Research Article

Some Inequalities of Simpson Type for \( h \)-Convex Functions via Fractional Integrals

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Received 21 March 2015; Accepted 1 July 2015

Academic Editor: Alberto Fiorenza

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We establish some inequalities of Simpson type involving Riemann-Liouville fractional integrals for mappings whose first derivatives are \( h \)-convex.

1. Introduction

The following inequality is well known in the literature as Simpson’s inequality.

**Theorem 1.** Let \( f : [a, b] \to \mathbb{R} \) be a four times continuously differentiable mapping on \( (a, b) \) and \( \|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty \).

Then, the following inequality holds:

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b - a)^4.
\]

In [1], Dragomir et al. proved the following inequality.

**Theorem 2.** Suppose \( f : [a, b] \to \mathbb{R} \) is a differentiable mapping whose derivative is continuous on \( (a, b) \) and \( f' \in L_1[a, b] \). Then the following inequality holds:

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{12} \left( \frac{1 + 2^{p+1}}{3(p+1)} \right)^{1/p} \left( \frac{3|f'(b)|^q + |f'(a)|^q}{4} \right)^{1/q} \]

where \( 1/p + 1/q = 1 \).

In [2], Sarikaya et al. obtained inequalities for differentiable convex mappings. The main inequality is as follows.

**Theorem 3.** Let \( f : I \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \( I^c \) such that \( f' \in L_1[a, b] \), where \( a, b \in I^c \) with \( a < b \). If \( |f'|^q \) is convex on \( [a, b] \), \( q > 1 \), then the following inequality holds:

\[
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{12} \left( \frac{1 + 2^{p+1}}{3(p+1)} \right)^{1/p} \left( \frac{3|f'(b)|^q + |f'(a)|^q}{4} \right)^{1/q}.
\]
Theorem 4. Let \( f : I \subset [0, \infty) \rightarrow \mathbb{R} \) be a differentiable mapping on \( I \) such that \( f' \in L^p(\mathbb{R}) \) for some fixed \( p \in [0, 1] \) and \( q > 1 \), then the following inequality holds:

\[
\frac{1}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f'(x) \, dx
\]

\[
\leq \frac{(b-a)}{12} \left( 1 + \frac{2^{p+1}}{3} \right)^{1/p} \cdot \left[ \left( \frac{1}{q+1} \right)^{1/q} \right] \left( \frac{1}{s+1} \right)^{1/q} \left( \frac{1}{s+1} \right)^{1/q} \left\{ \left( \frac{1}{s+1} \right)^{1/q} \right\},
\]

(4)

where \( 1/p + 1/q = 1 \).

For recent refinements, counterparts, generalizations, and inequalities of Simpson type, see [1-7]. In 2007, Varošanec in [8] introduced a large class of functions, the so-called h-convex functions. This class contains several well-known classes of functions such as nonnegative convex functions and s-convex functions. This class is defined in the following way: a function \( f : I \rightarrow \mathbb{R}, \emptyset \neq I \subset \mathbb{R} \) being an interval, is called h-convex if

\[
f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y)
\]

holds for all \( x, y \in I, t \in (0, 1) \), where \( h : J \rightarrow \mathbb{R}, h \neq 0 \), and \( J \) is an interval, \( (0, 1) \subseteq J \).

The aim of this paper is to establish inequalities of Simpson type for h-convex mappings via fractional integrals which are defined in the following way: left-sided and right-sided Riemann-Liouville fractional integrals of the order \( \alpha > 0 \) are defined by

\[
I^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt
\]

(0 \leq a < x \leq b),

\[
I^\alpha_b f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt
\]

(0 \leq a < x < b),

where \( \Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} \, du \) is the gamma function. Here is \( I^\alpha_a f(x) = I^\alpha_b f(x) = f(x) \).

2. Main Results

To prove our main results, we consider the following lemma.

Lemma 5. Let \( f : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be an absolutely continuous mapping on \( I \) such that \( f' \in L^p(\mathbb{R}) \) for some fixed \( p \in [0, 1] \) and \( q > 1 \). Then the following inequality holds:

\[
\int_0^1 \left( \frac{t^\alpha}{2} \right) f' \left( \frac{1 + t + 1 - t}{2} \right) \, dt
\]

\[
= \frac{2}{b-a} \left( \frac{t^\alpha}{2} \right) f' \left( \frac{1 + t + 1 - t}{2} \right) \, dt
\]

(7)

Proof. By integration by parts and by the change of the variables, we have

\[
\int_0^1 \left( \frac{t^\alpha}{2} \right) f' \left( \frac{1 + t + 1 - t}{2} \right) \, dt
\]

\[
= \frac{2}{b-a} \left( \frac{t^\alpha}{2} \right) f' \left( \frac{1 + t + 1 - t}{2} \right) \, dt
\]

