Research Article

Exact Solutions for Unsteady Free Convection Flow of Casson Fluid over an Oscillating Vertical Plate with Constant Wall Temperature

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The unsteady free flow of a Casson fluid past an oscillating vertical plate with constant wall temperature has been studied. The Casson fluid model is used to distinguish the non-Newtonian fluid behaviour. The governing partial differential equations corresponding to the momentum and energy equations are transformed into linear ordinary differential equations by using nondimensional variables. Laplace transform method is used to find the exact solutions of these equations. Expressions for shear stress in terms of skin friction and the rate of heat transfer in terms of Nusselt number are also obtained. Numerical results of velocity and temperature profiles with various values of embedded flow parameters are shown graphically and their effects are discussed in detail.

1. Introduction

Newtonian fluids described by Navier-Stokes equations are extensively studied in the literature for the past few decades. Largely, this is due to the fact that they are relatively simple and their solutions are convenient [1–10]. However, Newtonian fluids which have a linear relationship between the stress and the rate of strain are limited in view of their applications. They do not explain several phenomena observed for the fluids in industry and other technological applications. For example, many complex fluids such as blood, soap, clay coating, certain oils and greases, elastomers, suspensions, and many emulsions are noteworthy due to their various applications in industry. Unfortunately, Navier-Stokes equations are no more convincing to describe such fluids. In the literature, they are known as non-Newtonian fluids. These fluids are described by a nonlinear relationship between the stress and the rate of strain. The understanding of flow characteristics of non-Newtonian fluids is very important because they play a significant role in industry and engineering. Such interest is motivated because of extensive applications in diverse areas

of biorheology, geophysics, and chemical and petroleum industries [11, 12]. In view of these particular applications, the study and understanding of non-Newtonian fluids have now become an increasingly appealing topic of current research in this field.

Rheological properties of non-Newtonian fluids are described by their so-called constitutive equations. Due to complexity of fluids, several non-Newtonian fluid models/constitutive equations based on their empirical observations have been proposed. Amongst the different non-Newtonian fluids there is one known as Casson fluid which was originally introduced by Casson for the prediction of the flow behavior of pigment-oil suspensions [13]. The Casson model is based on a structure model of the interactive behavior of solid and liquid phases of a two-phase suspension. Casson fluid exhibits yield stress. It is a shear thinning liquid which has an infinite viscosity at zero rates of shear, a yield stress below which no flow occurs, and a zero viscosity at an infinite rate of shear [14]. More exactly, if a shear stress less than the yield stress is applied to the fluid, it behaves like a solid, whereas if a shear stress greater than yield

together with initial and boundary conditions

$$\begin{aligned} t < 0 : u &= 0, \quad T = T_\infty \quad \forall y > 0, \\ t \geq 0 : u &= UH(t) \cos(\omega t) \quad \text{or} \quad u = U \sin(\omega t), \\ T &= T_w \quad \text{at } y = 0, \\ u &\longrightarrow 0, \quad T \longrightarrow T_\infty \quad \text{as } y \longrightarrow \infty. \end{aligned} \quad (5)$$

We introduce the dimensionless variables

$$\begin{aligned} u^* &= \frac{u}{U}, \quad y^* = \frac{U}{\nu} y, \quad t^* = \frac{U^2}{\nu} t, \\ \theta &= \frac{T - T_\infty}{T_w - T_\infty}, \quad \omega^* = \frac{\omega \nu}{U^2}, \quad \tau^* = \frac{\tau}{\rho U^2} \end{aligned} \quad (6)$$

into (4)-(5), and we get (* symbols are dropped for simplicity)

$$\frac{\partial u}{\partial t} = \left(1 + \frac{1}{\gamma}\right) \frac{\partial^2 u}{\partial y^2} + \text{Gr}\theta, \quad (7)$$

$$\text{Pr} \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial y^2}, \quad (8)$$

