Research Article

New Pinning Synchronization of Complex Networks with Time-Varying Coupling Strength and Nondelayed and Delayed Coupling

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The pinning synchronization problem for a class of complex networks is studied by a stochastic viewpoint, in which both time-varying coupling strength and nondelayed and delayed coupling are included. Different from the traditionally similar methods, its interval is separated into two subintervals and described by a Bernoulli variable. Both bounds and switching probability of such subintervals are contained. Particularly, the nondelayed and delayed couplings occur alternately in which another independent Bernoulli variable is introduced. Then, a new kind of pinning controller without time-varying coupling strength signal is developed, in which only its bounds and probabilities are contained. When such probabilities are unavailable, two different kinds of adaption laws are established to make the complex network globally synchronous. Finally, the validity of the presented methods is proved through a numerical example.

1. Introduction

In the recent years, because of many systems in all fields of sciences and society, complex networks (CNs) have drawn a lot of attention. The property of CNs is that a large set of interconnected nodes are contained, and each node is an essential unit. Many natural and manmade systems can be found in our surroundings, such as ecosystems, internet, World Wide Web, social networks, and power grids, which are very important in our daily life. Due to the broad applications, the research of complex behaviors in complex networks has become a hot topic across many fields such as in [1–7].

In complex networks, such as ER random, small-world, and scale-free CNs [8–10], one of the most important collective behaviors is named as synchronization and has been studied extensively. Because of a large number of nodes existing in CNs, it is unrealistic to control CN by adding controllers to all nodes. Then, it is natural to exploit a practical control strategy reducing the number of controllers usually referred to as pinning control [11, 12]. Up to now, a lot of results on pinning synchronization of various CNs have emerged, for example, [13–21]. When there are some parameters unknown,

the problem of adaptive synchronization was reported in [22–24]. If there are constant or time-varying delays in coupling, some delay-dependent criteria for synchronization stability were established in [25–27]. As for CN with nondelayed and delayed coupling, the pinning synchronization problem was considered in [28]. It is seen that the underlying system in this reference has the nondelayed and delayed couplings occurring simultaneously. If such two couplings happen separately with some probability, how to consider the pinning synchronization problem will be necessary and interesting. On the other hand, by investigating the aforementioned works, it is known that the coupling strength (CS) in most of them is constant. When CS is time-varying in a limited range, the global synchronization was realized by adjusting time-varying coupling strength (TCS) [29]. Based on the proposed methods, it is concluded that the adjusted CS will vary in a range, whose lower and upper bounds are finite. Though the given adjusting TCS is time-varying, its bounds used in realizing global synchronization may be different from the ones of the original TCS. Moreover, even if such bounds can be equal in theory, they may not be realized in practice. That is because the ability of instruments or actuators is limited

in many practical applications. Thus, the method without considering the upper bound of CS may be conservative. In addition, the synchronization in [29] achieved by adjusting TCS needs every node state. It will make the realization not easy when CS has a great deal of nodes. Via exploiting the existing pinning algorithms, a state feedback controller will be constructed as one with TCS signal. In this case, the implementation of the designed pinning controller requires such a signal exactly available online. Unfortunately, from the point of application, this assumption is very ideal and hard to be satisfied. To the best of authors' knowledge, the pinning control problem of complex networks with TCS and nondelayed and delayed coupling has not yet been investigated, which motivates the current research.

In this paper, the pinning synchronization of complex networks with TCS and nondelayed and delayed coupling is studied by a stochastic viewpoint. Compared with the existing results of pinning control methods, the main contributions of this paper are as follows: (1) The pinning synchronization is realized by using a stochastic method. The original range of TCS is separated into two different subintervals, whose switching between such subintervals is described by a Bernoulli variable; (2) based on the proposed method, a kind of local state feedback controller is developed such that the resulting system is global synchronization in probability. It is seen that, without depending on TCS signal directly, the desired pinning controller is only related to the bounds and probabilities of such separated subintervals; (3) instead of the nondelayed and delayed couplings existing simultaneously, such couplings act on the underlying system asynchronously, which is modeled by another independent Bernoulli variable; (4) by the proposed results in this paper, it is concluded that such bounds and probabilities play important roles in the pinning synchronization, whose relationships are given and demonstrated in detail; (5) when such probabilities are unknown or inaccessible, two different adaptation laws for global synchronization are proposed, respectively.

Notations. $\mathbb{R}^n$ denotes the n dimensional Euclidean space, $\mathbb{R}^{m \times n}$ is the set of all $m \times n$ real matrices. $\mathbb{C}([-\tau, 0], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[\tau, 0]$ into $\mathbb{R}^n$ with the norm $|\phi| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$, where $|x|$ denotes the usual Euclidean norm of a vector. $\mathcal{E}\{\cdot\}$ is the expectation operator with respect to some probability measure. $\lambda_{\max}(M)$ denote largest eigenvalues of a matrix M . $\mathcal{L}$ is the infinitesimal generator of $V(x, t)$ from $\mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}^n$.

2. Problem Formulation

Consider a class of complex networks consisting of N identical linearly and diffusively coupled nodes; it is described as

$$\begin{aligned} \dot{x}_i(t) = & f(x_i(t)) + \beta(t) c(t) \sum_{j=1}^N a_{ij} x_j(t) \\ & + (1 - \beta(t)) c(t) \sum_{j=1}^N b_{ij} x_j(t - \tau), \quad i \in \mathbb{S}, \end{aligned} \quad (1)$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t), \dots, x_{in}(t))^T \in \mathbb{R}^n$, $f : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is a differentiable function and is continuous, $\tau > 0$ is the coupling delay, $c(t) \geq 0$ is the time-varying coupling strength and varies in $\Omega \triangleq [c_1, c_2]$ with $0 \leq c_1 \leq c_2 < \infty$, and matrices $A \triangleq (a_{ij})_{N \times N}$ and $B \triangleq (b_{ij})_{N \times N}$ are the coupling configuration matrices and irreducible, whose elements a_{ij} and b_{ij} are described as follows: $a_{ij} = a_{ji} \geq 0$, for $i \neq j$, $a_{ij} = -\sum_{j=1, i \neq j}^N a_{ij}$, and $b_{ij} = b_{ji} \geq 0$, for $i \neq j$, and $b_{ij} = -\sum_{j=1, i \neq j}^N b_{ij}$, $i, j \in \mathbb{S} \triangleq \{1, 2, \dots, N\}$. The initial conditions are given by $x_{i0}(t) = \phi_i(t) \in \mathbb{C}([-\tau, 0], \mathbb{R}^n)$, $\forall t \in [-\tau, 0]$. $\beta(t)$ is a Bernoulli variable and is described as follows:

$$\beta(t) = \begin{cases} 1, & \text{if nondelayed coupling acts,} \\ 0, & \text{if delayed coupling acts} \end{cases} \quad (2)$$

whose probability is

$$\begin{aligned} \mathbb{P}\{\beta(t) = 1\} &= \mathcal{E}\{\beta(t)\} = \beta^*, \\ \mathbb{P}\{\beta(t) = 0\} &= 1 - \beta^*. \end{aligned} \quad (3)$$

