Dynamic Processes, Fixed Points, Endpoints, Asymmetric Structures, and Investigations Related to Caristi, Nadler, and Banach in Uniform Spaces

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1. Introduction

The concepts of the symmetric and asymmetric structures became established and investigated in mathematics and in theoretical computer science and are some creative ideas in fixed point theory by which some fascinating results have been achieved. In the proofs of these results, some deep methods based on those symmetric and asymmetric structures do play very important roles. The range of important applications of these results is enormous.

Let \((X, \mathcal{D})\) be a uniform space with uniformity defined by a saturated family \(\mathcal{D} = \{d_\alpha : \alpha \in \mathcal{A}\}\) of pseudometrics \(d_\alpha : X^2 \to [0; \infty)\), \(\alpha \in \mathcal{A}\), uniformly continuous on \(X^2\) (\(\mathcal{D}\)-family, for short); here \(\mathcal{A}\) is a nonempty index set.

It was discovered that the \(\mathcal{J}\)-families of generalized pseudodistances defined below generalize: metrics \(d\), distances of Tataru [1], \(\omega\)-distances of Kada et al. [2], \(\tau\)-distances of Suzuki [3], and \(\tau\)-functions of Lin and Du [4] in metric spaces \((X, d)\) and also \(\mathcal{D}\)-families of pseudometrics and distances of Vályi [5] in uniform spaces \((X, \mathcal{D})\).

Definition 1 (see [6]). Let \((X, \mathcal{D})\) be a Hausdorff uniform space.

(a) The family \(\mathcal{J} = \{J_\alpha : \alpha \in \mathcal{A}\}\) of maps \(J_\alpha : X^2 \to [0; \infty)\), \(\alpha \in \mathcal{A}\), is said to be a \(\mathcal{J}\)-family of generalized pseudodistances on \(X\) (\(\mathcal{J}\)-family, for short) if the following two conditions hold:

\[
(\mathcal{J}1) \forall \alpha \in \mathcal{A} \forall x, y, z \in X \{J_\alpha(x, z) \leq J_\alpha(x, y) + J_\alpha(y, z)\}.
\]

(\(\mathcal{J}2\)) For any sequences \((x_m : m \in \mathbb{N})\) and \((y_m : m \in \mathbb{N})\) in \(X\) such that

\[
\forall \alpha \in \mathcal{A} \forall x, y \in X \left\{ \lim_{m \to \infty} J_\alpha(x, y) \right\} = 0
\]

and

\[
\forall \alpha \in \mathcal{A} \forall x_m, y_m \in X \left\{ \lim_{m \to \infty} J_\alpha(x_m, y_m) = 0 \right\}
\]

holds.

(b) Define \(J_{(X, \mathcal{D})} = \{\mathcal{J} : \mathcal{J} = \{J_\alpha : X^2 \to [0; \infty) : \alpha \in \mathcal{A}\}\}

\[\text{is a } \mathcal{J}\text{-family on } X\].

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Let \((X, \mathcal{D})\) be a uniform space with uniformity defined by a saturated family \(\mathcal{D} = \{d_\alpha : \alpha \in \mathcal{A}\}\) of pseudometrics \(d_\alpha : X^2 \to [0; \infty)\), \(\alpha \in \mathcal{A}\), uniformly continuous on \(X^2\) (\(\mathcal{D}\)-family, for short); here \(\mathcal{A}\) is a nonempty index set.

It was discovered that the \(\mathcal{J}\)-families of generalized pseudodistances defined below generalize: metrics \(d\), distances of Tataru [1], \(\omega\)-distances of Kada et al. [2], \(\tau\)-distances of Suzuki [3], and \(\tau\)-functions of Lin and Du [4] in metric spaces \((X, d)\) and also \(\mathcal{D}\)-families of pseudometrics and distances of Vályi [5] in uniform spaces \((X, \mathcal{D})\).
Definition 2. Let $(X,d)$ be a metric space.

(a) Then $J = \{J : X \to [0; \infty)\}$ is said to be a $J$-family on $X$ if $J$ is a generalized pseudodistance on $X$; that is, the following two conditions hold:

1. $\forall x, y, z \in X \quad J(x, z) \leq J(x, y) + J(y, z)$.
2. For any sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in $X$ such that $\lim_{m \to \infty} \sup_{n \in \mathbb{N}} d(x_m, x_n) = 0$ and $\lim_{m \to \infty} d(x_m, y_m) = 0$, the following holds $\lim_{m \to \infty} d(x, y_m) = 0$.

(b) Define $J_{(X,d)} = \{J : \mathcal{J} = \{J : X \times X \to [0; \infty)\} \}$ is a $J$-family on $X$.

In the following remark, we list some basic properties of $J_{(X,d)}$.

Remark 3. Let $(X, \mathcal{D})$ be a Hausdorff uniform space.

(a) $\mathcal{D} = \{d_{\alpha} : \alpha \in \mathcal{A}\} \in J_{(X,d)}$ and $\mathcal{D} \not= \emptyset$.

(b) $(\mathcal{T}, \text{Remark 1.1})$ Let $J = \{J_{\alpha} : X \times X \to [0; \infty), \alpha \in \mathcal{A}\} \in J_{(X,d)}$. If $x \neq y, x, y \in X$, then $\exists_{\alpha \in \mathcal{A}} \{J_{\alpha}(x,y) > 0 \lor J_{\alpha}(y,x) > 0\}$.

(c) Let $J = \{J_{\alpha} : X \times X \to [0; \infty), \alpha \in \mathcal{A}\} \in J_{(X,d)}$. If $\forall_{\alpha \in \mathcal{A}} \forall_{x \in X} \{J_{\alpha}(x,x) = 0\}$, then, for each $\alpha \in \mathcal{A}$, $J_{\alpha}$ is quasisynodometric; examples of $J = \{J_{\alpha} : X \times X \to [0; \infty), \alpha \in \mathcal{A}\}$ such that the maps $J_{\alpha}, \alpha \in \mathcal{A}$, are not quasisynodometric are given in Section 4.

Definition 4. Let $(X, \mathcal{D})$ be a Hausdorff uniform space. $J = \{J_{\alpha} : X \times X \to [0; \infty), \alpha \in \mathcal{A}\} \in J_{(X,d)}$ is said to be admissible if $X^0_{J} \not= \emptyset$ where

$$X^0_{J} = \{x \in X : \forall_{\alpha \in \mathcal{A}} \{J_{\alpha}(x,x) = 0\}\}.$$

Remark 5. It is a remarkable fact that $\mathcal{D}$-family is admissible and $X^0_{\mathcal{D}} = X$. Indeed, we have that $X = X^0_{\mathcal{D}} \cup X_{\mathcal{D}}^0$, where $X_{\mathcal{D}}^0 = \{x \in X : \exists_{\alpha \in \mathcal{A}} \{J_{\alpha}(x,x) > 0\}\}$. Therefore, by Definition 4, we get the following $X_{\mathcal{D}}^0 = \{x \in X : \forall_{\alpha \in \mathcal{A}} \{d_{\alpha}(x,x) = 0\}\} = X$.

Let $2^X$ denote the family of all nonempty subsets of a space $X$. A set-valued dynamic system is defined as a pair $(X, T)$, where $X$ is a certain space and $T$ is a set-valued map $T : X \to 2^X$; in particular, a set-valued dynamic system includes the usual dynamic system $(X, T)$ where $T : X \to X$ is a single-valued map.

Let $(X, T)$ be a set-valued dynamic system. By Fix$(T)$ and End$(T)$ we denote the sets of all fixed points and endpoints of $T$, respectively; that is, Fix$(T) = \{w \in X : w \in T(w)\}$ and End$(T) = \{w \in X : \{w\} = T(w)\}$. A dynamic process or a trajectory starting at $w_0 \in X$ or a motion of the system $(X, T)$ at $w_0$ is a sequence $(w_m : m \in \mathbb{N})$ defined by $w_m \in T(w_{m-1})$. Moreover, $\exists_{\alpha} = \{w_\alpha : \alpha \in \mathcal{A}\}$ is a family of finite positive numbers.

A system $(X, T)$ is a set-valued dynamic system.

(A4) For each $x \in X, Q_{f,\Omega,\gamma,T}(x)$ is a set defined by $Q_{f,\Omega,\gamma,T}(x) = \{y \in \gamma(T(x) \cap X^0_{J} : \forall_{\alpha \in \mathcal{A}} \{w_\alpha(y) + \varepsilon_\alpha I_{\mathcal{A}}(x,y) \in \omega_\alpha(x)\}\}.$

(A5) For each $x \in X^0_{J}$, the set $Q_{f,\Omega,\gamma,T}(x)$ is nonempty.

(A6) For each $x \in X^0_{J}$, the set $Q_{f,\Omega,\gamma,T}(x)$ is a closed subset in $X$.

Then, there exists $w \in D_{\Omega} \cap X^0_{J}$ such that $w \in T(w)$ (i.e., $\forall_{\alpha \in \mathcal{A}} \{I_{\alpha}(w,w) = 0\}$ and $w \in \text{Fix}(T)$).
(II) (Endpoint theorem) Assume, in addition, that

\[(A7) \text{ for each } x \in X^0_f, \text{ each dynamic process } (w_m : m \in \{0\} \cup \mathbb{N}) \text{ starting at } w_0 = x \text{ and satisfying } v_{m\in\{0\} \cup \mathbb{N}} \{w_{m+1} \in T(w_m)\} \text{ satisfies } v_m \in Q_f, \Omega, \mathcal{Y}, T(w_m). \]

Then, there exists \( w \in D_\Omega \times X^0_f \) such that \( \{w\} = T(w) \) (i.e., \( v_{\alpha \in \mathcal{A}} \{I_{\omega}(w, w) = 0\} \) and \( w \in \text{End}(T) \)).

It is known that a weaker condition than continuity is lower semicontinuity.

**Definition 8.** Let \( (X, \mathcal{A}) \) be a Hausdorff sequentially complete uniform space. Let \( E \subseteq X, E \neq \emptyset \) and let \( f : E \to [0; \infty)\). The map \( f \) is lower semicontinuous on \( E \) with respect to \( X \) (written: \( f \) is \( (E, X) \)-lsc) when \( E \neq X \) and \( f \) is lsc when \( E = X \) if the set \( \{y \in E : f(y) \leq c\} \) is a closed subset in \( X \) for each \( c \in [0; \infty)\).

The following alternative characterizations of lower semicontinuity hold.

