n - 2s) = 2^*_s\), and
\[
|h(t) - h(0)| = \left|\int_0^t h'(r) \, dr\right| \leq e_1 |t| + e_2 \int_0^t |t|^\rho + 1
\] (77)
\[
\leq e_1 t + e_2 + e_2 |t|^\rho + 1,
\]
\[
- f(x, t) t = - G(x) t^2 - h(t) t \geq - \sup_{t \in \mathbb{R}} h(t) t
\] (78)
\[> - \infty.
\]
Hence, (15)-(17) are satisfied for \(f(x, t) = G(x) t + h(t)\).

### 5. Proof of Theorem 4

Consider the product Hilbert space \(H = X_0(\Omega, K_1) \times X_0(\Omega, K_2)\) equipped with inner product
\[
\langle u, v \rangle_H = \langle (u_1, u_2), (v_1, v_2) \rangle_H
\] (80)
\[
= \langle u_1, v_1 \rangle_{X_0(\Omega, K_1)} + \langle u_2, v_2 \rangle_{X_0(\Omega, K_2)}
\]
for \(u = (u_1, u_2), v = (v_1, v_2) \in H\). The induced norm is
\[
\| (u, v) \|_H = \left( \| u \|_{X_0(\Omega, K_1)}^2 + \| v \|_{X_0(\Omega, K_2)}^2 \right)^{1/2}.
\]

Let \(\Omega = \cup_{k \in \mathbb{N}} \Omega_k\) and \(E = \cup_{m \in \mathbb{N}} E_m\) be as in Section 3. For every integer \(m \in \mathbb{N}\) let \(E_m = \cup_{k \in \mathbb{N}} (E_k \times E_k)\) and \(E = \cup_{m \in \mathbb{N}} E_m\). Denote by \(P_m\) the orthogonal projection from \(E\) onto \(E_m\). Corresponding to Lemma II, we have the following.

**Lemma 13.** (a) \(E\) is dense in \(H\).

(b) For each \(u \in C_0^\infty(\Omega) \times C_0^\infty(\Omega)\), there are \(k \in \mathbb{N}\) and a sequence \((u_n)_{n \in \mathbb{N}}\) in \(E\) such that the supports of all \(u_n\) are contained in \(\Omega_k \times \Omega_k\) and that \(u_n \to u\) in \(H\) as \(m \to \infty\).

(c) For every \(m \in \mathbb{N}\), and for every given \(u \in H\), there exists a unique \(T_m(u) \in H\) such that
\[
\langle T_m(u), v \rangle_H = -\int_{\Omega_m} f_1(x, u_2(x)) v_1(x) \, dx
\] (81)
\[
-\int_{\Omega_m} f_2(x, u_1(x)) v_2(x) \, dx
\]
for any \(v = (v_1, v_2) \in H\). Moreover, if \(v = (v_1, v_2) \in C_0^\infty(\Omega_k) \times C_0^\infty(\Omega_k)\), then \((T_m(u), v) \in \langle T_k(u), v \rangle_H \forall m \geq k\).
(d) Suppose that a sequence \( \{u_k\}_k \subset H \) weakly converges to \( u \) in \( H \). Then, \( \{T_m(u_k)\}_k \) weakly converges to \( T_m(u) \) in \( H \) for \( m = 1, 2, \ldots \).

(e) For every given \( u \in \mathcal{E} \) (the support of \( u \) must be contained in some \( \Omega_{m_0} \times \Omega_{m_0} \) by the construction of \( \mathcal{E} \)), there exists a unique \( T(u) \in H \) such that

\[
\langle T(u), v \rangle_H = -\int_{\Omega} f_1(x, u_2(x)) v_1(x) \, dx
-\int_{\Omega} f_2(x, u_1(x)) v_2(x) \, dx, \tag{81}
\]

\[
\langle T(u), v \rangle_H = \langle T_m(u), v \rangle \quad \forall v \in H(\Omega), \; \forall m \geq m_0.
\]

(f) \( P_m : \mathcal{E} \rightarrow \mathcal{E}_m \) is continuous for every \( m \in \mathbb{N} \).

(g) There exists a constant \( C > 0 \) such that

\[
\langle u + T(u), u \rangle_H \geq \|u\|_H^2 + C \left( \|f_1\|_{L^2(\Omega)}^2 + \|f_2\|_{L^2(\Omega)}^2 \right) \tag{82}
\]

for all \( u \in E \setminus \{0\} \).

Proof. (a) and (b) follow from Lemma 11(a) and (b) immediately.