In this paper, TCS $c(t)$ varies in $[c_1, c_2]$. That is, for any given constant $\hat{c} \in [c_1, c_2]$, $c(t)$ will take value in $[c_1, \hat{c}]$ or $(\hat{c}, c_2]$ randomly. Without loss of generality, $\hat{c}$ can be chosen as $\hat{c} = (1/2)(c_1 + c_2)$. So for any $c(t)$, two sets are described as

$$\begin{aligned} \Omega_1 &\triangleq \{c(t) : c(t) \in [c_1, \hat{c}]\}, \\ \Omega_2 &\triangleq \{c(t) : c(t) \in (\hat{c}, c_2]\}. \end{aligned} \quad (4)$$

A Bernoulli variable $\alpha(t)$ is described as

$$\alpha(t) = \begin{cases} 0, & c(t) \in \Omega_1, \\ 1, & c(t) \in \Omega_2 \end{cases} \quad (5)$$

with

$$\begin{aligned} \mathbb{P}\{\alpha(t) = 1\} &= \mathcal{E}\{\alpha(t)\} = \alpha^*, \\ \mathbb{P}\{\alpha(t) = 0\} &= 1 - \alpha^*, \end{aligned} \quad (6)$$

where $1 - \alpha^*$ and α^* denote the probabilities of $c(t)$ taking values in Ω_1 and Ω_2 , respectively. Moreover, it can be readily verified that

$$\mathcal{E}\{\alpha(t) - \alpha^*\} = 0. \quad (7)$$

Then, $c(t)$ is expressed as

$$c(t) = (1 - \alpha(t)) c_1(t) + \alpha(t) c_2(t), \quad (8)$$

where $c_1(t)$ and $c_2(t)$ are described as

$$\begin{aligned} c_1(t) &\triangleq \begin{cases} c(t), & \text{if } \alpha(t) = 0, \\ \hat{c}, & \text{if } \alpha(t) = 1, \end{cases} \\ c_2(t) &\triangleq \begin{cases} \hat{c}, & \text{if } \alpha(t) = 0, \\ c(t), & \text{if } \alpha(t) = 1. \end{cases} \end{aligned} \quad (9)$$

Then, system (1) is equivalent to

$$\begin{aligned} \dot{x}_i(t) = & f(x_i(t)) + \beta(t)\theta(t) \sum_{j=1}^N a_{ij}x_j(t) \\ & + (1 - \beta(t))\theta(t) \sum_{j=1}^N b_{ij}x_j(t - \tau), \quad i \in \mathbb{S}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} \theta(t) &= \theta_1(t) + \theta_2(t), \\ \theta_1(t) &= (\alpha(t) - \alpha^*)(c_2(t) - c_1(t)), \\ \theta_2(t) &= (1 - \alpha^*)c_1(t) + \alpha^*c_2(t). \end{aligned} \quad (11)$$

Let $s(t)$ be a solution to the isolated node of CN; we have

$$\dot{s}(t) = f(s(t)), \quad (12)$$

where $s(t)$ may be an equilibrium point, a nontrivial periodic orbit, or even a chaotic attractor. For complex network (10) with stochastic variable, some assumptions, definitions, and lemmas are needed to derive our main results.

Assumption 1 (see [15]). Supposing $P = \text{diag}(p_1, p_2, \dots, p_n)$ is a positive definite diagonal matrix and $\Delta = \text{diag}(\delta_1, \delta_2, \dots, \delta_n)$ is a diagonal matrix, then, for some $\eta > 0$, all $x, y \in \mathbb{R}^n$, and $t > 0$, $f(x, t)$ can make the following inequality establish:

$$\begin{aligned} (x - y)^T P (f(x) - f(y) - \Delta x + \Delta y) \\ \leq -\eta (x - y)^T (x - y). \end{aligned} \quad (13)$$

Definition 2. Complex network (5) is said to achieve globally asymptotical synchronization almost surely, for any initial conditions x_{i0} and s_0 , if the following condition

$$\mathbb{P} \left\{ \lim_{t \rightarrow \infty} |x_i(t; x_{i0}) - s(t, s_0)| = 0 \right\} = 1, \quad i \in \mathbb{S} \quad (14)$$

holds.

Definition 3. The solution $x(t)$ to complex network (5) is said to be bounded almost surely if

$$\sup_{t \geq 0} |x(t; x_0)| < \infty \quad \text{a.s.} \quad (15)$$

holds for all $x_0 \in \mathbb{R}^{nN}$ with $x(t) = (x_1^T(t), x_2^T(t), \dots, x_N^T(t))^T$.

Lemma 4 (see [15]). Suppose $A \in \mathbb{R}^{N \times N}$ is in accordance with the following conditions:

- (1) $a_{ij} \geq 0$, for $i \neq j$, and $a_{ij} = -\sum_{j=1, i \neq j}^N a_{ij}$, $\forall i, j \in \mathbb{S}$;
- (2) A is irreducible; then one has the following:

- (i) Real parts of the eigenvalues of A are all negative, in addition to an eigenvalue 0 with the multiplicity 1.

- (ii) A has the right eigenvector $(1, 1, \dots, 1)^T$ corresponding to the eigenvalue 0.
- (iii) Let $\xi = (\xi_1, \xi_2, \dots, \xi_N)^T$ be the left eigenvector of A corresponding to the eigenvalue 0; then one can let $\xi_i > 0$, $i \in \mathbb{S}$.

Lemma 5. If $A = (a_{ij})_{N \times N}$ is irreducible with $a_{ij} = a_{ji} \geq 0$ for $i \neq j$ and $\sum_{j=1}^N a_{ij} = 0$ for $i \in \mathbb{S}$, then all the eigenvalues of the matrix

$$\tilde{A} \triangleq (\tilde{a}_{ij})_{N \times N} = A - K \quad (16)$$

are negative, where $K \triangleq \text{diag}\{k_1, k_2, \dots, k_N\}$ with $k_i > 0$, $i = 1, 2, \dots, l$, and $k_i = 0$, $i \in \mathbb{S}_l \triangleq \{1, 2, \dots, l\}$.

Proof. Based on the result [12], it can be obtained directly. Thus it is omitted here.

If A is an irreducible coupling matrix, let $\xi = (\xi_1, \xi_2, \dots, \xi_N)^T$ be the left eigenvector of A corresponding to the eigenvalue 0 and with $\xi_i > 0$, one has that matrix $\Xi = \text{diag}(\xi_1, \xi_2, \dots, \xi_N)$ is positive definite. If A is irreducible, we have

$$\begin{aligned} (\Xi A + A^T \Xi) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} &= \Xi A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + A^T \Xi \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\ &= \Xi A \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + A^T \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix} = 0 \end{aligned} \quad (17)$$

which is irreducible. Letting $(\Xi \tilde{A})^s = (1/2)(\Xi \tilde{A} + \tilde{A}^T \Xi)$, it is known from Lemma 5 that $(\Xi \tilde{A})^s$ is negative definite. $\square$