**Theorem 9.** Let \( (X, \mathcal{A}) \) be a Hausdorff sequentially complete uniform space. Let \( E \subseteq X, E \neq \emptyset \) and let \( f : E \to [0; \infty)\). The following conditions are equivalent.

\[ \text{(Z1) The map } f \text{ is lower semicontinuous on } E \text{ with respect to } X. \]
\[ \text{(Z2) For each } x_0 \in E, \]
\[ f(x_0) \leq \liminf_{x \to x_0, x \in X} f(x) ; \]
\[ \text{here} \]
\[ \liminf_{x \to x_0, x \in X} f(x) = \sup \{ \inf \{ f(x) : x \in E \cup \{ U \backslash \{ x_0 \} \} \} : \]
\[ U \text{ open in } X, \ x_0 \in U, \]
\[ E \cap \{ U \backslash \{ x_0 \} \} \neq \emptyset \} . \]
\[ \text{(Z3) The map } f \text{ is sequentially lower semicontinuous on } E \text{ with respect to } X; \text{ that is, for each } x_0 \in E, \]
\[ f(x_0) \leq \liminf_{m \to \infty} f(x_m) \]
\[ \text{for any sequence } (x_m : m \in \mathbb{N}) \text{ in } X \text{ such that } \forall_{\alpha \in \mathcal{A}} \{ \lim_{m \to \infty} \alpha I_{\alpha}(x_m, x_0) = 0 \}; \text{ here} \]
\[ \liminf_{m \to \infty} f(x_m) = \sup \{ \inf \{ f(x_m) : m \geq n \} : n \in \mathbb{N} \} . \]

**Remark 10** (see [6, Remark 4.6]). The following hold.

(a) A special case of condition \((A6)\) is a condition \((A6')\) defined by

\[ (A6') \text{ for each } (x, \alpha) \in X^0_f \times \mathcal{A}, \text{ the map } \omega_{\alpha}(\cdot) + \epsilon_{\alpha} I_{\alpha}(x, \cdot) : T(x) \cap X^0_f \to [0; \infty] \text{ is } (T(x) \cap X^0_f, X)\text{-lsc}. \]

(b) If \( \mathcal{F} = \emptyset \), then a special case of condition \((A6)\) is a condition \((A6'')\) defined by

\[ (A6'') \text{ for each } (x, \alpha) \in X \times \mathcal{A}, \text{ the map } \omega_{\alpha}(\cdot) + \epsilon_{\alpha} d_{\alpha}(x, \cdot) : T(x) \to [0; \infty] \text{ is } (T(x), X)\text{-lsc}. \]

(c) Theorem 7(I) essentially generalizes Theorem 6(I) even in metric spaces.

A classic result of Banach [16], from 1922, is the milestone in the history of fixed point theory and its applications.

**Theorem 11** (Banach [16]). Let \( (X, d) \) be a complete metric space. Assume that the single-valued dynamic system \((X, T)\) is \((d, \lambda)\)-contraction; that is,
\[ \exists x_0 \in X, \forall x, y \in X \quad d(T(x), T(y)) \leq \lambda d(x, y) \].
Then \( T \) has a unique fixed point \( w \) in \( X \) (i.e., \( T(w) = w \) and \( \text{Fix}(T) = \{ w \} \)) and, for each \( w_0 \in X \), the sequence \( \{w_m = T^m(w_0) : m \in \mathbb{N}\} \) satisfies \( \lim_{m \to \infty} \omega_{\alpha} d(w, w_m) = 0 \).

In a slightly different direction is the elegant result of Nadler on set-valued dynamic systems.

**Theorem 12** (Nadler [17, Theorem 5]). Let \( (X, d) \) be a complete metric space, let \( CB(X) \) denotes the class of all nonempty closed and bounded subsets of \( X \), and let \( H^d : (CB(X))^2 \to [0; \infty) \) be defined by
\[ \forall_{A, B \in CB(X)} H^d(A, B) \]
\[ = \max \left\{ \sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A) \right\} , \]
where \( \forall_{u \in X} \forall_{V \in CB(X)} \{ d(u, V) = \inf_{z \in V} d(u, z) \} \). Assume that the set-valued dynamic system \((X, T)\) satisfying \( T : X \to CB(X) \) is \((H^d, \lambda)\)-contraction; that is,
\[ \exists x_0 \in X, \forall x, y \in X \quad \{ H^d(T(x), T(y)) \leq \lambda d(x, y) \}. \]
Then \( \text{Fix}(T) \neq \emptyset \) (i.e., there exists \( w \in X \) such that \( w \in T(w) \)).

**Remark 13.** Let \((X, d)\) be a complete metric space.

(a) It is well known that Caristi’s fixed point theorem [12] yields Banach’s [16] and Nadler’s [17, Theorem 5] results.

(b) Maps \( T : (X, d) \to (X, d) \) satisfying (3) are not necessarily continuous.

(c) It is well known that \((CB(X), H^d)\) is a complete metric space and that the continuity of maps \( T : (X, d) \to (X, d) \) and \( T : (X, d) \to (CB(X), H^d) \) satisfying conditions (9) and (11) plays an important role in the proofs of Theorems 11 and 12, respectively.

Contractions (3) of Caristi, (9) of Banach, (11) of Nadler, and others are among the most important notions in fixed point theory, as well as in its numerous applications. As one will see from the literature, the known results about them
have been achieved by employing complicated machineries from various branches of mathematics and the answers for many basic problems about them are still missing. Moreover, examples show that these fundamental results are not optimal even in metric spaces.

The several authors have made essential progress in these problems and have solved many cases, and similar methods and ideas have since been applied in greater generality; see for example [1–67] and the references cited therein. However, the complete solutions of some key open problems are still missing.

In this paper we show that there are complementary approaches to generalize the Nadler and Banach statements concerning uniform, locally convex, and metric spaces. They involve mixed properties of asymmetric structures and fixed point theory. One of the key ideas in this paper is that in \((X, \mathcal{D})\) the families \(\mathcal{J} = \{J_\alpha : X \rightarrow [0; \infty), \alpha \in \mathcal{A}\} \in \mathcal{J}(X, \mathcal{D})\) construct the symmetric and asymmetric structures on \(X\) which generalize the symmetric structure determined by \(\mathcal{D} = \{d_\alpha : \alpha \in \mathcal{A}\}\) on \(X\) and then, by subtle techniques, we may use stated above Theorem 7.

More precisely, let \((X, \mathcal{D})\) be a Hausdorff uniform space. For \(\mathcal{J} = \{J_\alpha : X \rightarrow [0; \infty), \alpha \in \mathcal{A}\} \in \mathcal{J}(X, \mathcal{D})\) and \(\nu \in \{1, 2\}\), let the distance \(\mathcal{H}_\nu\) on \(X\) be defined as in Definitions 15 and 23, and let the distance \(\mathcal{B}_\nu\) on \(X\) be defined as in Definitions 29 and 33.

This paper has two aims.

(1) To determine \(\mathcal{J} = \{J_\alpha : X \rightarrow [0; \infty), \alpha \in \mathcal{A}\} \in \mathcal{J}(X, \mathcal{D})\), various classes of not necessarily continuous set-valued dynamic systems \((X, T)\) satisfying \(T : (X, \mathcal{J}) \rightarrow (\text{Cl}(X), \mathcal{H}_\nu), \nu \in \{1, 2\}\), and the conditions guaranteeing that the maps \(x \rightarrow \inf_{\alpha \in \mathcal{A}} J_\alpha(x, z), \alpha \in \mathcal{A}\), attains its global optimal minimum at a point \(w\) (not necessarily unique) satisfying \(\forall \alpha \in \mathcal{A} : J_\alpha(w, T(w)) = 0\) and \(w \in \text{Fix}(T)\) or \(w \in \text{End}(T)\).

(2) To determine \(\mathcal{J} = \{J_\alpha : X \rightarrow [0; \infty), \alpha \in \mathcal{A}\} \in \mathcal{J}(X, \mathcal{D})\), various classes of not necessarily continuous single-valued dynamic systems \((X, T)\) satisfying \(T : (X, \mathcal{J}) \rightarrow (X, \mathcal{B}_\nu), \nu \in \{1, 2\}\), and the conditions guaranteeing that the maps \(x \rightarrow J_\alpha(x, T(x)), \alpha \in \mathcal{A}\), attains its unique global optimal approximate minimum at \(w\) satisfying \(J_\alpha(w, T(w)) = 0, \alpha \in \mathcal{A}, T(w) = w\) and \(\forall \alpha \in \mathcal{A} : \lim_{m \to \infty} I_\alpha(w, w_m) = \lim_{m \to \infty} J_\alpha(w_m, w) = \sup_{\alpha \in \mathcal{A}} J_\alpha(w_0) = 0\), where \((w_m = \text{sup}(w_0) : m \in \mathbb{N})\) and \(w_0 \in X\) is arbitrary.

Remark 14. (a) The methods of this paper provide a way to compute the fixed point and endpoint theorems in uniform, locally convex and metric spaces with structures determined by \(\mathcal{J} = \{J_\alpha : X \rightarrow [0; \infty), \alpha \in \mathcal{A}\} \in \mathcal{J}(X, \mathcal{D})\).

(b) Theorems 17, 20, 21, 22, 25, 26, 27, 31, 34 and Examples 1–4 and 5–7 show that our fixed point and endpoint results are new in uniform and locally convex spaces and even in metric spaces, are different from fixed point and endpoint results given in the literature, and their proofs are simpler.

2. Fixed Point and Endpoint Theorems for Set-Valued Contractions (of Nadler Type) in Uniform and Metric Spaces

The following definitions will be much used in the sequel.

Definition 15. Let \((X, \mathcal{D})\) be a Hausdorff sequentially complete uniform space, assume that \(\mathcal{J} = \{J_\alpha : X \rightarrow [0; \infty), \alpha \in \mathcal{A}\} \in \mathcal{J}(X, \mathcal{D})\), let

\[
\forall \alpha \in \mathcal{A} : \forall x \in X : \forall v \in \text{Cl}(X) \\{ J_\alpha(u, v) = \inf \{ J_\alpha(u, z) : z \in v \} \},
\]

and let \(\nu \in \{1, 2\}\).

(a) Define on \(\text{Cl}(X)\) the distance \(\mathcal{H}_\nu : \text{Cl}(X) \rightarrow [0; \infty), \alpha \in \mathcal{A}\), as follows:

\[
\mathcal{H}_\nu(A, B) = \max \left\{ \sup_{u \in A} J_\alpha(u, B), \sup_{z \in B} J_\alpha(z, A) \right\}
\]

if \(\nu = 1\),

\[
\mathcal{H}_2(A, B) = \sup_{u \in A} J_\alpha(u, B)
\]

if \(\nu = 2\).

(b) Let a set-valued dynamic system \((X, T)\) satisfy \(T : X \rightarrow \text{Cl}(X)\). If \((X, T)\) satisfies

\[
\forall \alpha \in \mathcal{A} : \exists \lambda_\alpha \in [0; 1) : \forall x, y \in X \\{ J_\alpha(T(x), T(y)) \leq \lambda_\alpha J_\alpha(x, y) \},
\]

then we say that \((X, T)\) is a \((\mathcal{H}_\nu, \lambda_\alpha)\)-contraction on \(X\) for \(\lambda = \{\lambda_\alpha \in [0; 1) : \alpha \in \mathcal{A}\}\).