(c) By (24), we can write \( f_j(x,t) = b_j(x) + f_{0_j}(x,t) \forall (x, t) \in \Omega \times \mathbb{R} \) as in the proof of Lemma 11(c); then, \( f_{0_j}(x, 0) = 0 \) \( \forall x \in \Omega \), and

\[
|f_{0_j}(x,t)| \leq a_j(x) + b_j(x) + \alpha_j|t|^{\gamma_j-1}
\]

\( \forall (x, t) \in \Omega \times \mathbb{R}, \; j = 1, 2. \tag{83} \)

Moreover, for every given \( m \in \mathbb{N} \) and \( j = 1, 2 \), we have continuous mappings

\[
L^{(\gamma_j-1)}(\Omega) \ni u \mapsto \mathfrak{F}_{m,j}(u) \in L^r(\Omega_m), \tag{84}
\]

where \( \mathfrak{F}_{m,j}(u)(x) = f_{0_j}(x, u(x)) \) for \( x \in \Omega_m, \; j = 1, 2 \). Recall that \( \gamma_j^* = 2_{j}^* \). For \( u_1, v_1 \in X_0^*(\Omega, K_1) \), \( u_2, v_2 \in X_0^*(\Omega, K_2) \), by Lemma 9(c) and (c), we have

\[
v_j \in L^{2^*}(\Omega_m), \tag{85}
\]

\[
u_j \in L^{(\gamma_j-1)}(\Omega),
\]

\( i, j = 1, 2. \)

Thus, \( \mathfrak{F}_{m,1}(u_2) \in L^r(\Omega_m) \), \( \mathfrak{F}_{m,2}(u_1) \in L^r(\Omega_m) \), and

\[
-\int_{\Omega_m} f_{0_1}(x, u_2(x)) v_1(x) \, dx
= \int_{\Omega_m} \mathfrak{F}_{m,1}(u_2) v_1(x) \, dx
\]

\[
\leq \|\mathfrak{F}_{m,1}(u_2)\|_{L^r(\Omega_m)} \|v_1\|_{L^r}^r, \tag{86}
\]

Using Lemma 9(c) again, there exist constants \( C > 0 \) such that \( \|v_j\|_{L^r} \leq C \|v_j\|_{0, \Omega, K_j} \) for all \( v = (v_1, v_2) \in H \), \( j = 1, 2 \). It follows that

\[
-\int_{\Omega_m} f_{0_2}(x, u_1(x)) v_2(x) \, dx
= \int_{\Omega_m} \mathfrak{F}_{m,2}(u_1) v_2(x) \, dx
\]

\[
\leq \|\mathfrak{F}_{m,2}(u_1)\|_{L^r(\Omega_m)} \|v_2\|_{L^r}^r. \tag{87}
\]

By the Riesz representative theorem, for each \( j \in \{1, 2\} \), we have a unique \( T_m(u) \in X_{0, \Omega, K_j} \) such that

\[
\langle T_{m_j}(u), v_j \rangle_{0, \Omega, K_j} = -\int_{\Omega_m} f_1(x, u_2(x)) v_1(x) \, dx, \tag{88}
\]

\[
\langle T_{m_2}(u), v_j \rangle_{0, \Omega, K_j} = -\int_{\Omega_m} f_2(x, u_1(x)) v_2(x) \, dx. \tag{89}
\]

Setting \( T_m(u) := (T_{m_1}(u), T_{m_2}(u)) \), we obtain

\[
\langle T_m(u), v \rangle_H = \langle T_{m_1}(u), v_1 \rangle_{0, \Omega, K_1}
+ \langle T_{m_2}(u), v_2 \rangle_{0, \Omega, K_2}
= -\int_{\Omega_m} f_1(x, u_2(x)) v_1(x) \, dx
-\int_{\Omega_m} f_2(x, u_1(x)) v_2(x) \, dx. \tag{90}
\]

Another claim can be proved as that of Lemma 11(c).

(d) Let \( u_k = (u_{k,1}, u_{k,2}) \) for each \( k \in \mathbb{N} \). Then, \( \{u_{k,1}\}_k \subset X_{0, \Omega, K_1} \) weakly converges to \( u_1^* \in X_{0, \Omega, K_1} \) and \( \{u_{k,2}\}_k \subset X_{0, \Omega, K_2} \) weakly converges to \( u_2^* \in X_{0, \Omega, K_2} \). For each \( m \in \mathbb{N} \), by Lemma 11(d), \( \{T_{m_1}(u_{k,1})\}_k \) weakly converges to \( T_{m_1}(u) \) in \( X_0^*(\Omega, K_1) \) and \( \{T_{m_2}(u_{k,2})\}_k \) weakly converges to \( T_{m_2}(u) \) in \( X_0^*(\Omega, K_2) \).
Thus, \{T_m(u_k)\}_k weakly converges to T_m(u) in H for m = 1, 2, ..., (e) Since \( f_{g,j}(x, 0) = 0 \) \( \forall x \in \Omega, j = 1, 2 \), and \( \text{Supp}(u) \subset \Omega_{m_j} \times \Omega_{m_j} \), for any integer \( m \geq m_0 \) and \( v = (v_1, v_2) \in H \), we derive from (89) that

\[
- \int_{\Omega} f_1(x, u_2(x)) v_1(x) \, dx = - \int_{\Omega} f_{0,1}(x, u_2(x)) v_1(x) \, dx \\
- \int_{\Omega} b_1(x) v_1(x) \, dx \\
= - \int_{\Omega} f_{0,1}(x, u_2(x)) v_1(x) \, dx \\
- \int_{\Omega} b_1(x) v_1(x) \, dx = \langle T_{m_1}(u), v \rangle_{0,\Omega,K_1},
\]