Lemma 6. For any given appropriate matrices Ω , Φ_1 , and Φ_2 and scalars $\underline{\lambda}_1 \leq \bar{\lambda}_1$ and $\underline{\lambda}_2 \leq \bar{\lambda}_2$, then

$$\begin{aligned} \Omega + \lambda_1(t) \Phi_1 + \lambda_2(t) \Phi_2 &< 0, \\ \lambda_1(t) &\in [\underline{\lambda}_1, \bar{\lambda}_1], \quad \lambda_2(t) \in [\underline{\lambda}_2, \bar{\lambda}_2] \end{aligned} \quad (18)$$

if and only if

$$\Omega + \underline{\lambda}_1 \Phi_1 + \underline{\lambda}_2 \Phi_2 < 0, \quad (19)$$

$$\Omega + \underline{\lambda}_1 \Phi_1 + \bar{\lambda}_2 \Phi_2 < 0, \quad (20)$$

$$\Omega + \bar{\lambda}_1 \Phi_1 + \underline{\lambda}_2 \Phi_2 < 0, \quad (21)$$

$$\Omega + \bar{\lambda}_1 \Phi_1 + \bar{\lambda}_2 \Phi_2 < 0. \quad (22)$$

Proof. The necessity is obvious, since (18) holds. Next, we prove the sufficiency. From $\lambda_1(t) \in [\underline{\lambda}_1, \bar{\lambda}_1]$ and $\lambda_2(t) \in [\underline{\lambda}_2, \bar{\lambda}_2]$, it is concluded that

$$\begin{aligned} \lambda_1(t) &= t_1 \underline{\lambda}_1 + (1 - t_1) \bar{\lambda}_1, \quad \forall t_1 \in [0, 1], \\ \lambda_2(t) &= t_2 \underline{\lambda}_2 + (1 - t_2) \bar{\lambda}_2, \quad \forall t_2 \in [0, 1]. \end{aligned} \quad (23)$$

Then, from (19) $\times t_1$ plus (21) $\times (1 - t_1)$, one gets

$$\Omega + \underline{\lambda}_1(t) \Phi_1 + \underline{\lambda}_2 \Phi_2 \leq 0. \quad (24)$$

Similarly,

$$\Omega + \underline{\lambda}_1(t) \Phi_1 + \bar{\lambda}_2 \Phi_2 \leq 0. \quad (25)$$

By applying similar method to (24) and (25), we obtain

$$\Omega + \lambda_1(t) \Phi_1 + \lambda_2(t) \Phi_2 \leq 0, \quad (26)$$

where “=” is satisfied if and only if $t_1 = 0$ or $t_1 = 1$ and $t_2 = 0$ or $t_2 = 1$ hold simultaneously. However, without loss of generality, if $t_1 = 0$ and $t_2 = 0$, it is seen that

$$\Omega + \bar{\lambda}_1 \Phi_1 + \bar{\lambda}_2 \Phi_2 \leq 0 \quad (27)$$

which is contradicted with (22). This completes the proof. $\square$

Here, the objective of pinning synchronization is to achieve $\mathbb{P}\{\lim_{t \rightarrow \infty} |x_i(t; x_{i0}) - s(t; s_0)| = 0\} = 1$, $i \in \mathbb{S}$, by using pinning control strategy on a small fraction δ ($0 < \delta \ll 1$) of the total nodes in (5). Suppose that nodes $i_1, i_2, \dots, i_l$ are selected, where $l = \lfloor \delta N \rfloor$ is the smaller, but a nearest integer to the real number δN . Without loss of generality, assume that the first l nodes are selected to be pinned. A kind of local pinning controller without $c(t)$ but containing its bounds and probability distributions is designed as

$$\begin{aligned} \dot{x}_i(t) = & f(x_i(t)) + \beta(t) \theta(t) \sum_{j=1}^N a_{ij} x_j(t) \\ & + (1 - \beta(t)) \theta(t) \sum_{j=1}^N b_{ij} x_j(t - \tau) + u_i(t), \\ & i \in \mathbb{S}_l, \end{aligned} \quad (28)$$

$$\begin{aligned} \dot{x}_i(t) = & f(x_i(t)) + \beta(t) \theta(t) \sum_{j=1}^N a_{ij} x_j(t) \\ & + (1 - \beta(t)) \theta(t) \sum_{j=1}^N b_{ij} x_j(t - \tau), \quad i \in \bar{\mathbb{S}}, \end{aligned}$$

where

$$\begin{aligned} u_i(t) = & -\beta^* \bar{c} k_i (x_i(t) - s(t)), \quad k_i > 0, \\ \bar{c} = & (1 - \alpha^*) \bar{c} + \alpha^* c_2, \\ \bar{\mathbb{S}} = & \mathbb{S} - \mathbb{S}_l. \end{aligned} \quad (29)$$

Letting $e_i(t) = x_i(t) - s(t)$, we have the error system as

$$\begin{aligned} \dot{e}_i(t) = & f(x_i(t)) - f(s(t)) \\ & + (1 - \beta(t)) \theta(t) \sum_{j=1}^N b_{ij} x_j(t - \tau) \\ & + \beta(t) \theta(t) \sum_{j=1}^N a_{ij} x_j(t) - k e_i(t), \quad i \in \mathbb{S}, \end{aligned} \quad (30)$$

where $k_i = 0$, $i \in \bar{\mathbb{S}}$.

Remark 7. It is worth mentioning that, compared with some existing results such as [5, 11, 18, 28, 29], complex network (28) in this paper is more general and has its properties. Firstly, the underlying system is more general in terms of having time-varying coupling strength, where nondelayed and delayed couplings take place synchronously. Secondly, a new kind of controller depending on its lower and upper bounds in addition to probabilities will be designed to realize the pinning synchronization, which is without TCS signal. However, in order to achieve this aim, new problems will be confronted, which will be mentioned below and should be dealt with.

3. Main Results

Theorem 8. Suppose Assumption 1 holds and A is a irreducible coupling matrix. If $k_1, k_2, \dots, k_l$ are positive constants and

$$\begin{aligned} 2\delta_j \Xi + 2\beta^* \bar{c} (\Xi \bar{A})^s + I_N + (1 - \beta^*)^2 \bar{c}^2 \lambda_{\max}^2(\Xi B) \\ < 0, \quad j \in \mathbb{T} \\ 2\delta_j \Xi + 2\beta^* \bar{c} (\Xi \bar{A})^s + I_N + (1 - \beta^*)^2 \bar{c}^2 \lambda_{\max}^2(\Xi B) \\ < 0, \quad j \in \mathbb{T}, \end{aligned} \quad (31)$$

where $\lambda_{\max}(\Xi B)$ is the spectral radius of ΞB , $\bar{c} = (1 - \alpha^*) c_1 + \alpha^* \bar{c}$, and $\mathbb{T} \triangleq \{1, \dots, n\}$, complex network (28) achieves globally asymptotical synchronization almost surely.