Remark 16. Each \((\mathcal{H}_1, \lambda_\alpha)\)-contraction on \(X\) is \((\mathcal{H}_2, \lambda_\alpha)\)-contraction on \(X\) but converse does not hold. One can prove the following characterizations of \((\mathcal{H}_1, \lambda_\alpha)\)-contractions \((X, T)\):

Theorem 17. Let \((X, \mathcal{D})\) be a Hausdorff sequentially complete uniform space, \(\mathcal{J} = \{J_\alpha : X \rightarrow [0; \infty), \alpha \in \mathcal{A}\} \in \mathcal{J}(X, \mathcal{D})\) and \(\nu \in \{1, 2\}\). Suppose also the following.

(I) A set-valued dynamic system \((X, T)\) satisfies \(T : X \rightarrow \text{Cl}(X)\).

(II) There exists a family \(\lambda = \{\lambda_\alpha \in [0; 1) : \alpha \in \mathcal{A}\}\) such that \((X, T)\) is a \((\mathcal{H}_\nu, \lambda_\alpha)\)-contraction on \(X\).

(III) The family \(\Gamma = \{\gamma_\alpha \in [0; 1) : \alpha \in \mathcal{A}\}\) satisfies

\[
\forall \alpha \in \mathcal{A} : \gamma_\alpha < \lambda_\alpha.
\]

The following hold.

(B1) \(\forall \alpha \in \mathcal{A} : \forall x \in X \\{ y \in T(x) : J_\alpha(y, T(y)) \leq \lambda_\alpha J_\alpha(x, y) = T(x) \} \)

(B2) \(\forall \alpha \in \mathcal{A} : \forall x \in X \\{ U_{\Gamma, \alpha}(x) \neq \emptyset \} \)

\[
U_{\Gamma, \alpha}(x) = \{ y \in T(x) : J_\alpha(x, y) \leq J_\alpha(x, T(x)) \},
\]

\(\alpha \in \mathcal{A}, \ x \in X.\)
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(B3) \( \forall \alpha \in \mathcal{A} \forall x \in X \{ \Omega_{\Gamma, \alpha}(x) \subset V_{\Gamma, \alpha}(x) \} \) where
\[
V_{\Gamma, \alpha}(x) = \{ y \in T(x) : I_{\alpha}(y, T(y)) + (\gamma_{\alpha} - \lambda_{\alpha}) I_{\alpha}(x, y) \leq I_{\alpha}(x, T(x)) \}, \quad \alpha \in \mathcal{A}, \ x \in X.
\] (16)

(B4) \( \forall \alpha \in \mathcal{A} \forall x \in X \forall y \in T(x) \{ 0 \leq I_{\alpha}(x, T(x)) - I_{\alpha}(y, T(y)) \leq (1 + \lambda_{\alpha}) I_{\alpha}(x, y) \}. \)

(B5) \( \forall \alpha \in \mathcal{A} \forall x \in X \forall y \in T(x) \{ J_{\alpha}(y, T(y)) + I_{\alpha}(x, y) \geq I_{\alpha}(x, T(x)) \}. \)

Proof. Let the family \( \Omega_{\alpha} = \{ \omega_{\alpha} : X \to [0, \infty) \mid \alpha \in \mathcal{A} \} \) be defined by
\[
\forall \alpha \in \mathcal{A} \forall x \in X \left\{ \omega_{\alpha}(x) = I_{\alpha}(x, T(x)) \right\}.
\] (17)

Proof of (B1). By assumption (II) and Definitions 15(a) and 15(b),
\[
\forall \alpha \in \mathcal{A} \forall x, y \in X \left\{ \sup_{u \in T(x)} I_{\alpha}(u, T(y)) \leq \max \left\{ \sup_{z \in T(x)} I_{\alpha}(z, T(y)) \right\} \right\} \text{ if } \nu = 1,
\]
\[
\forall \alpha \in \mathcal{A} \forall x, y \in X \left\{ \sup_{u \in T(x)} I_{\alpha}(u, T(y)) \leq \lambda_{\alpha} I_{\alpha}(x, y) \right\} \text{ if } \nu = 2.
\] (18)

Using this, we may thus conclude that
\[
\forall \alpha \in \mathcal{A} \forall x \in X \forall y \in T(x) \left\{ \omega_{\alpha}(y) \leq \sup_{u \in T(x)} I_{\alpha}(u, T(y)) \right\} \leq \lambda_{\alpha} I_{\alpha}(x, y) \right\}, \quad \alpha \in \mathcal{A}, \ x \in X.
\] (19)

and hence
\[
\forall \alpha \in \mathcal{A} \forall x \in X \left\{ T(x) \subset \{ y \in T(x) : \omega_{\alpha}(y) \leq \lambda_{\alpha} I_{\alpha}(x, y) \} \right\}.
\] (20)

On the other hand it is clear that
\[
\forall \alpha \in \mathcal{A} \forall x \in X \left\{ \{ y \in T(x) : \omega_{\alpha}(y) \leq \lambda_{\alpha} I_{\alpha}(x, y) \} \subset T(x) \right\}.
\] (21)

By applying (20) and (21), we obtain (B1).

Proof of (B2). By (12), we have
\[
\forall \alpha \in \mathcal{A} \forall x \in X \left\{ \omega_{\alpha}(x) = \inf_{y \in T(x)} I_{\alpha}(x, y) \right\}.
\] (22)

Further, by assumption (III), \( \forall \alpha \in \mathcal{A} \{ \gamma_{\alpha} \in (0; 1) \} \). Hence, for arbitrary and fixed \( \alpha \in \mathcal{A} \) and \( x \in X \), by (22) and definition of infimum, we obtain that
\[
\exists y_{\alpha} \in T(x) \left\{ y_{\alpha} I_{\alpha}(x, y_{\alpha}) \leq \inf_{y \in T(x)} I_{\alpha}(x, y) = \omega_{\alpha}(x) \right\}.
\] (23)

Consequently,
\[
\forall \alpha \in \mathcal{A} \forall x \in X \forall y_{\alpha} \in T(x) \{ y_{\alpha} \in U_{\Gamma, \alpha}(x) \}.
\] (24)

So we have proved (B2).

Proof of (B3). Let \( \alpha \in \mathcal{A} \), \( x \in X \), and \( y_{\alpha} \in U_{\Gamma, \alpha}(x) \) be arbitrary and fixed. Then, by (B2), we have \( y_{\alpha} \in T(x) \) and
\[
\gamma_{\alpha} I_{\alpha}(x, y_{\alpha}) \leq \omega_{\alpha}(x).
\] (25)

Clearly, by (B1), property \( y_{\alpha} \in T(x) \) implies \( \omega_{\alpha}(y_{\alpha}) \leq \lambda_{\alpha} I_{\alpha}(x, y_{\alpha}) \). Thus
\[
-\lambda_{\alpha} I_{\alpha}(x, y_{\alpha}) \leq -\omega_{\alpha}(y_{\alpha}).
\] (26)

Using (25) and (26) we obtain
\[
(\gamma_{\alpha} - \lambda_{\alpha}) I_{\alpha}(x, y_{\alpha}) \leq \omega_{\alpha}(x) - \omega_{\alpha}(y_{\alpha}).
\] (27)

We proved that
\[
\forall \alpha \in \mathcal{A} \forall x \in X \left\{ U_{\Gamma, \alpha}(x) \subset \{ y \in T(x) : \gamma_{\alpha} I_{\alpha}(x, y) \leq \omega_{\alpha}(x) \} \right\}.
\] (28)

Therefore, (B3) holds.

Proof of (B4). Let \( \alpha \in \mathcal{A} \), \( x \in X \), and \( y \in T(x) \) be arbitrary and fixed. Then, by (B1), since \( y \in T(x) \), we obtain \( \omega_{\alpha}(y) \leq \lambda_{\alpha} I_{\alpha}(x, y) \). This and (22) imply
\[
-\omega_{\alpha}(y) \geq -\lambda_{\alpha} I_{\alpha}(x, y) \geq -\lambda_{\alpha} \omega_{\alpha}(x) \geq -\omega_{\alpha}(x).
\] (29)

Therefore,
\[
\forall \alpha \in \mathcal{A} \forall x \in X \forall y \in T(x) \{ 0 \leq \omega_{\alpha}(x) - \omega_{\alpha}(y) \}
\] (30)
holds. Next, it follows from (22) and (B1) that
\[
\forall \alpha \in \mathcal{A} \forall x \in X \forall y \in T(x) \{ \omega_{\alpha}(x) - \omega_{\alpha}(y) \leq \omega_{\alpha}(x) + \omega_{\alpha}(y) \}
\] (31)
\leq (1 + \lambda_{\alpha}) I_{\alpha}(x, y) \right\}.

This shows that (B4) holds.

Proof of (B5). By (30) and (22),
\[
\forall \alpha \in \mathcal{A} \forall x \in X \forall y \in T(x) \{ \omega_{\alpha}(x) - \omega_{\alpha}(y) \leq \omega_{\alpha}(x) \leq I_{\alpha}(x, y) \}.
\] (32)

Therefore, (B5) holds.

□
Definition 18. Let \((X, \mathcal{D})\) be a Hausdorff sequentially complete uniform space and let \(\mathcal{J} = \{\lambda_\alpha : X^2 \to [0, \infty), \alpha \in \mathcal{A}\} \subseteq J_{(X, \mathcal{D})}\). We say that the family \(\mathcal{J}\) is continuous in \(X\) if, for each \(x_0 \in X\) and for each sequence \((x_m : m \in \mathbb{N})\) in \(X\) such that

\[
\forall \alpha \in \mathcal{A} \delta \left\{ \lim_{m \to \infty} d_\alpha(x_m, x_0) = 0 \right\},
\]

we have

\[
\forall \alpha \in \mathcal{A} \delta \left\{ \lim \inf_{m \to \infty} d_\alpha(x_m, x_0) = \lim \inf_{m \to \infty} d_\alpha(x_0, x_m) = 0 \right\}.
\]

Remark 19. The family \(\mathcal{D}\) is continuous in \(X\).