with associated initial and boundary conditions

$$\begin{aligned} t < 0 : u &= 0, \quad \theta = 0 \quad \forall y > 0, \\ t \geq 0 : u &= H(t) \cos(\omega t) \quad \text{or} \quad u = \sin(\omega t), \\ \theta &= 1 \quad \text{at } y = 0, \\ u &\longrightarrow 0, \quad \theta \longrightarrow 0 \quad \text{as } y \longrightarrow \infty, \end{aligned} \quad (9)$$

where

$$\text{Pr} = \frac{\mu c_p}{k}, \quad \text{Gr} = \frac{\nu g \beta (T_w - T_\infty)}{U^3}, \quad \gamma = \frac{\mu_B \sqrt{2\pi c}}{p_y} \quad (10)$$

are the Prandtl, Grashof numbers, and the Casson parameter.

3. Solution of the Problem

Applying Laplace transforms to (7) and (8), using initial and boundary conditions (9), we get the following solutions in the transformed (y, q) plane:

$$\bar{\theta}(y, q) = \frac{1}{q} e^{-y\sqrt{\text{Pr}q}}, \quad (11)$$

$$\bar{u}(y, q) = \frac{q}{q^2 + \omega^2} e^{-y\sqrt{aq}} + \frac{b}{q^2} e^{-y\sqrt{aq}} + \frac{b}{q^2} e^{-y\sqrt{\text{Pr}q}}. \quad (12)$$

The inverse Laplace transforms of (11) and (12) are obtained as follows:

$$\theta(y, t) = \text{erfc} \left(\frac{y}{2} \sqrt{\frac{\text{Pr}}{t}} \right), \quad (13)$$

$$\begin{aligned} u_c(y, t) &= \frac{H(t)}{4} e^{-i\omega t} \left[e^{-y\sqrt{-i\omega a}} \text{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{-i\omega t} \right) \right. \\ &\quad \left. + e^{y\sqrt{-i\omega a}} \text{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{-i\omega t} \right) \right] \\ &\quad + \frac{H(t)}{4} e^{i\omega t} \left[e^{-y\sqrt{i\omega a}} \text{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{i\omega t} \right) \right. \\ &\quad \left. + e^{y\sqrt{i\omega a}} \text{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{i\omega t} \right) \right] \\ &\quad + b \left[\left(t + \frac{ay^2}{2} \right) \text{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} \right) - y\sqrt{a} \sqrt{\frac{t}{\pi}} e^{-ay^2/4t} \right] \\ &\quad - b \left[\left(t + \frac{\text{Pr}y^2}{2} \right) \text{erfc} \left(\frac{y}{2} \sqrt{\frac{\text{Pr}}{t}} \right) \right. \\ &\quad \left. - y\sqrt{\text{Pr}} \sqrt{\frac{t}{\pi}} e^{-\text{Pr}y^2/4t} \right]. \end{aligned} \quad (14)$$

The subscript “c” on the left side of (14) stands for the cosine oscillations of the plate. Similarly, the velocity corresponding to the sine oscillations of the plate is given by

$$\begin{aligned} u_s(y, t) &= -\frac{1}{4i} e^{-i\omega t} \\ &\quad \times \left[e^{-y\sqrt{-i\omega a}} \text{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{-i\omega t} \right) \right. \\ &\quad \left. + e^{y\sqrt{-i\omega a}} \text{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{-i\omega t} \right) \right] \\ &\quad + \frac{1}{4i} e^{i\omega t} \left[e^{-y\sqrt{i\omega a}} \text{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{i\omega t} \right) \right. \\ &\quad \left. + e^{y\sqrt{i\omega a}} \text{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{i\omega t} \right) \right] \\ &\quad + b \left[\left(t + \frac{ay^2}{2} \right) \text{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} \right) - y\sqrt{a} \sqrt{\frac{t}{\pi}} e^{-ay^2/4t} \right] \\ &\quad - b \left[\left(t + \frac{\text{Pr}y^2}{2} \right) \text{erfc} \left(\frac{y}{2} \sqrt{\frac{\text{Pr}}{t}} \right) \right. \\ &\quad \left. - y\sqrt{\text{Pr}} \sqrt{\frac{t}{\pi}} e^{-\text{Pr}y^2/4t} \right], \end{aligned} \quad (15)$$