Proof. The Lyapunov function is defined as

$$\begin{aligned} V(t) = & \frac{1}{2} \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) \\ & + \frac{1}{2} \sum_{j=1}^n p_j \int_{t-\tau}^t \bar{e}_j^T(s) \bar{e}_j(s) ds, \end{aligned} \quad (32)$$

where $\xi_i > 0$ is the element of the left eigenvector of A corresponding the eigenvalue 0, and $\bar{e}_j(t) = (e_{1j}(t), e_{2j}(t), \dots, e_{Nj}(t))^T \in \mathbb{R}^N$, $j \in \mathbb{T}$. Then we have

$$\begin{aligned} \mathcal{L}V(t) = & \sum_{i=1}^N \xi_i e_i^T(t) P [f(x_i(t)) - f(s(t)) - \Delta e_i(t)] \\ & + \sum_{i=1}^N \xi_i e_i^T P \Delta e_i(t) + \sum_{i=1}^N \xi_i \theta_2(t) e_i^T(t) \\ & \cdot P \left[\beta^* \sum_{j=1}^N a_{ij} e_j(t) + (1 - \beta^*) \sum_{j=1}^N b_{ij} e_j(t - \tau) \right] \\ & - \bar{c} \beta^* \sum_{q=1}^l \xi_q k_q e_q^T(t) P e_q(t) + \frac{1}{2} \\ & \cdot \sum_{j=1}^n p_j [\bar{e}_j^T(t) \bar{e}_j^T(t) - \bar{e}_j^T(t - \tau) \bar{e}_j^T(t - \tau)] \end{aligned}$$

$$\begin{aligned}
 & \leq \sum_{i=1}^N \xi_i e_i^T(t) P [f(x_i(t)) - f(s(t)) - \Delta e_i(t)] \\
 & + \sum_{i=1}^N \xi_i e_i^T P \Delta e_i(t) + \sum_{i=1}^N \xi_i \theta_2(t) e_i^T(t) \\
 & \cdot P \left[\beta^* \sum_{j=1}^N a_{ij} e_j(t) + (1 - \beta^*) \sum_{j=1}^N b_{ij} e_j(t - \tau) \right] \\
 & - \theta_2(t) \beta^* \sum_{q=1}^l \xi_q k_q e_q^T(t) P e_q(t) + \frac{1}{2} \\
 & \cdot \sum_{j=1}^n p_j [\tilde{e}_j^T(t) \tilde{e}_j^T(t) - \tilde{e}_j^T(t - \tau) \tilde{e}_j^T(t - \tau)] \\
 & \leq -\eta \sum_{i=1}^N \xi_i e_i^T(t) e_i(t) + \sum_{j=1}^n p_j \delta_j \tilde{e}_j^T(t) \Xi \tilde{e}_j(t) \\
 & + \beta^* \theta_2(t) \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi \tilde{A} \tilde{e}_j^T(t) + (1 - \beta^*) \theta_2(t) \\
 & \cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j^T(t - \tau) + \frac{1}{2} \\
 & \cdot \sum_{j=1}^n p_j [\tilde{e}_j^T(t) \tilde{e}_j^T(t) - \tilde{e}_j^T(t - \tau) \tilde{e}_j^T(t - \tau)] \\
 & \leq -\eta \sum_{i=1}^N \xi_i e_i^T(t) e_i(t) + \sum_{j=1}^n p_j \delta_j \tilde{e}_j^T(t) \Xi \tilde{e}_j(t) \\
 & + \beta^* \theta_2(t) \sum_{j=1}^n p_j \tilde{e}_j^T(t) (\Xi \tilde{A})^s \tilde{e}_j^T(t) + (1 - \beta^*) \theta_2(t) \\
 & \cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j^T(t - \tau) + \frac{1}{2} \\
 & \cdot \sum_{j=1}^n p_j [\tilde{e}_j^T(t) \tilde{e}_j^T(t) - \tilde{e}_j^T(t - \tau) \tilde{e}_j^T(t - \tau)] \\
 & = -\eta \sum_{i=1}^N \xi_i e_i^T(t) e_i(t) + \frac{1}{2} \sum_{j=1}^n p_j (\tilde{e}_j^T(t), \tilde{e}_j^T(t - \tau)) \\
 & \cdot Z_j(t) (\tilde{e}_j(t), \tilde{e}_j(t - \tau)) < 0,
 \end{aligned} \tag{33}$$

where

$$\begin{aligned}
 & Z_j(t) \\
 & = \begin{bmatrix} 2\delta_j \Xi + 2\beta^* \theta_2(t) (\Xi \tilde{A})^s + I_N & (1 - \beta^*) \theta_2(t) \Xi B \\ (1 - \beta^*) \theta_2(t) (\Xi B)^T & -I_N \end{bmatrix}, \tag{34} \\
 & j \in \mathbb{T}.
 \end{aligned}$$

By Lemma 6, it is concluded that $Z_j(t) < 0$ if

$$\begin{aligned}
 & 2\delta_j \Xi + 2\beta^* \tilde{c} (\Xi \tilde{A})^s + I_N + (1 - \beta^*)^2 \tilde{c}^2 (\Xi B) (\Xi B)^T \\
 & < 0, \quad j \in \mathbb{T}, \\
 & 2\delta_j \Xi + 2\beta^* \tilde{c} (\Xi \tilde{A})^s + I_N + (1 - \beta^*)^2 \tilde{c}^2 (\Xi B) (\Xi B)^T \\
 & < 0, \quad j \in \mathbb{T}
 \end{aligned} \tag{35}$$

hold, which could be obtained by (31). Then, (33) holds. Thus, there is always a small scalar $\mu > 0$ such that

$$\begin{aligned}
 & \mathcal{E} \{V(e_t, t)\} - \mathcal{E} \{V(e_0, 0)\} \\
 & \leq -\mu \mathcal{E} \left\{ \int_0^t e^T(s) e(s) ds \right\}
 \end{aligned} \tag{36}$$

which implies

$$\mathcal{E} \left\{ \int_0^t e^T(s) e(s) ds \right\} \leq \frac{1}{\mu} \mathcal{E} \{V(e_0, 0)\} < \infty. \tag{37}$$

By Dynkins formula, one gets

$$\lim_{t \rightarrow \infty} \mathcal{E} \{|e(t)|^2\} = 0 \tag{38}$$

which implies

$$\mathbb{P} \left\{ \lim_{t \rightarrow \infty} |e(t)| = 0 \right\} = 1. \tag{39}$$

This completes the proof. $\square$

Remark 9. From Theorem 8, it is seen that both the lower bounds of the separated subintervals and the probabilities α^* and β^* are involved. By using the existing methods dealing with TCS $c(t)$, it is known that only its lower bound is considered. Thus, for any given K , it is concluded that the lower bound c_1 obtained by Theorem 8 will be smaller than one obtained by the traditional methods only using its lower bound. In this sense, it is said that our result is less conservative. In other words, the range of $c(t)$ determined by (28) is usually larger than one obtained by considering c_1 only. Particularly, it is worth mentioning that such Bernoulli variables can also be replaced with Markov processes such as [30], since Bernoulli process is a special case of Markov process.

It is worth mentioning that the pinning controller in (28) needs α^* and β^* to be known. When either of α^* and β^* is unavailable, how to synchronize the complex network via using a pinning controller without $c(t)$ signal and to estimate such probabilities should be considered. In the following, adaptive pinning control laws will be developed to deal with them separately.