(91)

These show that

\[
X_0(\Omega, K_1) \ni v_1 \mapsto - \int_{\Omega} f_1(x, u_2(x)) v_1(x) \, dx,
\]

\[
X_0(\Omega, K_2) \ni v_2 \mapsto - \int_{\Omega} f_2(x, u_1(x)) v_2(x) \, dx
\]

are two continuous linear functionals. Using the Riesz representative theorem again we obtain a unique \( T_1(u) \in X_0(\Omega, K_1), T_2(u) \in X_0(\Omega, K_2) \) such that

\[
\langle T_1(u), v \rangle_{0,\Omega,K_1} = - \int_{\Omega} f_1(x, u_2(x)) v_1(x) \, dx,
\]

(93)

\[
\langle T_2(u), v \rangle_{0,\Omega,K_2} = - \int_{\Omega} f_2(x, u_1(x)) v_2(x) \, dx
\]

for all \( v = (v_1, v_2) \in H \). Set \( T(u) := (T_1(u), T_2(u)) \); then,

\[
\langle T(u), v \rangle_H = - \int_{\Omega} f_1(x, u_2(x)) v_1(x) \, dx \\
- \int_{\Omega} f_2(x, u_1(x)) v_2(x) \, dx,
\]

(94)

\[ \forall v \in H. \]

Clearly, \( \langle T(u), v \rangle_H = \langle T_m(u), v \rangle_H \) \( \forall m \geq m_0 \).

\( f \) follows the above (e) and Lemma II(f) directly.

(g) Since \( 1 \leq p_j, p_j < 2 < 2' \), by Lemma 9(b) and (c) there is a constant \( C_0 > 0 \) such that

\[
\|u\|_{L^{2'}(\Omega)} + \|u\|_{L^{p_j}(\Omega)} \leq C_0 \|u\|_{0,\Omega,K_j} \forall u = (u_1, u_2) \in H, i, j = 1, 2.
\]

(95)

It follows from this and (25) that

\[
\langle u + T(u), u \rangle_H = \|u\|_H^2 - \int_{\Omega} f_1(x, u_2(x)) u_1(x) \, dx \\
- \int_{\Omega} f_2(x, u_1(x)) u_2(x) \, dx \geq \|u\|_H^2 \\
- \int_{\Omega} \left( \beta_1 |u_1|^{2p_j/2} |u_1|^{p_j} + |c_1| \right) \, dx \\
- \int_{\Omega} \left( \beta_2 |u_2|^{2p_j/2} |u_2|^{p_j} + |c_2| \right) \, dx \geq \|u\|_H^2 - \left( \beta_1 + \beta_2 \right) \\
\cdot \left( \|u_1\|_{L^{p_j}} + \|u_2\|_{L^{p_j}} \right) \\
+ \|u_1\|_{L^{p_j}}^2 + \|u_2\|_{L^{p_j}}^2 + \|u_2\|_{L^{p_j}}^2 \geq \|u\|_H^2 - \left( \beta_1 + \beta_2 \right) \\
\cdot \left( \|u_1\|_{L^{p_j}}^2 + \|u_2\|_{L^{p_j}}^2 \right)
\]

(96)

Here, \( C = (\beta_1 + \beta_2) \max\{C_0, C_0, C_0, C_0, C_0 \} \). This leads to (g).

\[ \square \]

Proof of Theorem 4. We replace the space \( X_0(\Omega, K) \) in the proof of Theorem 1 by \( H \). Since \( p_j < 2, \hat{p}_j < 2 \) for \( j = 1, 2 \), as in (54), we have \( R > 0 \) such that

\[
1 - C \left( R^{p_j-2} + \hat{p}_j^{p_j-2} + R^{p_j-2} + \hat{p}_j^{p_j-2} \right) \\
\geq \left( \|c_1\|_{L^{2'}} + \|c_2\|_{L^{2'}} \right) R^{-2} > \frac{1}{4}
\]

(97)

Then repeating the proof of Theorem 1, we get a \( u = (u_1, u_2) \in H \) such that \( \langle u + T(u), v \rangle_H = 0 \) for any \( v = (\phi, \psi) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega) \); namely, (22) holds.

\[ \square \]

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.
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