where

$$a = \frac{\gamma}{1 + \gamma}, \quad b = \frac{aGr}{Pr - 1}. \quad (16)$$

Note that the above solutions for velocity are only valid for $Pr \neq 1$. Moreover the solution for $Pr = 1$ can be easily obtained by putting $Pr = 1$ into (8) and can follow a similar procedure as discussed above. The obtained solutions for cosine and sine oscillations of the plate when $Pr = 1$ are

$$\begin{aligned} u_c(y, t) &= \frac{H(t)}{4} e^{-i\omega t} \\ &\times \left[e^{-y\sqrt{-i\omega a}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{-i\omega t} \right) \right. \\ &\quad \left. + e^{y\sqrt{-i\omega a}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{-i\omega t} \right) \right] \\ &+ \frac{H(t)}{4} e^{i\omega t} \left[e^{-y\sqrt{i\omega a}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{i\omega t} \right) \right. \\ &\quad \left. + e^{y\sqrt{i\omega a}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{i\omega t} \right) \right] \\ &+ \frac{aGr\gamma}{2} \left[2\sqrt{\frac{t}{\pi}} e^{-y^2/4t} - y \operatorname{erfc} \left(\frac{y}{2\sqrt{t}} \right) \right], \end{aligned} \quad (17)$$

$$\begin{aligned} u_s(y, t) &= -\frac{1}{4i} e^{-i\omega t} \\ &\times \left[e^{-y\sqrt{-i\omega a}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{-i\omega t} \right) \right. \\ &\quad \left. + e^{y\sqrt{-i\omega a}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{-i\omega t} \right) \right] \\ &+ \frac{1}{4i} e^{i\omega t} \left[e^{-y\sqrt{i\omega a}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{i\omega t} \right) \right. \\ &\quad \left. + e^{y\sqrt{i\omega a}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{i\omega t} \right) \right] \\ &+ \frac{aGr\gamma}{2} \left[2\sqrt{\frac{t}{\pi}} e^{-y^2/4t} - y \operatorname{erfc} \left(\frac{y}{2\sqrt{t}} \right) \right]. \end{aligned} \quad (18)$$

Note that, in (14) and (15), the first two terms in each equation account for the contribution from mechanical parts while the last two terms show the thermal effects. On the other hand in (17) and (18), the last term in each equation shows the contribution from the thermal part.

The nondimensional skin friction is calculated from the velocity field (14), using the relation

$$\begin{aligned} \tau &= -\mu \left(1 + \frac{1}{\gamma} \right) \frac{\partial u}{\partial y} \Big|_{y=0}, \\ \tau &= \frac{1}{2a\sqrt{\pi}} \\ &\times \left[e^{-it\omega} \left[2e^{it\omega} \left\{ H\sqrt{\frac{a}{t}} + \sqrt{ab}\sqrt{t} - ab\sqrt{Pr}\sqrt{t} \right. \right. \right. \\ &\quad \left. \left. + b\sqrt{\frac{a}{t}} + ab\sqrt{\frac{Pr}{t}} \right\} \right. \\ &\quad \left. + H\sqrt{\pi} \left\{ \sqrt{-ia\omega} \operatorname{erf}(\sqrt{-it\omega}) \right. \right. \\ &\quad \left. \left. + e^{2it\omega} \sqrt{ia\omega} \operatorname{erf}(\sqrt{it\omega}) \right\} \right] \Big]. \end{aligned} \quad (19)$$

The expression of Nusselt number is given by

$$Nu = -\frac{\gamma}{U_0(T - T_\infty)} \frac{\partial T^*}{\partial y^*} \Big|_{y^*=0} = \sqrt{\frac{Pr}{\pi t}}. \quad (20)$$

4. Limiting Cases

The following solutions from the literature appear as the limiting cases of our general solutions.