Theorem 10. Suppose Assumption 1 holds and the coupling matrix A is irreducible. If $k_1, \dots, k_l$ are positive constants and

$$\begin{aligned} & 2\delta_j \Xi + 2\beta^* c_1 (\Xi \tilde{A})^s + I_N + (1 - \beta^*)^2 c_1^2 \lambda_{\max}^2 (\Xi B) I_N \\ & + c_2 I_N < 0, \\ & 2\delta_j \Xi + 2\beta^* \hat{c} (\Xi \tilde{A})^s + I_N + (1 - \beta^*)^2 \hat{c}^2 \lambda_{\max}^2 (\Xi B) I_N \quad (40) \\ & + c_2 I_N < 0, \\ & \beta^* (\Xi \tilde{A})^s + \frac{1}{4} (1 - \beta^*)^2 \lambda_{\max}^2 (\Xi B) I_N < 0 \end{aligned}$$

hold for $\forall j \in \mathbb{T}$, complex network (28) achieves globally asymptotical synchronization almost surely under the adaptive pinning controller

$$u_i(t) = -\beta^* \hat{c} k_i e_i(t) + v_i(t), \quad i \in \mathbb{S}_l, \quad (41)$$

where $v_i(t) = -\hat{\alpha}(t) e_i(t)$, and the updating law

$$\hat{\alpha}(t) = \begin{cases} 0, & \text{if } \hat{\alpha}(t) = 1 \\ \rho \sum_{i=1}^l \xi_i e_i^T(t) P e_i(t), & \text{otherwise} \end{cases} \quad (42)$$

with $\forall \rho > 0$ and $\hat{\alpha}_0 \in [0, 1]$.

Proof. Choose a Lyapunov function

$$\begin{aligned} V(t) &= \frac{1}{2} \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) \\ &+ \frac{1}{2} \sum_{j=1}^n p_j \int_{t-\tau}^t \tilde{e}_j^T(s) \tilde{e}_j(s) ds + \frac{1}{2\rho} \tilde{\alpha}(t) \tilde{\alpha}(t), \end{aligned} \quad (43)$$

where $\tilde{\alpha}(t) = \hat{\alpha}(t) - \alpha^*$. Then, for $\forall \zeta(t) = [e^T(t), \tilde{\alpha}^T(t)]^T \neq 0$, we have

$$\begin{aligned} \mathcal{L}V(t) &= \sum_{i=1}^N \xi_i e_i^T(t) P [f(x_i(t)) - f(s(t)) - \Delta e_i(t)] \\ &+ \sum_{i=1}^N \xi_i e_i^T(t) P \Delta e_i(t) + \beta^* c_1(t) \sum_{i=1}^N \xi_i e_i^T(t) P \\ &\cdot \sum_{j=1}^N a_{ij} e_j(t) + \beta^* \tilde{\theta}(t) \sum_{i=1}^N \xi_i e_i^T(t) P \sum_{j=1}^N a_{ij} e_j(t) \\ &+ (1 - \beta^*) c_1(t) \sum_{i=1}^N \xi_i e_i^T(t) P \sum_{j=1}^N b_{ij} e_j(t - \tau) \\ &+ \frac{1}{\rho} (\hat{\alpha}(t) - \alpha^*) \dot{\hat{\alpha}}(t) + (1 - \beta^*) \tilde{\theta}(t) \sum_{i=1}^N \xi_i e_i^T(t) P \\ &\cdot \sum_{j=1}^N b_{ij} e_j(t - \tau) - \beta^* \hat{c} \sum_{q=1}^l \xi_q k_q e_q^T(t) P e_q(t) + \frac{1}{2} \\ &\cdot \sum_{j=1}^n p_j [\tilde{e}_j^T(t) \tilde{e}_j(t) - \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau)] \end{aligned}$$

$$\begin{aligned} &+ \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) \leq -\eta \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) \\ &+ \sum_{j=1}^n p_j \delta_j \tilde{e}_j^T(t) \Xi \tilde{e}_j(t) + \beta^* c_1(t) \\ &\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi \tilde{A} \tilde{e}_j(t) + (1 - \beta^*) c_1(t) \\ &\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) + \beta^* \tilde{\theta}(t) \\ &\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi \tilde{A} \tilde{e}_j(t) + (1 - \beta^*) \tilde{\theta}(t) \\ &\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) + \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) + \frac{1}{2} \\ &\cdot \sum_{j=1}^n p_j [\tilde{e}_j^T(t) \tilde{e}_j(t) - \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau)] \\ &+ \frac{1}{\rho} (\hat{\alpha}(t) - \alpha^*) \dot{\hat{\alpha}}(t) \leq -\eta \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) + \beta^* \\ &+ \sum_{j=1}^n p_j \delta_j \tilde{e}_j^T(t) \Xi \tilde{e}_j(t) + \beta^* c_1(t) \\ &\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) (\Xi \tilde{A})^s \tilde{e}_j(t) + (1 - \beta^*) c_1(t) \\ &\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) \tilde{\theta}(t) \\ &\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) (\Xi \tilde{A})^s \tilde{e}_j(t) + (1 - \beta^*) \tilde{\theta}(t) \\ &\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) + \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) + \frac{1}{2} \\ &\cdot \sum_{j=1}^n p_j [\tilde{e}_j^T(t) \tilde{e}_j(t) - \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau)] \\ &+ \frac{1}{\rho} (\hat{\alpha}(t) - \alpha^*) \dot{\hat{\alpha}}(t) = -\eta \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) \\ &+ \sum_{j=1}^n p_j \tilde{e}_j^T(t) \left[\delta_j \Xi + \beta^* c_1(t) (\Xi \tilde{A})^s + \frac{1}{2} I_N \right] \tilde{e}_j(t) \\ &+ (1 - \beta^*) c_1(t) \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) - \frac{1}{2} \\ &\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) + \beta^* \tilde{\theta}(t) \end{aligned}$$

$$\begin{aligned}
& \cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) (\Xi \tilde{A})^s \tilde{e}_j(t) + (1 - \beta^*) \tilde{\theta}(t) \\
& \cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) + \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) \\
& + \frac{1}{\rho} (\hat{\alpha}(t) - \alpha^*) \dot{\hat{\alpha}}(t) - \tilde{\theta}(t) \\
& \cdot \sum_{i=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) + \tilde{\theta}(t) \\
& \cdot \sum_{i=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) \leq -\eta \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) \\
& + \sum_{j=1}^n p_j \tilde{e}_j^T(t) \left[\delta_j \Xi + \beta^* c_1(t) (\Xi \tilde{A})^s + \frac{1}{2} I_N \right] \tilde{e}_j(t) \\
& + (1 - \beta^*) c_1(t) \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) - \frac{1}{2} \\
& \cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) + \beta^* \tilde{\theta}(t) \\
& \cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) (\Xi \tilde{A})^s \tilde{e}_j(t) + (1 - \beta^*) \tilde{\theta}(t) \\
& \cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) + \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) \\
& + \frac{1}{\rho} (\hat{\alpha}(t) - \alpha^*) \dot{\hat{\alpha}}(t) - \tilde{\theta}(t) \\
& \cdot \sum_{i=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) \\
& + c_2 \sum_{i=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) = -(\eta + \alpha^*) \\
& \cdot \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) + \frac{1}{2} \\
& \cdot \sum_{j=1}^n p_j (\tilde{e}_j^T(t), \tilde{e}_j^T(t - \tau)) Z_j(t) (\tilde{e}_j(t), \tilde{e}_j(t - \tau)) \\
& + \tilde{\theta}(t) \\
& \cdot \sum_{j=1}^n p_j (\tilde{e}_j^T(t), \tilde{e}_j^T(t - \tau)) M_j (\tilde{e}_j(t), \tilde{e}_j(t - \tau)) \\
& \leq 0,
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
& Z_j(t) \\
& = \begin{bmatrix} 2\delta_j \Xi + 2\beta^* c_1(t) (\Xi \tilde{A})^s + I_N + c_2 I_N & (1 - \beta^*) c_1(t) \Xi B \\ (1 - \beta^*) c_1(t) (\Xi B)^T & -I_N \end{bmatrix}, \\
& j \in \mathbb{T} \tag{45}
\end{aligned}$$