Assertion (B5) says that, for each \(x \in X\), the set

\[ Q_{\mathcal{J}, \Gamma}(x) = \{ y \in T(x) : \forall \alpha \in \mathcal{A} \delta \{ I_{\alpha}(y, T(y)) \}
+ I_{\alpha}(x, y) \geq I_{\alpha}(x, T(x)) \} \]

has the property

\[
\emptyset \neq Q_{\mathcal{J}, \Gamma}(x) = T(x) \in \text{Cl}(X).
\]

Let \(Y = \{\varepsilon_\alpha, \alpha \in \mathcal{A}\}\) be a family of positive numbers satisfying

\[
\forall \alpha \in \mathcal{A} \delta \{ \varepsilon_\alpha \in (0, 1) \}
\]

and, for each \(x \in X\), let the set \(Q_{\mathcal{J}, \Gamma, \Gamma'}(x)\) be defined by

\[
Q_{\mathcal{J}, \Gamma, \Gamma'}(x) = \{ y \in T(x) \cap X_{\mathcal{J}}^0 : \forall \alpha \in \mathcal{A} \delta \{ I_{\alpha}(y, T(y))
+ \varepsilon_\alpha I_{\alpha}(x, y) \leq I_{\alpha}(x, T(x)) \} \}.
\]

Now, for \((\mathcal{J}, \Lambda)\)-contractions \((X, T)\), we can give the following characterizations of the sets \(Q_{\mathcal{J}, \Gamma, \Gamma'}(x), x \in X\), defined in (37).

Theorem 20. Let \((X, \mathcal{D})\) be a Hausdorff sequentially complete uniform space, \(\mathcal{J} = \{\lambda_\alpha : X^2 \to [0, \infty), \alpha \in \mathcal{A}\} \subseteq J_{(X, \mathcal{D})}\) and \(v \in \{1, 2\}\). Suppose also the following.

(I) \(\mathcal{J}\) is admissible.

(II) A set-valued dynamic system \((X, T)\) satisfies \(T : X \to \text{Cl}(X)\).

(III) There exists a family \(\Lambda = \{\lambda_\alpha \in [0, 1), \alpha \in \mathcal{A}\}\) such that \((\mathcal{J}, \Lambda)\) is a \((\mathcal{J}, \Lambda)\)-contraction on \(X\).

(IV) For each family \(\Gamma = \{\gamma_\alpha \in (0, 1), \alpha \in \mathcal{A}\}\) satisfying \(\forall \alpha \in \mathcal{A} \delta \{ \lambda_\alpha < \gamma_\alpha \}\) and for each \(x \in X\), let the set \(Q_{\mathcal{J}, \Gamma, \Gamma'}(x)\) be defined by

\[
Q_{\mathcal{J}, \Gamma, \Gamma'}(x) = \{ y \in T(x) \cap X_{\mathcal{J}}^0 : \forall \alpha \in \mathcal{A} \delta \{ I_{\alpha}(y, T(y))
+ (\gamma_\alpha - \lambda_\alpha) I_{\alpha}(x, y)
\leq I_{\alpha}(x, T(x)) \} \}.
\]

The following hold.

(C1) If there exists a family \(\Gamma^0 = \{\gamma_\alpha^0 \in (0, 1), \alpha \in \mathcal{A}\}\) satisfying \(\forall \alpha \in \mathcal{A} \delta \{ \lambda_\alpha < \gamma_\alpha^0 \}\) and such that

\[
\forall x \in X_{\mathcal{J}}^0 \delta \{ y \in T(x) \cap X_{\mathcal{J}}^0 : \forall \alpha \in \mathcal{A} \delta \{ I_{\alpha}(y, T(y))
\leq I_{\alpha}(x, T(x)) \} \}
\]

then \(\forall x \in X_{\mathcal{J}}^0 \delta \{ Q_{\mathcal{J}, \Gamma^0, \Gamma'}(x) \neq \emptyset \} \).

(C2) If there exists a family \(\Gamma^0 = \{\gamma_\alpha^0 \in (0, 1), \alpha \in \mathcal{A}\}\) satisfying \(\forall \alpha \in \mathcal{A} \delta \{ \lambda_\alpha < \gamma_\alpha^0 \}\) and such that, for each \((x, \alpha) \in X_{\mathcal{J}}^0 \times \mathcal{A}\), the map

\[
I_{\alpha}(x, \cdot) + (\gamma_\alpha^0 - \lambda_\alpha) I_{\alpha}(x, \cdot) : T(x) \cap X_{\mathcal{J}}^0 \to [0, \infty)
\]

is \((T(x) \cap X_{\mathcal{J}}^0, X, \Gamma')\)-Iscc, then, for each \(x \in X_{\mathcal{J}}^0\),

\(Q_{\mathcal{J}, \Gamma^0, \Gamma'}(x)\) is a closed subset in \(X\).

(C3) Let the family \(\mathcal{J}\) be continuous in \(X\). Then, for each family \(\Gamma = \{\gamma_\alpha \in (0, 1), \alpha \in \mathcal{A}\}\) satisfying \(\forall \alpha \in \mathcal{A} \delta \{ \lambda_\alpha < \gamma_\alpha \}\) and for each \(x \in X_{\mathcal{J}}^0\), \(Q_{\mathcal{J}, \Gamma, \Gamma'}(x)\) is a closed subset in \(X\).

(C4) Let \(\mathcal{J} = \mathcal{D}\). If there exists a family \(\Gamma^0 = \{\gamma_\alpha^0 \in (0, 1), \alpha \in \mathcal{A}\}\) satisfying \(\forall \alpha \in \mathcal{A} \delta \{ \lambda_\alpha < \gamma_\alpha^0 \}\) and such that

\[
\forall x \in X \delta \{ y \in T(x) : \forall \alpha \in \mathcal{A} \delta \{ I_{\alpha}^0(y, x, T(x)) \leq I_{\alpha}(x, T(x)) \} \}
\]

then \(\forall x \in X \delta \{ Q_{\mathcal{D}, \Gamma^0, \Gamma'}(x) \neq \emptyset \} \).

(C5) Let \(\mathcal{J} = \mathcal{D}\). Then, for each family \(\Gamma = \{\gamma_\alpha \in (0, 1), \alpha \in \mathcal{A}\}\) satisfying \(\forall \alpha \in \mathcal{A} \delta \{ \lambda_\alpha < \gamma_\alpha \}\) and for each \(x \in X\), \(Q_{\mathcal{D}, \Gamma, \Gamma'}(x)\) is a closed subset in \(X\).

Proof. Let the family \(\Omega^0 = \{\omega_\alpha^0 : X \to [0, \infty), \alpha \in \mathcal{A}\}\) be defined by

\[
\forall x \in X \delta \{ \omega_\alpha^0(x) = I_{\alpha}(x, T(x)) \}.
\]

Proof of (C1). Denote

\[
\forall x \in X \delta \{ U_{\mathcal{J}, \Gamma'}(x) = \bigcup_{\alpha \in \mathcal{A} \delta} U_{\mathcal{J}, \Gamma'}(x) \}
\]

\[
\forall x \in X \delta \{ V_{\mathcal{J}, \Gamma'}(x) = \bigcup_{\alpha \in \mathcal{A} \delta} V_{\mathcal{J}, \Gamma'}(x) \}
\]

Then, by (B2), (B3), and (IV),

\[
\forall x \in X \delta \{ U_{\mathcal{J}, \Gamma'}(x) \cap X_{\mathcal{J}}^0 \subseteq V_{\mathcal{J}, \Gamma'}(x) \cap X_{\mathcal{J}}^0 \}
\]

\[
Q_{\mathcal{J}, \Gamma', \Gamma'}(x) \}.
\]
Hence, we conclude that, for each $x \in X^0_J$, the set $Q_{J,\Gamma-\Lambda,T}(x)$ is nonempty whenever $\forall x \in X^0_J \cup U_{\text{tr}_x}(x) \cap X^0_J \neq \emptyset$.

**Proof of (C2).** The assertion follows immediately from Remark 10(a).

**Proof of (C3).** The assertion also follows from Remark 10(a).

Indeed, let $x \in X$ be arbitrary and fixed and let a sequence $(x_m : m \in \mathbb{N})$ in $X$ be convergent to $x_0$; that is, let

$$\forall_{x \in A} \lim_{m \to \infty} d(x_0, x_m) = 0.$$ 

If $m \in \mathbb{N}$, $z \in T(x_m)$ and $\alpha \in A$ are arbitrary and fixed, then, by $(\mathcal{F}1)$,

$$\forall_{x \in X} \left\{ \omega^\alpha_J(x_0) = I_a(x_0, T(x_0)) \leq I_a(x_0, u) \leq I_a(x_0, x_m) + I_a(x_m, z) + I_a(z, u) \right\}.$$ 

(45)

This gives

$$\omega^\alpha_J(x_0) \leq I_a(x_0, x_m) + I_a(x_m, z) + I_a(z, T(x_0)).$$ 

(46)

Hence

$$\omega^\alpha_J(x_0) \leq I_a(x_0, x_m) + I_a(x_m, z) + \sup_{u \in T(x_m)} I_a(u, T(x_0)).$$ 

(47)

Furthermore, this holds for each $z \in T(x_m)$ and, thus, by (12),

$$\omega^\alpha_J(x_0) \leq I_a(x_0, x_m) + \omega^\alpha_J(x_m) + \sup_{u \in T(x_m)} I_a(u, T(x_0)).$$ 

(48)

However, $(X, T)$ is $(\mathcal{P}_T, \Lambda)$-contraction on $X$. Therefore,

$$\sup_{u \in T(x_m)} I_a(u, T(x_0)) \leq \max \left\{ \sup_{u \in T(x_m)} I_a(u, T(x_0)), \sup_{z \in T(x_m)} I_a(z, T(x_m)) \right\}$$

$$\leq \lambda_a I_a(x_m, x_0) \text{ if } v = 1,$$

$$\sup_{u \in T(x_m)} I_a(u, T(x_0)) \leq \lambda_a I_a(x_m, x_0) \text{ if } v = 2.$$ 

(49)

Consequently, we obtain that

$$\omega^\alpha_J(x_0) \leq I_a(x_0, x_m) + \omega^\alpha_J(x_m) + \sup_{u \in T(x_m)} I_a(u, T(x_0))$$

$$\leq I_a(x_0, x_m) + \omega^\alpha_J(x_m) + \lambda_a I_a(x_m, x_0).$$ 

(50)

Since the family $\mathcal{F}$ is continuous, this implies

$$\omega^\alpha_J(x_0) \leq \liminf_{m \to \infty} \omega^\alpha_J(x_m).$$ 

(51)

Therefore, for each $\alpha \in A$, $\omega^\alpha_J(\cdot)$ is lsc in $X$.

Moreover, if $m \in \mathbb{N}$, $x \in X$, and $\alpha \in A$ are arbitrary and fixed, then, by $(\mathcal{F}1)$,

$$J_a(x, x_0) \leq J_a(x, x_m) + J_a(x_m, x_0).$$ 

(52)

Since $\mathcal{F}$ is continuous, this gives

$$J_a(x, x_0) \leq \liminf_{m \to \infty} J_a(x, x_m).$$ 

(53)

that is, for each $(x, \alpha) \in X \times A$, the map $J_a(x, \cdot)$ is lsc in $X$.