- (i) By taking $\gamma \rightarrow \infty$ into (14) and (15), the corresponding solutions for viscous fluid can be obtained as a special case:

$$\begin{aligned} u_c(y, t) &= \frac{H(t)}{4} e^{-i\omega t} \\ &\times \left[e^{-y\sqrt{-i\omega}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{1}{t}} - \sqrt{-i\omega t} \right) \right. \\ &\quad \left. + e^{y\sqrt{-i\omega}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{1}{t}} + \sqrt{-i\omega t} \right) \right] \\ &+ \frac{H(t)}{4} e^{i\omega t} \left[e^{-y\sqrt{i\omega}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{1}{t}} - \sqrt{i\omega t} \right) \right. \\ &\quad \left. + e^{y\sqrt{i\omega}} \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{1}{t}} + \sqrt{i\omega t} \right) \right] \\ &+ b \left[\left(t + \frac{y^2}{2} \right) \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{1}{t}} \right) - y \sqrt{\frac{t}{\pi}} e^{-y^2/4t} \right] \\ &+ b \left[\left(t + \frac{Pr y^2}{2} \right) \operatorname{erfc} \left(\frac{y}{2} \sqrt{\frac{Pr}{t}} \right) \right. \\ &\quad \left. - y \sqrt{Pr} \sqrt{\frac{t}{\pi}} e^{-Pr y^2/4t} \right], \end{aligned}$$

$$\begin{aligned}
u_s(y, t) &= -\frac{1}{4i}e^{-i\omega t} \\
&\times \left[e^{-y\sqrt{-i\omega}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{1}{t}} - \sqrt{-i\omega t} \right) \right. \\
&\quad \left. + e^{y\sqrt{-i\omega}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{1}{t}} + \sqrt{-i\omega t} \right) \right] \\
&+ \frac{1}{4i}e^{i\omega t} \left[e^{-y\sqrt{i\omega}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{1}{t}} - \sqrt{i\omega t} \right) \right. \\
&\quad \left. + e^{y\sqrt{i\omega}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{1}{t}} + \sqrt{i\omega t} \right) \right] \\
&+ b \left[\left(t + \frac{y^2}{2} \right) \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{1}{t}} \right) - y \sqrt{\frac{t}{\pi}} e^{-y^2/4t} \right] \\
&+ b \left[\left(t + \frac{\operatorname{Pr} y^2}{2} \right) \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{\operatorname{Pr}}{t}} \right) \right. \\
&\quad \left. - y \sqrt{\operatorname{Pr}} \sqrt{\frac{t}{\pi}} e^{-\operatorname{Pr} y^2/4t} \right].
\end{aligned} \tag{21}$$

(ii) By taking $\omega = 0$, which corresponds to impulsive motion of the plate, then (14) and (17) yield

$$\begin{aligned}
u_c(y, t) &= H(t) \left[\operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{a}{t}} \right) \right. \\
&+ ab \left[\left(t + \frac{ay^2}{2} \right) \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{a}{t}} \right) - y \sqrt{a} \sqrt{\frac{t}{\pi}} e^{-ay^2/4t} \right] \\
&+ ab \left[\left(t + \frac{\operatorname{Pr} y^2}{2} \right) \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{\operatorname{Pr}}{t}} \right) \right. \\
&\quad \left. - y \sqrt{\operatorname{Pr}} \sqrt{\frac{t}{\pi}} e^{-\operatorname{Pr} y^2/4t} \right]; \quad \operatorname{Pr} \neq 1,
\end{aligned} \tag{22}$$

$$\begin{aligned}
u_c(y, t) &= H(t) \left[\operatorname{erf} c \left(\frac{1}{2} \sqrt{\frac{a1}{t}} y \right) \right. \\
&+ \frac{a\operatorname{Gr}y}{2} \left[2 \sqrt{\frac{t}{\pi}} e^{-y^2/4t} - y \operatorname{erf} c \left(\frac{y}{2\sqrt{t}} \right) \right]; \quad \operatorname{Pr} = 1,
\end{aligned} \tag{23}$$

respectively, and (22) and (23) describe the corresponding solution of Stokes' first problem for Casson fluid. It is important to note that exact solutions of Stokes' first problem for Casson fluid (22) and (23) are also not reported in the literature and hence are new.