$$M_j = \begin{bmatrix} \beta^* (\Xi \tilde{A})^s & \frac{1}{2} (1 - \beta^*) \Xi B \\ \frac{1}{2} (1 - \beta^*) (\Xi B)^T & -I_N \end{bmatrix}, \quad j \in \mathbb{T}$$

$$\tilde{\theta}(t) = \alpha^* (c_2(t) - c_1(t)).$$

By Lemma 6, it is known that $Z_j(t) < 0$ if

$$\begin{aligned}
& 2\delta_j \Xi + 2\beta^* c_1(t) (\Xi \tilde{A})^s + I_N + (1 - \beta^*)^2 c_1^2(t) (\Xi B) (\Xi B)^T \\
& + c_2 I_N < 0, \\
& 2\delta_j \Xi + 2\beta^* \hat{c} (\Xi \tilde{A})^s + I_N + (1 - \beta^*)^2 \hat{c}^2 (\Xi B) (\Xi B)^T \\
& + c_2 I_N < 0
\end{aligned} \tag{46}$$

hold for $j \in \mathbb{S}$. Similarly, $M_j < 0$ is equivalent to

$$\beta^* (\Xi \tilde{A})^s + \frac{1}{4} (1 - \beta^*)^2 (\Xi B) (\Xi B)^T < 0. \tag{47}$$

From conditions (31)-(32), it is obtained that (46)-(47) hold. Then, one has (44) which implies

$$\mathcal{E}\{V(\zeta(t), t)\} \leq \mathcal{E}\{V(\zeta_0, 0)\}. \tag{48}$$

For any $t \geq 0$, $k > 0$, when $\sup_{0 \leq s \leq t} |\zeta(s)| \geq k$, and from (48), one obtains

$$\begin{aligned}
& \mathbb{P} \left\{ \sup_{0 \leq s \leq t} |\zeta(s)| \geq k \right\} \leq \inf_{|\zeta| \geq k, t \geq 0} V(\zeta(t), t) \\
& \leq \mathcal{E}\{V(\zeta(t), t)\} \leq \mathcal{E}\{V(\zeta_0, 0)\}
\end{aligned} \tag{49}$$

which implies

$$\mathbb{P} \left\{ \sup_{0 \leq s \leq t} |\zeta(s)| \geq k \right\} \leq \frac{\mathcal{E}\{V(\zeta_0, 0)\}}{\inf_{|\zeta| \geq k, t \geq 0} V(\zeta(t), t)}. \tag{50}$$

From (43), it is concluded that $\inf_{|\zeta| \geq k, t \geq 0} V(\zeta(t), t) \rightarrow \infty$, as $k \rightarrow \infty$. Based on this and taking into account (50), one concludes that $\zeta(t)$ is bounded almost surely. On the other hand, because of (44), there is $\mu > 0$ such that

$$\begin{aligned}
& \mathcal{E}\{V(\zeta(t), t)\} - \mathcal{E}\{V(\zeta_0, 0)\} \\
& \leq -\mu \mathcal{E} \left\{ \int_0^t e^T(s) e(s) ds \right\}.
\end{aligned} \tag{51}$$

Then, it is directly obtained that

$$\mathcal{E} \left\{ \int_0^t e^T(s) e(s) ds \right\} \leq \frac{1}{\mu} \mathcal{E}\{V(\zeta_0, 0)\} < \infty. \tag{52}$$

Similar to the proof of (37), it is concluded that complex network (28) plus adaptive pinning controller (41) is asymptotical synchronization almost surely and the unknown probability α^* is estimated. This completes the proof. $\square$

Remark 11. Compared with some existing references, the adaptive pinning controller (33) has its properties which are listed as follows. Firstly, the adaptive controller is used to estimate a scalar in a range and having an upper finite bound, while the other results such as [20, 22, 28] are used to estimate control gains and without this restriction. Secondly, when $c(t)$ varies in a range, a new class of pinning controller in this paper is designed without signal $c(t)$, which is different from that in [29]. Moreover, instead of $c(t)$ without an upper bound, it is more practical that it is in a finite range. Thirdly, some adaption controllers such as in [17, 22, 24] should be added to all the nodes of complex network. It is impossible to realize these controllers for complex network with large nodes. In addition, the adaption controller in [17] can cause chattering as the error state can reach the discontinuity region before the complex network achieves the global synchronization. However, this is not true for adaption

controller (33) which is also on a small fraction of nodes in complex networks.

Theorem 12. Suppose Assumption 1 holds and the coupling matrix A is irreducible. If $k_1, \dots, k_l$ are positive constants and satisfy

$$\begin{aligned} 2\delta_j \Xi + (1 - 2\tilde{c}) I_N + \tilde{c}^2 \lambda_{\max}^2(\Xi B) I_N &< 0, \quad j \in \mathbb{T} \\ 2\delta_j \Xi + (1 - 2\tilde{c}) I_N + \tilde{c}^2 \lambda_{\max}^2(\Xi B) I_N &< 0, \quad j \in \mathbb{T} \quad (53) \\ (\Xi \tilde{A})^s + \frac{1}{4} (\Xi B) (\Xi B)^T &< 0 \end{aligned}$$

the complex network achieves globally asymptotical synchronization almost surely under the adaptive pinning controller

$$u_i(t) = -\tilde{c}k_i e_i(t) + v_i(t), \quad i \in \mathbb{S}_l,$$

$$v_i(t) = \begin{cases} 0, & \text{if } \sum_{i=1}^l |e_i(t)|^2 = 0, \\ -\frac{P^{-1}e_i(t)}{\sum_{i=1}^l \xi_i |e_i(t)|^2} \tilde{c} [\Phi(e(t)) + \hat{\beta}(t) \Phi(e(t-\tau))], & \text{otherwise} \end{cases} \quad (54)$$

with $\Phi(e(t)) = \sum_{i=1}^N e_i^T(t) P e_i(t)$ and the updating laws

$$\dot{\hat{\beta}}(t) = \begin{cases} 0, & \text{if } \hat{\beta}(t) = 1 \\ \rho \tilde{c} \sum_{i=1}^N e_i^T(t-\tau) P e_i(t-\tau), & \text{otherwise} \end{cases} \quad (55)$$

with $\forall \rho > 0$ and $\hat{\beta}_0 \in [0, 1]$.