Using these two facts, in particular, we have that, for each $(x, \alpha) \in X^0_J \times A$, the map

$$\omega^\alpha_J(\cdot) + (\gamma_a - \lambda_a) J_a(x, \cdot) : T(x) \cap X^0_J \to [0; \infty]$$ 

(54)

is $(T(x) \cap X^0_J)$-lsc; that is, $(A6')$ holds.

**Proof of (C4).** This follows from (C1).

**Proof of (C5).** This follows from (C3) and Remarks 3(a) and 19.

We use notations and auxiliary Theorems 17 and 20 above in proving the following basic fixed point and endpoint theorem for set-valued contractions with respect to $J \in J_{X, A_D}$ (of Nadler-type) in uniform spaces $(X, D)$.

**Theorem 21.** Let $(X, D)$ be a Hausdorff sequentially complete uniform space, $\mathcal{F} = \{ J_a : X^2 \to [0; \infty), \alpha \in A \}$ satisfying $\mathcal{F} \in J_{X, A_D}$ of uniform spaces $(X, D)$ and $v \in \{1, 2\}$. Suppose also the following.

(I) $\mathcal{F}$ is admissible.

(II) A set-valued dynamic system $(X, T)$ satisfies $T : X \to \text{Cl}(X)$.

(III) There exists a family $\Lambda = \{ \lambda_a \in [0; 1), \alpha \in A \}$ such that $(X, T)$ is a $(\mathcal{P}_T, \Lambda)$-contraction on $X$.

(IV) For each family $\Gamma = \{ \gamma_a \in (0; 1), \alpha \in A \}$ satisfying $\forall_{a \in A} (\lambda_a < \gamma_a)$ and for each $x \in X$, let the set $Q_{J,\Gamma-\Lambda,T}(x)$ be defined by

$$Q_{J,\Gamma-\Lambda,T}(x) = \{ y \in T(x) \cap X^0_J : \forall_{a \in A} (|J_a(y, T(x)) + (\gamma_a - \lambda_a) J_a(x, y) \leq J_a(x, T(x))) \}.$$ 

(55)

(V) There exists a family $\Gamma^0 = \{ \gamma_a^0 \in (0; 1), \alpha \in A \}$ satisfying $\forall_{a \in A} (\lambda_a < \gamma_a^0)$ and such that, for each $x \in X^0_J$, $Q_{J,\Gamma^0-\Lambda,T}(x)$ is a nonempty closed subset in $X$.

**The following hold**

(El) (Fixed point theorem) $\text{Fix}(T) \neq \emptyset$ and there exists $w \in \text{Fix}(T)$ satisfying $\forall_{a \in A} (|J_a(w, w)| = 0)$.
(E2) (Endpoint theorem) If, for each \( x \in X \), each dynamic process \( (w_m : m \in [0] \cup N) \) starting at \( w_0 = x \), then \( \forall x \in X \), \( \exists \lambda \in [0;1) \), \( \forall x, y \in X \) \( H_J \left( T(x), T(y) \right) \leq \lambda J(x, y) \), then \( \text{End} (T) \neq \emptyset \).

Proof. The proof will be broken into five steps.

Step 1. Let the family \( \Omega^\mathcal{F} \) be defined by
\[
\forall \alpha \in \mathcal{F}, \forall x \in X \quad \left\{ \alpha^\mathcal{F}_\alpha(x) = J_\alpha(x, T(x)) \right\}.
\]

Step 2. The assumptions (A5) and (A6) of Theorem 7 hold if \( Y = \{ \alpha \} \), \( \alpha \in \mathcal{F} \) and \( \Omega^\mathcal{F} \) is defined in Step 1.

Step 3. There exists \( w \in X \) such that \( w = T(w) \).

Step 4. We now observe that \( \forall \alpha \in \text{End}(T) \), \( \forall x \in X \), \( \exists \lambda \in [0,1) \).

Step 5. The assertions hold.

As a corollary of the above Theorems 17, 20, and 21 we have the following new fixed point and endpoint theorem for set-valued contractions with respect to \( \mathcal{D} \)-families of Nadler-type in uniform spaces \( (X, \mathcal{D}) \).

**Theorem 22.** Let \( (X, \mathcal{D}) \) be a Hausdorff sequentially complete uniform space and let \( \nu \in \{1,2\} \). Suppose also the following.

(I) A set-valued dynamic system \( (X, T) \) satisfies \( T: X \to \text{Cl} (X) \).

(II) There exists a family \( \Lambda = \{ \lambda \} \), \( \alpha \in \mathcal{F} \) such that \( T(x, T(x)) \) is a \( \mathcal{H}_\nu^\mathcal{F}, \lambda \)-contraction on \( X \).

(III) For each family \( \Gamma = \{ \gamma \} \), \( \alpha \in \mathcal{F} \) satisfying \( \forall \alpha \in \mathcal{F}, \exists \gamma \in \mathcal{F} \) such that \( \forall x \in X \), \( \text{set} \ Q_{\mathcal{D}, \mathcal{G}, \mathcal{A}, \mathcal{D}}(x) \) be defined by
\[
\forall \alpha \in \mathcal{F}, \|d_\alpha (y, T(\gamma)) + (\gamma \lambda) - d_\alpha (x, y)\| \leq \lambda d_\alpha (x, T(\gamma)) \\text{for each } x \in X.
\]

The following hold.

(F1) \( (\text{Closedness property}) \) For each \( x \in X \), \( Q_{\mathcal{D}, \mathcal{G}, \mathcal{A}, \mathcal{D}}(x) \) is a closed subset in \( X \).

(F2) \( (\text{Fixed point theorem}) \) \( \text{Fix}(T) \neq \emptyset \).

(F3) \( (\text{Endpoint theorem}) \) If, for each \( x \in X \), each dynamic process \( (w_m : m \in [0] \cup N) \) starting at \( w_0 = x \) and satisfying \( \forall m \in [0] \cup N \), \( \|w_{m+1} - T(w_m)\| \leq \lambda_{m+1} \), then \( \text{End} (T) \neq \emptyset \).

We now state consequences of the above in metric spaces.

**Definition 23.** Let \( (X, d) \) be a complete metric space, let \( \mathcal{F} = \{ J : X \to [0, \infty) \} \) and let \( \nu \in \{1,2\} \).

(a) Let
\[
\forall \mathcal{F} \in \text{End}(T) \forall \mathcal{F} \in \text{Cl}(X) \quad \left( J(u, V) = \inf \{ J(u, z) : z \in V \} \right).
\]

(b) Let a set-valued dynamic system \( (X, T) \) satisfy \( T: X \to \text{Cl} (X) \). If \( (X, T) \) satisfies
\[
\exists \lambda \in [0,1) \forall x \in X \quad H_T (T(x), T(y)) \leq \lambda J(x, y),
\]
then we say that \( (X, T) \) is a \( \mathcal{H}_\nu^\mathcal{F}, \lambda \)-contraction on \( X \).
(c) Let $X_f^0 = \{x \in X : f(x, x) = 0\}$. $F$ is said to be admissible if $X_f^0 \neq \emptyset$.

(d) We say that $F$ is continuous in $X$ if, for each $x_0 \in X$ and for each sequence $(x_m : m \in \mathbb{N})$ in $X$ such that $\lim_{m \to \infty} d(x_m, x_0) = 0$, we have
\[
\lim_{m \to \infty} f(x_m, x_0) = \lim_{m \to \infty} f(x_0, x_m) = 0. \tag{64}
\]

Remark 24. Let $\mathcal{D} = \{d : X^2 \to [0; \infty)\}$. It is clear that $\mathcal{D}$ is $F$-family; $\mathcal{D}$ is admissible; $\mathcal{D}$ is continuous; and $X_{\mathcal{D}}^0 = X$.

As corollaries from Theorems 17, 20, and 21 and their proofs we get the following three theorems concerning contractions with respect to $F \in \mathcal{J}(X, d)$.

Theorem 25. Let $(X, d)$ be a complete metric space, $F = \{f : X^2 \to [0; \infty)\} \in \mathcal{J}(X, d)$, and $\nu \in \{1, 2\}$. Suppose also the following.

(I) $F$ is admissible.
(II) A set-valued dynamic system $(X, T)$ satisfies $T : X \to \text{Cl}(X)$.
(III) There exists $\lambda \in (0, 1)$ such that $(X, T)$ is a $(H_f^T, \lambda)$-contraction on $X$.
(IV) For each $y \in (0, 1)$ satisfying $\lambda < \gamma$ and for each $x \in X$ let the set $Q_{X_f^\gamma, \lambda, T}(x)$ be defined by
\[
Q_{X_f^\gamma, \lambda, T}(x) = \left\{y \in T(x) \cap X_f^0 : f(x, y) + (\gamma - \lambda) f(x, T(x)) \leq f(x, T(x))\right\}. \tag{65}
\]

The following hold.

(G1) If there exists $y^0 \in (0, 1)$ satisfying $\lambda < \gamma^0$ and such that
\[
\forall x \in X_f^0 \quad \left\{\left\{y \in T(x) \cap X_f^0 : f(x, y) \leq f(x, T(x))\right\} \neq \emptyset\right\}, \tag{66}
\]
then $\forall x \in X_f^0 \quad [Q_{X_f^{\gamma^0, \lambda, T}}(x) \neq \emptyset]$.

(G2) If there exists $y^0 \in (0, 1)$ satisfying $\lambda < \gamma^0$ and such that, for each $x \in X_f^0$, the map
\[
J(\cdot, T(\cdot)) + (\gamma^0 - \lambda) J(x, \cdot) : T(x) \cap X_f^0 \to [0; \infty) \tag{67}
\]
is $(T(x) \cap X_f^0, X, \text{ls})$-lsC, then, for each $x \in X_f^0$, $Q_{X_f^{\gamma^0, \lambda, T}}(x)$ is a closed subset in $X$.

(G3) Let $F$ be continuous in $X$. Then, for each $y \in (0, 1)$ satisfying $\lambda < \gamma$ and for each $x \in X_f^0$, $Q_{X_f^\gamma, \lambda, T}(x)$ is a closed subset in $X$.

Theorem 26. Let $(X, d)$ be a complete metric space, $F = \{f : X^2 \to [0; \infty)\} \in \mathcal{J}(X, d)$, and $\nu \in \{1, 2\}$. Suppose also the following.