(iii) In this last case, we assume that the flow is induced only due to bounding plate and the corresponding buoyancy forces are zero equivalently; it shows the absence of free convection ($\operatorname{Gr} = 0$) due to the differences in temperature gradient. This shows that the thermal parts of velocities in (14) and (15) are zero. Hence the flow is only governed by the corresponding mechanical parts given by

$$\begin{aligned}
u_c(y, t) &= \frac{H(t)}{4} e^{-i\omega t} \left[e^{-y\sqrt{-i\omega a}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{-i\omega t} \right) \right. \\
&\quad \left. + e^{y\sqrt{-i\omega a}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{-i\omega t} \right) \right] \\
&+ \frac{H(t)}{4} e^{i\omega t} \left[e^{-y\sqrt{i\omega a}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{i\omega t} \right) \right. \\
&\quad \left. + e^{y\sqrt{i\omega a}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{i\omega t} \right) \right],
\end{aligned} \tag{24}$$

$$\begin{aligned}
u_s(y, t) &= -\frac{1}{4i} e^{-i\omega t} \left[e^{-y\sqrt{-i\omega a}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{-i\omega t} \right) \right. \\
&\quad \left. + e^{y\sqrt{-i\omega a}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{-i\omega t} \right) \right] \\
&+ \frac{1}{4i} e^{i\omega t} \left[e^{-y\sqrt{i\omega a}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{a}{t}} - \sqrt{i\omega t} \right) \right. \\
&\quad \left. + e^{y\sqrt{i\omega a}} \operatorname{erf} c \left(\frac{y}{2} \sqrt{\frac{a}{t}} + \sqrt{i\omega t} \right) \right].
\end{aligned}$$

Note that (24) are identical to those obtained by Fetecau et al. [9]; see (9) and (11). This fact is also shown in Figure 8.

5. Graphical Results and Discussion

In this section, the obtained exact solutions are studied numerically in order to determine the effects of several involved parameters such as Prandtl number Pr , Grashof number Gr , Casson parameter γ , phase angle ωt , and time t . Numerical values of skin friction and Nusselt number are computed and presented in tables for different parameters. For the sake of correctness and verification, we have compared our results with those of Fetecau et al. [9]. This comparison is shown in Figure 2. It is found that our limiting solutions (24) are identical to (9) and (11) obtained by Fetecau et al. [9]. This confirms the accuracy of our obtained results.

The velocity profiles for different values of Prandtl number Pr are shown in Figure 3, when the other parameters are fixed. It is observed that velocity of the fluid decreases with increasing Prandtl number. Figure 4 illustrates the profiles of velocity for different values of Gr . It is observed that velocity increases with increasing values of Gr .

FIGURE 2: Comparison of the present results (see (24)) with those obtained by Fetecau et al. [9] (see (9) and (11)), when $t = 0.2$, $\omega = 0$, $a = 1$, $U = 1$, and $\nu = 1$.

FIGURE 3: Profiles of velocity for different values of Pr , when $\omega = \pi/4$, $t = 0.2$, and $Gr = 3$.

FIGURE 4: Profiles of velocity for different values of Gr , when $Pr = 0.3$, $\gamma = 0.6$, $t = 0.3$, and $\omega = \pi/4$.

FIGURE 5: Profiles of velocity for different values of γ , when $Pr = 0.3$, $Gr = 0$, $t = 0.3$, and $\omega = \pi/4$.

FIGURE 6: Profiles of velocity for different values of ωt , when $Gr = 3$, $Pr = 0.3$, $t = 1$, and $\gamma = 0.5$.