Proof. Define the Lyapunov function as

$$\begin{aligned} V(t) &= \frac{1}{2} \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) \\ &\quad + \frac{1}{2} \sum_{j=1}^n p_j \int_{t-\tau}^t \tilde{e}_j^T(s) \tilde{e}_j(s) ds + \frac{1}{2\rho} \tilde{\beta}(t) \tilde{\beta}(t), \end{aligned} \quad (56)$$

where ξ_i is the element of the left eigenvector of A corresponding to the eigenvalue 0 and $\tilde{\beta}(t) = \hat{\beta}(t) - \beta^*$. Then, for $\forall \zeta(t) = [e^T(t), \tilde{\alpha}^T(t)]^T \neq 0$, we have

$$\begin{aligned} \mathcal{L}V(t) &= \sum_{i=1}^N \xi_i e_i^T(t) P [f(x_i(t)) - f(s(t)) - \Delta e_i(t)] \\ &\quad + \sum_{i=1}^N \xi_i e_i^T(t) P \Delta e_i(t) + \beta^* \theta_2(t) \sum_{i=1}^N \xi_i e_i^T(t) P \\ &\quad \cdot \sum_{j=1}^N a_{ij} e_j(t) + (1 - \beta^*) \theta_2(t) \sum_{i=1}^N \xi_i e_i^T(t) P \end{aligned}$$

$$\begin{aligned} &\cdot \sum_{j=1}^N b_{ij} e_j(t-\tau) - \tilde{c} \sum_{q=1}^l \xi_q k_q e_q^T(t) P e_q(t) + \frac{1}{2} \\ &\cdot \sum_{j=1}^n p_j [\tilde{e}_j^T(t) \tilde{e}_j(t) - \tilde{e}_j^T(t-\tau) \tilde{e}_j(t-\tau)] \\ &+ \frac{1}{\rho} (\hat{\beta}(t) - \beta^*) \dot{\hat{\beta}}(t) + \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) \\ &\leq \sum_{i=1}^N \xi_i e_i^T(t) P [f(x_i(t)) - f(s(t)) - \Delta e_i(t)] \\ &+ \sum_{i=1}^N \xi_i e_i^T(t) P \Delta e_i(t) + \beta^* \theta_2(t) \sum_{i=1}^N \xi_i e_i^T(t) P \\ &\cdot \sum_{j=1}^N a_{ij} e_j(t) + (1 - \beta^*) \theta_2(t) \sum_{i=1}^N \xi_i e_i^T(t) P \\ &\cdot \sum_{j=1}^N b_{ij} e_j(t-\tau) - \beta^* \theta_2(t) \sum_{q=1}^l \xi_q k_q e_q^T(t) P e_q(t) + \frac{1}{2} \\ &\cdot \sum_{j=1}^n p_j [\tilde{e}_j^T(t) \tilde{e}_j(t) - \tilde{e}_j^T(t-\tau) \tilde{e}_j(t-\tau)] \\ &+ \frac{1}{\rho} (\hat{\beta}(t) - \beta^*) \dot{\hat{\beta}}(t) + \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) \end{aligned}$$

$$\begin{aligned}
&\leq -\eta \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) + \sum_{j=1}^n p_j \delta_j \tilde{e}_j^T(t) \Xi \tilde{e}_j(t) \\
&+ \beta^* \theta_2(t) \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi \tilde{A} \tilde{e}_j(t) + (1 - \beta^*) \theta_2(t) \\
&\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) + \frac{1}{2} \\
&\cdot \sum_{j=1}^n p_j [\tilde{e}_j^T(t) \tilde{e}_j(t) - \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau)] \\
&+ \frac{1}{\rho} (\hat{\beta}(t) - \beta^*) \dot{\hat{\beta}}(t) + \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) \\
&= -\eta \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) + \beta^* \theta_2(t) \\
&\cdot \left[\sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi \tilde{A} \tilde{e}_j(t) - \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) \right] \\
&+ \sum_{j=1}^n p_j \tilde{e}_j^T(t) \left(\delta_j \Xi + \frac{1}{2} I_N \right) \tilde{e}_j(t) + \theta_2(t) \\
&\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) - \frac{1}{2} \\
&\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) + \frac{1}{\rho} (\hat{\beta}(t) - \beta^*) \dot{\hat{\beta}}(t) \\
&+ \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) = -\eta \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) + \frac{1}{2} \\
&\cdot \sum_{j=1}^n p_j (\tilde{e}_j^T(t), \tilde{e}_j^T(t - \tau)) \bar{Z}_j(t) (\tilde{e}_j(t), \tilde{e}_j(t - \tau)) \\
&+ \beta^* \theta_2(t) \\
&\cdot \left[\sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi \tilde{A} \tilde{e}_j(t) - \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) \right] \\
&+ \frac{1}{\rho} (\hat{\beta}(t) - \beta^*) \dot{\hat{\beta}}(t) + \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) \\
&= -\eta \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) + \frac{1}{2} \\
&\cdot \sum_{j=1}^n p_j (\tilde{e}_j^T(t), \tilde{e}_j^T(t - \tau)) \bar{Z}_j(t) (\tilde{e}_j(t), \tilde{e}_j(t - \tau)) \\
&+ \beta^* \theta_2(t) \\
&\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi \tilde{A} \tilde{e}_j(t) - \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) \\
&+ \beta^* \theta_2(t) \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi \tilde{A} \tilde{e}_j(t) - \beta^* \theta_2(t)
\end{aligned}$$

$$\begin{aligned}
&\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) + \beta^* \theta_2(t) \\
&\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) - \beta^* \theta_2(t) \\
&\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) + \frac{1}{\rho} (\hat{\beta}(t) - \beta^*) \dot{\hat{\beta}}(t) \\
&+ \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) \leq -\eta \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) + \frac{1}{2} \\
&\cdot \sum_{j=1}^n p_j (\tilde{e}_j^T(t), \tilde{e}_j^T(t - \tau)) \bar{Z}_j(t) (\tilde{e}_j(t), \tilde{e}_j(t - \tau)) \\
&+ \beta^* \theta_2(t) \sum_{j=1}^n p_j \tilde{e}_j^T(t) (\Xi \tilde{A})^s \tilde{e}_j(t) - \beta^* \theta_2(t) \\
&\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t) \Xi B \tilde{e}_j(t - \tau) + \beta^* \theta_2(t) \\
&\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) - \beta^* \theta_2(t) \\
&\cdot \sum_{j=1}^n p_j \tilde{e}_j^T(t - \tau) \tilde{e}_j(t - \tau) + \frac{1}{\rho} (\hat{\beta}(t) - \beta^*) \dot{\hat{\beta}}(t) \\
&+ \sum_{i=1}^N \xi_i e_i^T(t) P v_i(t) = -\eta \sum_{i=1}^N \xi_i e_i^T(t) P e_i(t) + \frac{1}{2} \\
&\cdot \sum_{j=1}^n p_j (\tilde{e}_j^T(t), \tilde{e}_j^T(t - \tau)) \bar{Z}_j(t) (\tilde{e}_j(t), \tilde{e}_j(t - \tau)) \\
&+ \beta^* \theta_2(t) \\
&\cdot \sum_{j=1}^n p_j (\tilde{e}_j^T(t), \tilde{e}_j^T(t - \tau)) \bar{M}_j(\tilde{e}_j(t), \tilde{e}_j(t - \tau)) \\
&\leq 0,
\end{aligned}$$