(I) $F$ is admissible.
(II) A set-valued dynamic system $(X, T)$ satisfies $T : X \to \text{Cl}(X)$.
(III) There exists $\lambda \in (0, 1)$ such that $(X, T)$ is a $(H_f^T, \lambda)$-contraction on $X$.
(IV) For each $y \in (0, 1)$ satisfying $\lambda < \gamma$ and for each $x \in X$ let the set $Q_{X_f^\gamma, \lambda, T}(x)$ be defined by
\[
Q_{X_f^\gamma, \lambda, T}(x) = \left\{y \in T(x) \cap X_f^\lambda : f(y, T(y)) + (\gamma - \lambda) f(x, T(x)) \leq f(x, T(x))\right\}. \tag{68}
\]

(V) There exists $y^0 \in (0, 1)$ satisfying $\lambda < \gamma^0$ such that, for each $x \in X_f^0$, $Q_{X_f^{\gamma^0, \lambda, T}}(x)$ is a nonempty closed subset in $X$.

The following hold.

(K1) (Fixed point theorem) $\text{Fix}(T) \neq \emptyset$ and there exists $w \in \text{Fix}(T)$ such that $f(w, w) = 0$.

(K2) (Endpoint theorem) If, for each $x \in X_f^0$, each dynamic process $(w_m : m \in \mathbb{N})$ starting at $w_0 = x$ and satisfying $\forall m \in \mathbb{N} w_{m+1} \in T(w_m)$ satisfies $\forall m \in \mathbb{N} \{w_{m+1} \in Q_{X_f^{\gamma^0, \lambda, T}}(w_m)\}$, then $\text{End}(T) \neq \emptyset$.

Theorem 27. Let $(X, d)$ be a complete metric space, $\mathcal{D} = \{d : X^2 \to [0; \infty)\}$, and $\nu \in \{1, 2\}$. Suppose also the following.

(I) A set-valued dynamic system $(X, T)$ satisfies $T : X \to \text{Cl}(X)$.
(II) There exists $\lambda \in (0, 1)$ such that $(X, T)$ is a $(H_f^T, \lambda)$-contraction on $X$.
(III) For each $y \in (0, 1)$ satisfying $\lambda < \gamma$ and for each $x \in X$ let the set $Q_{X_f^{\gamma, \lambda, T}}(x)$ be defined by
\[
Q_{X_f^{\gamma, \lambda, T}}(x) = \left\{y \in T(x) : d(y, T(y)) + (\gamma - \lambda) d(x, T(x)) \leq d(x, T(x))\right\}. \tag{69}
\]

The following hold.

(L1) (Nonemptiness and closedness property) For each $y \in (0, 1)$ satisfying $\lambda < \gamma$ and for each $x \in X$, $Q_{X_f^{\gamma, \lambda, T}}(x)$ is a nonempty closed subset in $X$.

(L2) (Fixed point theorem) $\text{Fix}(T) \neq \emptyset$.

(L3) (Endpoint theorem) If there exists $y^0 \in (0, 1)$ satisfying $\lambda < \gamma^0$ and such that, for each $x \in X$, each dynamic process $(w_m : m \in \mathbb{N})$ starting at $w_0 = x$ and satisfying $\forall m \in \mathbb{N} w_{m+1} \in T(w_m)$ satisfies $\forall m \in \mathbb{N} \{w_{m+1} \in Q_{X_f^{\gamma^0, \lambda, T}}(w_m)\}$, then $\text{End}(T) \neq \emptyset$.

Remark 28. Theorem 27(L2) generalizes Theorem 12 (see Examples 5 and 6).
3. Fixed Point Theorems for Single-Valued Contractions (of Banach-Type) in Uniform and Metric Spaces

**Definition 29.** Let \((X, \mathcal{D})\) be a Hausdorff sequentially complete uniform space, assume that \(\mathcal{F} = \{I_\alpha : X^2 \to [0;\infty), \alpha \in \mathcal{A}\} \subseteq J_{(X,\mathcal{D})}\), and let \(v \in \{1, 2\}\).

(a) Define on \(X\) the distance \(\mathcal{B}_v^{\mathcal{F}}, \mathcal{B}_v^{\mathcal{F}} = \{B_{v,\alpha}^{\mathcal{F}} : X^2 \to [0;\infty), \alpha \in \mathcal{A}\}\), as follows:

\[
\forall_{\alpha \in \mathcal{A}} \forall_{x, y \in X} B_{v,\alpha}^{\mathcal{F}}(x, y)
= \begin{cases} 
\max \{I_\alpha(x, y), I_\alpha(y, x)\}, & \text{if } v = 1, \\
I_\alpha(x, y), & \text{if } v = 2.
\end{cases}
\]

(b) Let \((X, T)\) be a single-valued dynamic system, \(T : X \to X\). If \((X, T)\) satisfies

\[
\forall_{\alpha \in \mathcal{A}} \exists \lambda_{\alpha, \mathcal{D}}(0;1) \forall_{x, y \in X} \left\{ B_{v,\alpha}^{\mathcal{F}}(T(x), T(y)) \leq \lambda_{\alpha} I_\alpha(x, y) \right\},
\]

then we say that \((X, T)\) is a \((\mathcal{B}_v^{\mathcal{F}}, \mathcal{A})\)-contraction on \(X\) for \(\Lambda = \{\lambda_{\alpha}, \alpha \in \mathcal{A}\}\).

**Remark 30.** Each \((\mathcal{B}_v^{\mathcal{F}}, \mathcal{A})\)-contraction on \(X\) is \((\mathcal{B}_v^{\mathcal{F}}, \mathcal{A})\)-contraction on \(X\) but converse does not hold.

We use notations above and Theorem 21 in proving the following new fixed point theorem for single-valued contractions with respect to \(\mathcal{F} \in J_{(X,\mathcal{D})}\) (of Banach-type) in uniform spaces \((X, \mathcal{D})\).

**Theorem 31.** Let \((X, \mathcal{D})\) be a Hausdorff sequentially complete uniform space, let \(\mathcal{F} = \{I_\alpha : X^2 \to [0;\infty), \alpha \in \mathcal{A}\} \subseteq J_{(X,\mathcal{D})}\), and let \(v \in \{1, 2\}\). Suppose also the following.

(I) \(\mathcal{F}\) is admissible.

(II) \((X, T)\) is a single-valued dynamic system, \(T : X \to X\).

(III) There exists a family \(\Lambda = \{\lambda_{\alpha} \in (0;1), \alpha \in \mathcal{A}\}\) such that \((X, T)\) is a \((\mathcal{B}_v^{\mathcal{F}}, \mathcal{A})\)-contraction on \(X\) for \(\Lambda\).

(IV) \(T(X_0^v) \subseteq X_0^v\).

The following hold.

(M1) \(T\) has a unique fixed point \(w\) in \(X\); that is, \(T(w) = w\) and \(\text{Fix}(T) = \{w\}\).

(M2) \(\forall_{\alpha \in \mathcal{A}} \{I_\alpha(w, w) = 0\}\).

(M3) For each \(w_0 \in X\), the sequence \((w_m = T^m(w_0) : m \in \mathbb{N})\) satisfies

\[
\forall_{\alpha \in \mathcal{A}} \lim_{m \to \infty} I_\alpha(w, w_m) = \lim_{m \to \infty} I_\alpha(w_m, w) = 0,
\]

\[
\forall_{\alpha \in \mathcal{A}} \lim_{m \to \infty} d_\alpha(w, w_m) = 0.
\]

Proofs of (M1) and (M2). By Remark 30, Definition 29, and the assumptions (I)–(IV) of Theorem 31, we see that

\[
\forall_{x \in X^v} \forall_{\alpha \in \mathcal{D}} \left\{ I_\alpha(T(x), T[2] (x)) \leq \lambda_{\alpha} I_\alpha(x, T(x)) \right\},
\]

\[
\forall_{x \in X^v} \left\{ y = T(x) \in X^v_\alpha \right\}.
\]

Let now \(\Lambda = \{\delta_\alpha \in (0;1), \alpha \in \mathcal{A}\}\) satisfying \(\forall_{\alpha \in \mathcal{A}} \{\lambda_{\alpha} < \delta_\alpha\}\) be arbitrary and fixed. One then immediately finds that

\[
\forall_{x \in X^v} \forall_{\alpha \in \mathcal{D}} \left\{ I_\alpha(T(x), T[2] (x)) \leq \lambda_{\alpha} I_\alpha(x, T(x)) \right\},
\]

\[
+(1 - \delta_\alpha) I_\alpha(x, T(x))
\]

or, equivalently,

\[
\forall_{x \in X^v} \forall_{\alpha \in \mathcal{D}} \left\{ I_\alpha(y, T(y)) + (\delta_\alpha - \lambda_{\alpha}) I_\alpha(x, y) \right\},
\]

where \(y = T(x) \in X^v_\alpha\). Consequently, for each \(x \in X^v_\alpha\), the singleton set

\[
\forall_{x \in X^v} \forall_{\alpha \in \mathcal{D}} \left\{ I_\alpha(y, T(y)) \leq I_\alpha(x, T(x)) \right\}
\]

\[
\forall_{x \in X^v} \forall_{\alpha \in \mathcal{D}} \left\{ I_\alpha(y, T(y)) \leq I_\alpha(x, T(x)) \right\}
\]

is a nonempty closed subset in \(X\).

From the above and Theorem 21 it follows that \(T\) has a fixed point \(w\) in \(X\) (i.e., \(w = T(w)\)) and \(\forall_{\alpha \in \mathcal{D}} \{I_\alpha(w, w) = 0\}\).

It remains to verify that \(\text{Fix}(T) = \{w\}\). Suppose that \(\{u, w\} \subset \text{Fix}(T)\). By Definition 29 and assumptions of Theorem 31, we obtain that, if \(v = 1\), then

\[
\forall_{\alpha \in \mathcal{A}} \exists \lambda_{\alpha, \mathcal{D}}(0;1) \left\{ \left\{ I_\alpha(u, w) \leq \max \{I_\alpha(u, w), I_\alpha(w, u)\} \right\} \right.
\]

\[
= \max \{I_\alpha(T(u), T(w))\},
\]

\[
I_\alpha(T(w), T(u)) \leq \lambda_{\alpha} I_\alpha(u, w) \right\}
\]

\[
\wedge \left\{ I_\alpha(u, w) \leq \max \{I_\alpha(u, w), I_\alpha(w, u)\} \right\}
\]

\[
= \max \{I_\alpha(T(u), T(w))\},
\]

\[
I_\alpha(T(w), T(u)) \leq \lambda_{\alpha} I_\alpha(u, w) \right\}\right.
\]

\[
\wedge \left\{ I_\alpha(u, w) \leq \max \{I_\alpha(u, w), I_\alpha(w, u)\} \right\}
\]

\[
\leq \lambda_{\alpha} I_\alpha(w, u) \right\}\right.
\]

\[
\text{and, if } v = 2, \text{ then}
\]

\[
\forall_{\alpha \in \mathcal{A}} \exists \lambda_{\alpha, \mathcal{D}}(0;1) \left\{ \left\{ I_\alpha(u, w) = I_\alpha(T(u), T(w)) \leq \lambda_{\alpha} I_\alpha(u, w) \right\} \right.
\]

\[
\wedge \left\{ I_\alpha(u, w) = I_\alpha(T(u), T(w)) \leq \lambda_{\alpha} I_\alpha(u, w) \right\}
\]

\[
\leq \lambda_{\alpha} I_\alpha(w, u) \right\}\right.
\]

\[
\text{.}
\]
Hence \( \forall a \in A \{ J_a(u, w) = J_a(w, u) = 0 \} \). From this information, by Remark 3(b), we deduce that \( u = w \).