The influence of Casson fluid parameter on velocity profiles is shown in Figure 5. It is found that velocity decreases with increasing values of γ . It is important to note that an increase in Casson parameter makes the velocity boundary layer thickness shorter. It is further observed from this graph that when the Casson parameter γ is large enough, that is, $\gamma \rightarrow \infty$, the non-Newtonian behaviours disappear and the fluid purely behaves like a Newtonian fluid. Thus, the velocity boundary layer thickness for Casson fluid is larger than the Newtonian fluid. It occurs because of plasticity of Casson fluid. When Casson parameter decreases, the plasticity of the fluid increases, which causes the increment in velocity boundary layer thickness. The graphical results for the phase angle, ωt , are shown in Figure 6. It is observed that the fluid is oscillating between -1 and 1 . These fluctuations near the plate are maximum and decrease for further values of independent variable y . This figure can easily help us to check the accuracy of our results. For illustration of such results we have concentrated more on the values of $\omega t = 0, \pi/2$ and π . We can see that, for these values of ωt , the velocity shows its values either $1, 0$, or -1 which are identical with the

FIGURE 7: Profiles of velocity for different values of t , when $Pr = 0.3$, $Gr = 3$, $\gamma = 0.5$, and $\omega = 0$.

FIGURE 8: Profiles of temperature for different values of Pr , when $t = 0.4$.

imposed boundary conditions of velocity in (11). Hence, both the graphical and mathematical results are found in excellent agreement. In Figure 7 the influence of dimensionless time t on the velocity profiles is shown. It is found that the velocity is an increasing function of time t .

It is depicted from Figure 8 that the temperature profile decreases with increasing values of Prandtl number Pr . The specific values of Prandtl number such as $Pr = 0.71$, $Pr = 1.0$, and $Pr = 7.0$ are important in the sense that, physically, they correspond to air, electrolytic solution, and water, respectively. It is observed that the thermal boundary layer thickness is maximum near the plate and decreases with increasing distance from the leading edge and finally approaches to zero. It is also noticed from this figure that the magnitude of temperature is greater for air compared to electrolytic solution and water. It is justified due to the fact that thermal conductivity of the fluid decreases with increasing Prandtl number Pr and hence decreases the thermal boundary layer thickness and finally the temperature profiles.

Figure 9 is plotted to show the effects of the dimensionless time t on the temperature profiles. Four different values of time $t = 0.1$, $t = 0.2$, $t = 0.3$, and $t = 0.4$ are

FIGURE 9: Profiles of temperature for different values of t , when $Pr = 0.71$.

TABLE 1: Skin friction variations.

Pr	Gr	γ	ωt	t	τ
0.3	3	0.5	$\pi/4$	0.3	3.384043
0.71	3	0.5	$\pi/4$	0.3	0.437932
0.3	5	0.5	$\pi/4$	0.3	0.200932
0.3	3	1.0	$\pi/4$	0.3	0.536355
0.3	3	0.5	$\pi/2$	0.3	0.422417
0.3	3	0.5	$\pi/4$	0.4	0.307634

TABLE 2: Nusselt number variations.

Pr	t	Nu
0.3	0.3	0.564
0.71	0.3	0.867
0.3	0.6	0.398

chosen. Obviously the temperature increases with increasing time t . This graphical behaviour of temperature is in good agreement with the corresponding boundary conditions of temperature profiles as shown in (9). Results for skin friction and Nusselt number are computed in Tables 1 and 2. The computations of skin friction give complex results. Therefore, for the sake of convenience we have considered in Table 1 only its real part. Table 1 shows that skin friction increases with increasing values of Pr , Gr , γ , and ωt , whereas it decreases with increasing values of t . On the other hand, it is found from Table 2 that Nusselt number increases with increasing Pr whereas it decreases with increasing t .

6. Conclusion

In this paper an exact analysis is performed to investigate the unsteady boundary layer flow of a Casson fluid past an oscillating vertical plate with constant wall temperature. The dimensionless governing equations are solved by using the Laplace transform technique. The results for velocity and temperature are obtained and plotted graphically. The numerical results for skin friction and Nusselt number are

computed in tables. The main conclusions of this study are as follows.

- (1) Velocity increases with increasing Gr and t , whereas it decreases with increasing values of Pr , γ , and ωt .
- (2) Temperature increases with increasing t , whereas it decreases when Pr is increased.
- (3) Skin friction is increasing with increasing values of Pr , γ , and ωt , whereas it decreases with increasing values of Gr and t .
- (4) Nusselt number increases with increasing Pr , whereas it decreases with increasing t .
- (5) Solutions (24) are found in excellent agreement with those obtained by Fetecau et al. [9].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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