(57)

where

$$\bar{Z}_j(t) = \begin{bmatrix} 2\delta_j \Xi + (1 - 2\tilde{c}) I_N & \theta_2(t) \Xi B \\ \theta_2(t) (\Xi B)^T & -I_N \end{bmatrix}, \quad j \in \mathbb{T},$$

(58)

$$\bar{M}_j = \begin{bmatrix} (\Xi \tilde{A})^s & -\frac{1}{2} \Xi B \\ -\frac{1}{2} (\Xi B)^T & -I_N \end{bmatrix}, \quad j \in \mathbb{T}.$$

Similarly, one has that $\bar{Z}_j(t) < 0$ is guaranteed by

$$2\delta_j \Xi + (1 - 2\tilde{c}) I_N + \tilde{c}^2 (\Xi B) (\Xi B)^T < 0, \quad j \in \mathbb{T},$$

(59)

$$2\delta_j \Xi + (1 - 2\tilde{c}) I_N + \tilde{c}^2 (\Xi B) (\Xi B)^T < 0, \quad j \in \mathbb{T},$$

where $\overline{M}_j < 0$ is obtained by

$$(\Xi \tilde{A})^s + \frac{1}{4} (\Xi B) (\Xi B)^T < 0. \quad (60)$$

Based on (53), we have (59)-(60). The next is similar to the proof of Theorem 10. This completes the proof. $\square$

4. Numerical Examples

In this section, a simulation example is used to illustrate the usefulness of our main results.

Example 1. Consider that a dynamical node of complex network is a Lorenz system. It is described as

$$\begin{aligned} \dot{x}_1(t) &= a(x_2(t) - x_1(t)), \\ \dot{x}_2(t) &= cx_1(t) - x_1(t)x_3(t) - x_2(t), \\ \dot{x}_3(t) &= x_1(t)x_2(t) - bx_3(t). \end{aligned} \quad (61)$$

When the parameters are given as $a = 10$, $b = 8/3$, and $c = 28$, Lorenz system has a chaotic attractor shown in Figure 1. We take $P = \text{diag}\{24, 0.002, 12\}$ and $\Delta = \text{diag}\{3, 3, 3\}$ such that Lorenz system (61) satisfies Assumption 1. Suppose an undirected E-R random network consisting of $N = 10$ nodes with $p = 0.8$, where the coupling matrix is

$$A = \begin{pmatrix} -6 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & -6 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -6 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -7 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -7 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & -6 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & -7 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & -6 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -5 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -6 \end{pmatrix}, \quad (62)$$

$$B = \begin{pmatrix} -5 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & -5 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -4 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & -5 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -4 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -5 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -3 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -4 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & -6 \end{pmatrix}.$$

FIGURE 1: Chaotic orbit of Lorenz's circuit.

FIGURE 2: The curve of c_1 along with α^* .

In this example, we randomly choose $l = \lfloor \delta N \rfloor$ nodes with a fraction $\delta = 0.2$. For this E-R network with $\beta^* = 0.4$ and given control gains $k_1 = k_2 = \dots = k_l = 0.1$, when the upper bound $c_2 = 3$ and $\hat{c} = (1/2)(c_1 + c_2)$, by Theorem 8, we could get the curve of c_1 along with α^* varying in $[0, 1]$. Based on Figure 2, it is known that the larger α^* results in the smaller c_1 . This conclusion is consistent with constant coupling strength case that the larger coupling strength results in the easier synchronization of complex network. In order to establish the correlation between $\hat{c}$ and α^* , we select $c_1 = 20$ and $c_2 = 100$. Under the similar conditions and by Theorem 8, one can have the correlation between $\hat{c}$ and α^* , which is illustrated in Figure 3. It is also concluded that the smaller α^* leads to the larger $\hat{c}$. Moreover, when α^* is smaller, $\alpha^* \leq 0.2$, the value of $\hat{c}$ changes actually and is very near to the upper bound c_2 . Particularly, it is worth mentioning that when α^* is very small, $\alpha^* \ll 0.1$, there will be no solvable solution to $\hat{c}$ in terms of $\hat{c}$ being larger than its upper bound $c_2 = 100$. In this sense, it is seen that such probabilities play important roles in pinning synchronization of complex networks. Next, we will demonstrate the correlation between probabilities α^* and β^* . Letting $c(t) \in [c_1, c_2]$ with $c_1 = 15$ and $c_2 = 55$, a simulation of

FIGURE 3: The curve of $\hat{c}$ along with α^* .

FIGURE 4: The curve of β^* along with α^* .

probabilities α^* and β^* will be obtained by Theorem 8, which is shown in Figure 4. Based on Figure 4, one knows that the larger β^* results in the larger α^* . In other words, the larger β^* means that the probability of delayed coupling taking place is larger, which needs the desired pinning controller to be added with a higher probability. When such probabilities are unavailable, we will exploit the adaptive control to realize the purpose of pinning synchronization. Without loss of generality, here it is assumed that $\beta^* = 0.3$ is known, while $\alpha^* = 0.2$ is inaccessible. If α^* is unknown, let the initial condition $x_0 = [-1 \ 0.5 \ 0.1]^T$; by Theorem 10, we have the state responses of synchronization error and estimation α^* shown in Figures 5 and 6. Particularly, Figure 5 is the simulation of state error $|e_i(t)|$, $i = 1, 2, \dots, 10$, instead of $e_i(t)$. The reason is we want to demonstrate the state response of $e_i(t)$ in detail. Unfortunately, there are 10 nodes, where every node has 3 variables. Then, there will be 30 variables to be illustrated in one figure. This will make the state response of $e_i(t)$, $i = 1, 2, \dots, 10$, difficult to be distinguished. However, if we plot $|e_i(t)|$ instead of $e_i(t)$, there will be 10 lines in one figure. Then, we will identify them easily. But, the effects of $|e_i(t)|$ and $e_i(t)$ implying that $e(t)$ is stable are same. All the simulations show the utility of the proposed methods in this paper.

5. Conclusion

In this paper, based on the stochastic viewpoint, we have investigated the pinning synchronization problem of complex networks with time-varying coupling strength and nondelayed and delayed coupling. Here, the lower and upper bounds in addition to the probability distribution of each subinterval are firstly considered in designing a pinning controller. Moreover, the nondelayed and delayed coupling here are asynchronous, whose switchings are described by a Bernoulli variable. Then, a new kind of local state feedback controller is developed without using time-varying coupling strength signal, which is dependent on such bounds and probabilities. When such probabilities are unavailable, two

FIGURE 5: Synchronization error $|e_i(t)|$ by adaption controller (41).

different kinds of adaption laws are proposed to estimate the unknown parameters, respectively, and also guarantee the global synchronization of complex networks. Finally, a numerical example is given to show the utility of the presented theories. Future research could be considered to the pinning synchronization of complex networks with nondelayed and time-varying delayed couplings, stochastic pinning controller added to some nodes in term of Brownian noise, and such Bernoulli variables $\alpha(t)$ and $\beta(t)$ replaced by Markov process.

FIGURE 6: Response curve of the estimation of α^* .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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