Therefore, the assertions (M1) and (M2) hold.

**Proof of (M3).** Let now \( w_0 \in X \) be arbitrary and fixed and put \( (w_m) = T^{[m]}(w_0) : m \in \mathbb{N} \). By Definition 29, assumptions of Theorem 31 and the fact that \( T^{[m]}(w) = w \) for \( m \in \mathbb{N} \), we obtain that, if \( v = 1 \), then

\[
\forall a \in A \exists \lambda_a \in (0, 1) \forall m \in \mathbb{N} \left\{ \begin{array}{l}
\{ J_a(w, w_{m+1}) \\
\leq \lambda_a J_a(w, w_m) \\
\wedge \{ J_a(w_{m+1}, w) \leq \lambda_a J_a(w_m, w) \} 
\end{array} \right.. \tag{80}
\]

Hence

\[
\forall a \in A \exists \lambda_a \in (0, 1) \forall m \in \mathbb{N} \left\{ \begin{array}{l}
\{ J_a(w, w_{m+1}) \\
\leq \lambda_a J_a(w, w_m) \\
\wedge \{ J_a(w_{m+1}, w) \leq \lambda_a J_a(w_m, w) \} 
\end{array} \right.. \tag{81}
\]

This gives the assertion (72), since, by Definition 1,

\[
\forall a \in A \{ \begin{array}{l}
\{ J_a(w, w_0) < +\infty \}, \{ J_a(w_0, w) < +\infty \} \}
\end{array} \right.. \tag{83}
\]

Finally, let \( w_0 \in X \) be arbitrary and fixed and put \( (w_m) = T^{[m]}(w_0) : m \in \mathbb{N} \), \( (x_m = w : m \in \mathbb{N}) \), and \( (y_m = w_m : m \in \mathbb{N}) \). Using assertion (M2), we then have

\[
\forall a \in A \left\{ \begin{array}{l}
\{ \lim_{m \to \infty} J_a(x_n, x_m) = 0 \}
\end{array} \right., \tag{84}
\]

and, using assertion (72), we get

\[
\forall a \in A \left\{ \begin{array}{l}
\{ \lim_{m \to \infty} J_a(x_m, y_m) = 0 \}
\end{array} \right.. \tag{85}
\]

Hence, using Definition 1 (J2), we find

\[
\forall a \in A \left\{ \begin{array}{l}
\{ \lim_{m \to \infty} d_a(x_m, y_m) = \lim_{m \to \infty} d_a(w, w_m) = 0 \}
\end{array} \right.. \tag{86}
\]

Thus (73) holds.

**Remark 32.** (a) Theorem 31 includes Theorem II [16] and the result of [52]. Theorem 31 is different from Theorem II [16] and the result of [52] even in metric spaces and in uniform spaces, respectively (see Examples 4 and 7).

(b) Let \( v \in \{1, 2\} \). Assumptions (III) and (IV) imply that \( (X_0^v, T) \) is also a \( (\mathcal{B}_v^p, \Lambda) \)-contraction on \( X_0^v \). However, the dynamic systems \( (X, T) \) and \( (X_0^v, T) \) are not necessarily \( (\mathcal{B}_v^p, \Lambda) \)-contractions on \( X \) or \( X_0^v \), respectively (see Examples 4 and 7).

(c) Assumptions (II) and (IV) and assertions (M1) and (M2) imply that \( w \in X_0^v \) is a unique fixed point of \( (X, T) \) and \( (X_0^v, T) \). Assertion (M3) implies, in particular, that, for each starting point \( w_0 \) of the space \( X \), the dynamic process of the system \( (X, T) \) converges to \( w \).

The above has interesting implications for metric spaces.

**Definition 33.** Let \( (X, \mathcal{D}) \) be a complete metric space, assume that \( \mathcal{J} = \{ J : \mathcal{X}^2 \to [0, +\infty) \} \in \mathcal{J}(X, d) \) and let \( v \in \{1, 2\} \).

(a) Define on \( X \) the distance \( B_v^\mathcal{J} : \mathcal{X}^2 \to [0, +\infty) \) as follows:

\[
\forall x, y \in X \quad B_v^\mathcal{J}(x, y) = \begin{cases} \max \{ J(x, y), J(y, x) \}, & \text{if } v = 1, \\ J(x, y), & \text{if } v = 2. \end{cases} \tag{87}
\]

(b) Let \( (X, T) \) be a single-valued dynamic system, \( T : X \to X \). If \( (X, T) \) satisfies

\[
\exists \lambda \in (0, 1) \forall x, y \in X \quad \{ B_v^\mathcal{J}(T(x), T(y)) \leq \lambda J(x, y) \} \tag{88}
\]

then we say that \( (X, T) \) is a \( (B_v^\mathcal{J}, \lambda) \)-contraction on \( X \) for \( \lambda \).

As a corollary from Theorem 31 and its proof we get the following fixed point theorem for single-valued contractions with respect to \( \mathcal{J} \in \mathcal{J}(X, d) \) (of Banach-type) in metric spaces \( (X, d) \).

**Theorem 34.** Let \( (X, d) \) be a complete metric space, \( \mathcal{J} = \{ J : \mathcal{X}^2 \to [0, +\infty) \} \in \mathcal{J}(X, d) \) and \( v \in \{1, 2\} \). Suppose also the following.

(I) \( \mathcal{J} \) is admissible.

(II) \( (X, T) \) is a single-valued dynamic system, \( T : X \to X \).

(III) There exists \( \lambda \in (0; 1) \) such that \( (X, T) \) is a \( (B_v^\mathcal{J}, \lambda) \)-contraction on \( X \) for \( \lambda \).

(IV) \( T(X_0^v) \subseteq X_0^v \).

The following hold.

(S1) \( T \) has a unique fixed point \( w \) in \( X \) (i.e., \( T(w) = w \) and \( \text{Fix}(T) = \{w\} \)).

(S2) \( J(w, w) = 0 \).

(S3) For each \( w_0 \in X \), the sequence \( (w_m = T^{[m]}(w_0) : m \in \mathbb{N}) \) satisfies \( \lim_{m \to \infty} J(w_0, \ldots, w_m) = \lim_{m \to \infty} J(w_m, w) = 0 \) and \( \lim_{m \to \infty} d(w, w_m) = 0 \).
Remark 35. Theorem 34 generalizes Theorem 11 (see Example 7).

4. Examples Illustrating the Results

The following example describes some \( J \)-family in metric spaces.

Example 1. Let \((X, d)\) be a metric space. Let the set \( E \subset X \), containing at least two different points, be arbitrary and fixed and let \( c > 0 \) satisfy \( \delta(E) < c \) where \( \delta(E) = \sup \{ d(x, y) : x, y \in E \} \). Let \( J : X^2 \to [0; \infty) \) be defined by the formulae:

\[
J(x, y) = \begin{cases} 
  d(x, y), & \text{if } E \cap \{x, y\} = \{x, y\}, \\
  c, & \text{if } E \cap \{x, y\} \neq \{x, y\},
\end{cases}
\]

(89) \( x, y \in X \). Then \( J = \{ J : X^2 \to [0; \infty) \} \in J_{\{X, d\}} \) (see \([6, \text{Example 6.12}]\)).

The following example illustrates the Theorem 26(K1) in the case when \( J = \{ J : X^2 \to [0; \infty) \} \in J_{\{X, d\}}, J \neq d \).

Example 2. Let \( X = [0; 6] \) be a complete metric space with a metric \( d : X^2 \to [0; \infty), d(x, y) = |x - y|, x, y \in X \). Let \( T_1 : X \to \text{Cl}(X) \) be of the form:

\[
T_1(x) = \begin{cases} 
  [1; 2], & \text{if } x \in [0; 6), \\
  [4; 5], & \text{if } x = 6.
\end{cases}
\]

(90) \( E = [0; 3) \cup (3; 6) \) and let \( J \) be of the form:

\[
J(x, y) = \begin{cases} 
  d(x, y), & \text{if } E \cap \{x, y\} = \{x, y\}, \\
  8, & \text{if } E \cap \{x, y\} \neq \{x, y\}.
\end{cases}
\]

(91) Clearly, \( J \in J_{\{X, d\}} \) (Example 1).

We observe that \( X^0_J = [0; 3) \cup (3; 6) \neq \emptyset \).

Let \( \lambda = 3/4 \). We show that \((X, T_1)\) is a \((H^J_T, 3/4)\)-contraction on \( X \). Indeed, let \( x, y \in X \) be arbitrary and fixed. We consider three cases.

Case 1. If \( x, y \in [0; 6) \), then we have that \( T_1(x) = T_1(y) = [1; 2] \) and \( H^J_T(T_1(x), T_1(y)) = 0 \leq (3/4)J(x, y) = \lambda J(x, y) \).

Case 2. If \( x \in [0; 6) \) and \( y = 6 \), then \( y \neq E, J(x, y) = 8, T_1(x) = [1; 2] \) and \( T_1(y) = [4; 5] \). Hence, we calculate the following.

\[
H^J_T(T_1(x), T_1(y)) = \max \{ \sup \{ J(u, T_1(y)) : u \in T_1(x) \}, \sup \{ J(v, T_1(x)) : v \in T_1(y) \} \} 
\]

(92) \( 3 \leq 6 = \lambda \cdot 8 = \lambda J(x, y) \).

Case 3. If \( x = 6 \) and \( y \in X \setminus \{6\} \), then also (92) holds.

By Cases 1–3, \((X, T_1)\) is a \((H^J_T, \lambda)\)-contraction on \( X \).

Now, let \( y^0 = 7/8 \). We prove that, for each \( x \in X^0_J \), \( Q_{J, y^0-L,T_1}(x) \) is a nonempty closed subset in \( X \). Indeed, for each \( x \in X^0_J = [0; 3) \cup (3; 6) \), we have \( T_1(x) = [1; 2] \subset X^0_J \), \( V^y_{y^0}(x) = [1; 2] \), \( J(y, T_1(y)) = 0 \) and

\[
Q_{J, y^0-L,T_1}(x) = \{ y \in [1; 2] : J(y, T_1(y)) + \frac{1}{8} J(x, y) \leq J(x, T_1(x)) \} 
\]

(93) This implies the following.