(8)

Similarly, we have

\[
\int_0^1 \left( \frac{t^\alpha}{2} \right) f' \left( \frac{1 + t + 1 - t}{2} \right) \, dt
\]

\[
= \frac{2}{b-a} \left( \frac{t^\alpha}{2} \right) f' \left( \frac{1 + t + 1 - t}{2} \right) \, dt
\]

(9)

From (8) and (9), we get (7). This completes the proof. \( \square \)
The following theorems give a new result of Simpson’s inequality for h-convex functions via fractional integrals.

**Theorem 6.** Let $f : I \subset R \to R$ be a differentiable mapping on $I^*$ such that $f' \in L_1[a, b]$, where $a, b \in I^*$ with $a < b$. If $|f'|$ is h-convex on $[a, b]$, then the following inequality holds:

$$
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] + \frac{b-a}{3} \left[ |f'(a)| + |f'(b)| \right] \right| \leq \int_0^1 h(t) \, dt.
$$

**Corollary 7.** If in Theorem 6 one takes $h(t) = t^q$, then inequality (10) reduces to the following inequality for the convex function:

$$
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{b-a}{6} \left[ |f'(a)| + |f'(b)| \right] \right|
$$

**Theorem 6.** Let $f : I \subset R \to R$ be a differentiable mapping on $I^*$ such that $f' \in L_1[a, b]$, where $a, b \in I^*$ with $a < b$. If $|f'|$ is h-convex on $[a, b]$, then the following inequality holds:

$$
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right]
- \frac{b-a}{6} \left[ |f'(a)| + |f'(b)| \right] \right| \leq \int_0^1 h(t) \, dt.
$$

**Corollary 8.** If in Theorem 6 one takes $h(t) = t^\alpha$ then inequality (10) reduces to the following inequality for the $s$-convex function:

$$
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right]
- \frac{b-a}{6} \left[ |f'(a)| + |f'(b)| \right] \right| \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{5(b-a)}{72} \left[ |f'(a)| + |f'(b)| \right].
$$

**Corollary 9.** If in Theorem 6 one takes $h(t) = t$ and $\alpha = 1$ then from the proof of Theorem 6 it follows that the following inequality holds:

$$
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right]
- \frac{b-a}{6} \left[ |f'(a)| + |f'(b)| \right] \right| \leq \frac{b-a}{6} \left[ |f'(a)| + |f'(b)| \right].
$$

**Theorem 10.** Let $f : I \subset R \to R$ be a differentiable mapping on $I^*$ such that $f' \in L_1[a, b]$, where $a, b \in I^*$ with $a < b$. If $|f'|$ is h-convex on $[a, b]$ and $q > 1$, then the following inequality holds:

$$
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right]
- \frac{b-a}{6} \left[ |f'(a)| + |f'(b)| \right] \right|
\leq \frac{b-a}{6} \left[ |f'(a)| \right]^{1/q} \int_0^1 h \left( \frac{1+t}{2} \right) dt
+ |f'(a)|^{1/q} \int_0^1 h \left( \frac{1+t}{2} \right) dt
+ |f'(b)|^{1/q} \int_0^1 h \left( \frac{1+t}{2} \right) dt
$$

**Corollary 7.** If in Theorem 6 one takes $h(t) = t$ then inequality (10) reduces to the following inequality for the convex function:

$$
\left| \frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) + f(b) \right]
- \frac{b-a}{6} \left[ |f'(a)| + |f'(b)| \right] \right|
\leq \frac{b-a}{6} \left[ |f'(a)| + |f'(b)| \right].
$$
Proof. From Lemma 5 and the Hölder inequality, we have
\[
\left|\frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) \right] + f(b) \right|
- \frac{2^{n-1}(\alpha+1)}{(b-a)^\alpha} \left[ I^\alpha_a f \left( \frac{a+b}{2} \right) + I^\alpha_b f \left( \frac{a+b}{2} \right) \right]
\leq \frac{b-a}{2} \left[ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right]
\cdot \left( \int_0^1 \left| f' \left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \right|^q dt \right)^{1/q}
\leq \frac{b-a}{2} \left[ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right]
\cdot \left( \int_0^1 \left| f' \left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \right|^q dt \right)^{1/q}
\leq \left\{ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right\}^2.
\]
(16)

Using the fact that \(|a^2/2 - 1/3| = 1/3 - t^2/2| \leq 1/3 for all \ t \in [0, 1] and using the last two inequalities in (16) we obtain (15).