Case 1. If \( x \in [0; 1] \), then

\[
Q_{J, y^0-L,T_1}(x) = \{ y \in [1; 2] : y - x \leq 8d(x, [1; 2]) \} = \{ y \in [1; 2] : y \leq 8 - 7x \}. 
\]

(94) Case 2. If \( x \in [1; 2] \), then

\[
Q_{J, y^0-L,T_1}(x) = \{ y \in [1; 2] : \frac{1}{8} |y - x| \leq d(x, [1; 2]) \} = \{ x \}. 
\]

(95) Case 3. If \( x \in (2; 3) \cup (3; 6) \), then

\[
Q_{J, y^0-L,T_1}(x) = \{ y \in [1; 2] : y - x \leq 8(x - 2) \} = \{ y \in [1; 2] : y \geq 16 - 7x \}. 
\]

(96) Assumptions of Theorem 26(K1) hold for \( v = 1 \), \( \text{Fix}(T_1) = [1; 2] \), and, for each \( w \in \text{Fix}(T_1) \), \( J(w, w) = 0 \).

The following example illustrates the Theorem 26(K2) in the case when \( J = \{ J \}, J \neq d \).

Example 3. Let \( X, E, J, \lambda = 3/4 \), and \( y^0 = 7/8 \) be such as in Example 2 and let \( T_2 : X \to \text{Cl}(X) \) be of the form:

\[
T_2(x) = \begin{cases} 
  [1], & \text{for } x \in [0; 2) \cup [3; 5],
  [2], & \text{for } x \in [2; 3) \cup [5; 6],
  [4; 5], & \text{for } x = 6.
\end{cases}
\]

(97) Then \( X^0_J = [0; 3) \cup (3; 6) \neq \emptyset \) and, by analogous considerations as in Example 2, we obtain that \((X, T_2)\) is a \((H^J_T, 3/4)\)-contraction on \( X \).
Next, let us observe that, for \( x \in X^0_f \),
\[
Q_{f \cdot \rho - \lambda, T_2} (x) = \left\{ y \in T_2 (x) : J (y, T_2 (y)) + \left( \frac{1}{8} \right) J (x, y) \leq J (x, T_2 (x)) \right\}.
\]

Hence we have the following.

Case 1. If \( x \in \{ 0; 2 \} \cup \{ 3; 5 \} \), then \( T_2 (x) = \{ 1 \} \). Therefore, \( Q_{f \cdot \rho - \lambda, T_2} (x) = \{ x \} \).

Case 2. If \( x \in \{ 2; 3 \} \cup \{ 3; 5 \} \), then \( T_2 (x) = \{ 2 \} \). Therefore, \( Q_{f \cdot \rho - \lambda, T_2} (x) = \{ x \} \).

Case 3. If \( x \in \{ 0; 2 \} \cup \{ 3; 5 \} \), then \( T_2 (x) = \{ 1 \} \). Therefore, \( Q_{f \cdot \rho - \lambda, T_2} (x) = \{ x \} \).

Therefore, for each \( x \in X^0_f \), each dynamic process \( (w_m : m \in \{ 0 \} \cup \mathbb{N}) \) starting at \( w_0 = x \) and satisfying \( \forall m \in \{ 0 \} \cup \mathbb{N} \), \( w_{m+1} \in T(w_m) \) satisfies \( \forall m \in \{ 0 \} \cup \mathbb{N} \), \( w_{m+1} \in Q_{f \cdot \rho - \lambda, T} (w_m) \).

The following example illustrates the Theorem 34 in the case when \( \mathcal{F} = \{ f : X^2 \to [0; 1) \} \). Let \( \mathcal{F} = \{ f : X^2 \to [0; 1) \} \).

Example 4. Let \( X = \{ 1, 2, 3, 4 \} \cup \{ 5; 6 \} \). Then \( \forall x \in X^0_f \), \( f \cdot \rho - \lambda, T_2 \) is a \( \mathcal{F} \)-family on \( X \) (Example 1).

Therefore, \( \mathcal{F} = \{ f : X^2 \to [0; 1) \} \) is a \( \mathcal{F} \)-family on \( X \) (Example 1).

Let \( \lambda = 3/4 \) and let \( T_3 : X \to X \) be the form:
\[
T_3 (x) = \begin{cases} 
2, & \text{for } x \in \{ 1, 2, 3 \} \cup \{ 5; 6 \}, \\
1, & \text{for } x \in \{ 4, 5 \}.
\end{cases}
\]

Then \( X^0_f = \{ 1, 2, 4, 5 \} \neq \emptyset \) and \( T_3 : X^0_f \to X^0_f \). Thus assumption (IV) of Theorem 34 holds.

We see that
\[
\forall x, y \in X \quad \max \{ J (T_3 (x), T_3 (y)), J (T_3 (y), T_3 (x)) \} = J (T_3 (x), T_3 (y))
\]

that is, \( (X, T_3) \) is a \( (B^f_\rho, \lambda) \)-contraction on \( X \). Indeed, we have the following.

Case 1. If \( x \in \{ 1, 2, 3 \} \cup \{ 5; 6 \} \) and \( y \in \{ 4, 5 \} \), then
\[
\lambda J (x, y) = \begin{cases} 
3 \lambda = \frac{9}{4}, & \text{for } x = 1, y = 4, \\
4 \lambda = 3, & \text{for } x = 1, y = 5, \\
8 \lambda = 6, & \text{for } x \in \{ 3 \} \cup \{ 5; 6 \}, y \in \{ 4, 5 \}.
\end{cases}
\]

Case 2. If \( x, y \in \{ 1, 2, 3 \} \cup \{ 5; 6 \} \), then
\[
0 = d (2, 2) = J (2, 2) = J (T_3 (x), T_3 (y)).
\]

Case 3. If \( x \in \{ 4, 5 \} \), then
\[
0 = d (1, 1) = J (1, 1) = J (T_3 (x), T_3 (y)).
\]

Assumptions of Theorem 34 hold and the assertions (S1)–(S3) are as follows. Fix \( T_3 \) = \{ 2 \}, \( J (2, 2) = 0 \) and, for each \( w_0 \in X \), the sequence \( (w_m : T^{[m]} (w_0) : m \in \mathbb{N}) \) satisfies
\[
\forall x \in X \quad \lim_{m \to \infty} d (w_m, 2) = 0.
\]

5. Comparisons of Our Results with Nadler’s and Banach’s Results

It is worth noticing that our results in metric spaces include Nadler’s and Banach’s results. Clearly, it is not otherwise. More precisely we have the following.

(a) In Examples 5 and 6 below we show that, for each \( \lambda \in [0; 1) \), the set-valued dynamic systems \( (X, T_1) \) and \( (X, T_2) \) defined in Examples 2 and 3, respectively, are not \( (H^f_\rho, \lambda) \)-contractions on \( X \) and thus we cannot use Theorem 12.

(b) In Example 7 we show that, for each \( \lambda \in [0; 1) \), the single-valued dynamic system \( (X, T_3) \) defined in Example 4 is not \( (d, \lambda) \)-contractions on \( X \) and thus we cannot use Theorem 11.

Therefore, in our concepts of \( (R^f_\rho, \lambda) \)-contractive set-valued dynamic systems and \( (D^f_\rho, \lambda) \)-contractive single-valued dynamic systems, \( \nu \in \{ 1, 2 \} \), the existence of \( \mathcal{F} \)-family such that \( \mathcal{F} \neq \emptyset \) is essential.

Example 5. Let \( (X, d) \) and \( T_1 \) be such as in Example 2 and let \( \mathcal{F} = \{ d \} \). We observe that \( X^0_f = X \).
Next, we see that, for each $\lambda \in [0; 1)$, $(X, T_1)$ is not a $(H^d, \lambda)$-contraction on $X$. Indeed, suppose that

$$\exists \lambda \in [0; 1) \forall x, y \in X \quad \{d \left( T_1 \left( x \right), T_1 \left( y \right) \right) \leq \lambda d \left( x, y \right) \}. \quad (108)$$

Then, in particular, for $x_0 = 3 \in X$ and $y_0 = 6 \in X$, we obtain the following.

1. $T_1(x_0) = [1; 2]$ and $T_1(y_0) = [4; 5]$.
2. For $u \in T_1(x_0) = [1; 2]$, $d(u, T_1(y_0)) = d(u, [4; 5]) = 4 - u$ and, consequently,
   $$\sup \{ d \left( u, T_1 \left( y_0 \right) \right) : u \in T_1 \left( x_0 \right) \} = 3. \quad (109)$$
3. For $z \in T_1(y_0) = [4; 5]$, $d(z, T_1(x_0)) = d(z, [1; 2]) = 2 - z$ and, consequently,
   $$\sup \{ d \left( z, T_1 \left( x_0 \right) \right) : z \in T_1 \left( y_0 \right) \} = 3. \quad (110)$$
4. By (2) and (3),
   $$H^d \left( T_1 \left( x_0 \right), T_1 \left( y_0 \right) \right) = \max \left\{ \sup_{u \in T_1(x_0)} d \left( u, T_1 \left( y_0 \right) \right), \sup_{z \in T_1(y_0)} d \left( z, T_1 \left( x_0 \right) \right) \right\} = 3. \quad (111)$$

Hence, we get

$$\forall \lambda \in [0; 1) \quad \left\{ 3 = H^d \left( T_1 \left( x_0 \right), T_1 \left( y_0 \right) \right) \leq \lambda d \left( x_0, y_0 \right) \right\}, \quad (112)$$

which is absurd.

**Example 6.** Let $(X, d)$ and $T_2$ be such as in Example 3 and let $J = \{d\}$. By similar argumentation as in Example 5, we observe that, for each $\lambda \in [0; 1)$, $(X, T_2)$ is not a $(H^d, \lambda)$-contraction on $X$.

**Example 7.** Let $(X, d)$ and $T_3$ be such as in Example 4 and let $J = \emptyset = \{d\}$. Clearly, $X_{\emptyset, \emptyset} = X$.

We observe that, for each $\lambda \in [0; 1)$, $(X, T_3)$ is not a $(d, \lambda)$-contraction on $X$. Otherwise, by Definition 29 for $J = \{d\}$ (or by (9)), the following holds:

$$\exists \lambda \in [0; 1) \exists x, y \in X \quad \{d \left( T_3 \left( x \right), T_3 \left( y \right) \right) \leq \lambda d \left( x, y \right) \}. \quad (113)$$

However, in particular, for $x_0 = 3 \in X$ and $y_0 = 4 \in X$, we get $d(x_0, y_0) = 1$ and then

$$\forall \lambda \in [0; 1) \quad \left\{ 1 = d \left( 2, 1 \right) = d \left( T_3 \left( x_0 \right), T_3 \left( y_0 \right) \right) \leq \lambda d \left( x_0, y_0 \right) = \lambda < 1 \right\}, \quad (114)$$

which is absurd. This gives that the condition (113) does not hold.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.