This completes the proof of the theorem. □

Corollary 11. If in Theorem 10 one takes \( h(t) = t^3 \) then inequality (15) reduces to the following inequality for the convex function:
\[
\left|\frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) \right] + f(b) \right|
- \frac{2^{n-1}(\alpha+1)}{(b-a)^\alpha} \left[ I^\alpha_a f \left( \frac{a+b}{2} \right) + I^\alpha_b f \left( \frac{a+b}{2} \right) \right]
\leq \frac{b-a}{2} \left[ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right]
\cdot \left( \int_0^1 \left| f' \left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \right|^q dt \right)^{1/q}
\leq \frac{b-a}{2} \left[ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right]
\cdot \left( \int_0^1 \left| f' \left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \right|^q dt \right)^{1/q}
\leq \left\{ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right\}^2.
\]
(18)

Corollary 12. If in Theorem 10 one takes \( h(t) = t^3 \) then inequality (15) reduces to the following inequality for the \( s \)-convex function:
\[
\left|\frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) \right] + f(b) \right|
- \frac{2^{n-1}(\alpha+1)}{(b-a)^\alpha} \left[ I^\alpha_a f \left( \frac{a+b}{2} \right) + I^\alpha_b f \left( \frac{a+b}{2} \right) \right]
\leq \frac{b-a}{2} \left[ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right]
\cdot \left( \int_0^1 \left| f' \left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \right|^q dt \right)^{1/q}
\leq \left\{ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right\}^2.
\]
(19)

Theorem 13. Let \( f: I \subset R \rightarrow R \) be a differentiable mapping on \( I \) such that \( f' \in L^1[a,b] \), where \( a, b \in I \) with \( a < b \). If \( |f'|^q \) is \( h \)-convex on \( [a,b] \) and \( q \geq 1 \), then the following inequality holds:
\[
\left|\frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) \right] + f(b) \right|
- \frac{2^{n-1}(\alpha+1)}{(b-a)^\alpha} \left[ I^\alpha_a f \left( \frac{a+b}{2} \right) + I^\alpha_b f \left( \frac{a+b}{2} \right) \right]
\leq \frac{b-a}{2} \left[ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right]
\cdot \left( \int_0^1 \left| f' \left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \right|^q dt \right)^{1/q}
\leq \left\{ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right\}^2.
\]
(20)

Proof. From Lemma 5 and the power mean inequality, we have that the following inequality holds:
\[
\left|\frac{1}{6} \left[ f(a) + 4f \left( \frac{a+b}{2} \right) \right] + f(b) \right|
- \frac{2^{n-1}(\alpha+1)}{(b-a)^\alpha} \left[ I^\alpha_a f \left( \frac{a+b}{2} \right) + I^\alpha_b f \left( \frac{a+b}{2} \right) \right]
\leq \frac{b-a}{2} \left[ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right]
\cdot \left( \int_0^1 \left| f' \left( \frac{1+t}{2}a + \frac{1-t}{2}b \right) \right|^q dt \right)^{1/q}
\leq \left\{ \left( \int_0^1 \left| f' \left( \frac{1+t}{2}b + \frac{1-t}{2}a \right) \right|^q dt \right)^{1/q} \right\}^2.
\]
(21)
By the $h$-convexity of $|f'|^q$ and using the fact that $|t^{a/2} - 1/3| = |1/3 - t^{a/2}| \leq 1/3$ for all $t \in [0, 1]$, we have
\[
\begin{align*}
&\int_0^1 \left| t^{\frac{a}{2}} - \frac{1}{3} \right| |f'(\frac{1+t}{2}b + \frac{1-t}{2}a)|^q \, dt \\
&\quad \leq \frac{1}{3} \left( |f'(b)|^q \int_0^1 h\left( \frac{1+t}{2} \right) \, dt \\
&\quad \quad + |f'(a)|^q \int_0^1 h\left( \frac{1-t}{2} \right) \, dt \right), \\
&\int_0^1 \left| \frac{1}{3} - t^{\frac{a}{2}} \right| |f'(\frac{1+t}{2}a + \frac{1-t}{2}b)|^q \, dt \\
&\quad \leq \frac{1}{3} \left( |f'(a)|^q \int_0^1 h\left( \frac{1+t}{2} \right) \, dt \\
&\quad \quad + |f'(b)|^q \int_0^1 h\left( \frac{1-t}{2} \right) \, dt \right).
\end{align*}
\] (22)

Using the last two inequalities in (21) we obtain (20). This completes the proof.

\section*{Conflict of Interests}

The author declares that there is no conflict of interests regarding the publication of this paper.

\section*{References}


