Variational Approaches to Characterize Weak Solutions for Some Problems of Mathematical Physics Equations

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Received 10 November 2015; Accepted 3 January 2016

Academic Editor: Khalil Ezzinbi

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This paper is aimed at providing three versions to solve and characterize weak solutions for Dirichlet problems involving the $p$-Laplacian and the $p$-pseudo-Laplacian. In this way generalized versions for some results which use Ekeland variational principle, critical points for nondifferentiable functionals, and Ghousoub-Maurey linear principle have been proposed. Three sequences of generalized statements have been developed starting from the most abstract assertions until their applications in characterizing weak solutions for some mathematical physics problems involving the abovementioned operators.

1. Introduction

Obtaining and/or characterization of weak solutions for problems of mathematical physics equations involving $p$-Laplacian and $p$-pseudo-Laplacian is a subject matter previously discussed by the author through several approach methods in [1–6]. New similar results can be found by other authors, for instance, Amiri and Zivari-Rezapour in [7], El Khalil and Ouanan in [8], Rhouma and Sayeb in [9], Yoshida in [10]. The importance of these operators also devolves from their involvement in actual modelings of natural phenomena as thermal transfer by Lanchon-Ducaquis, Tulita, and Meuris in [11] or glacier sliding or flow by Partridge in [12] and Pelissier in [13]. In this paper three methods are proposed following three sequences of results, starting from abstract statements and finishing with their application to find weak solutions of Dirichlet problems for $p$-Laplacian and for $p$-pseudo-Laplacian as well. In the first succession of assertions, two results of Ghousoub from [14] have been generalized replacing the frame of Hilbert space by reflexive strictly convex Banach space and the condition imposed to the goal function to be of $C^1$-Fréchet class by the weaker condition to have a lower semicontinuous and Gâteaux differentiable function. These two theoretical statements were used, together with other results concerning Dirichlet problems for both cited generalized operators, in finding weak solutions for these problems. The second proposition sequence involves critical points for nondifferentiable functional. In this context, two results of Chang from [15] have been generalized changing the space $H^1(\Omega)$ in $W^{1,p}_0(\Omega)$ and introducing $\Delta_p$ and $\Delta^\gamma_p$ in some problems formulated for $\Delta$ by Costa and Gonçalves in [16]. The last series of assertions starts from Ghousoub-Maurey linear principle which is used here to characterize weak solutions for some mathematical physics equations. Moreover, the problems were discussed and solved using important findings for these operators obtained by the author in [5, 6] in connection with specific properties of the Sobolev spaces involved.

2. Critical Points and Weak Solutions for Elliptic Type Equations

2.1. Theoretical Support. In order to introduce the first result, a theoretical support will be given starting with the following.

Ekeland Principle (see [1, 17, 18]). Let $(X, d)$ be a complete metric space and $\varphi : X \to (-\infty, +\infty]$ bounded from below, lower semicontinuous, and proper. For any $\varepsilon > 0$ and $u$ of $X$ with

$$\varphi (u) \leq \inf \varphi (X) + \varepsilon$$

(1)
and for any \( \lambda > 0 \), there exists \( v_\varepsilon \) in \( X \) such that
\[
\varphi(v_\varepsilon) < \varphi(w) + \frac{\varepsilon}{\lambda} d(v_\varepsilon, w) \quad \forall w \in X \setminus \{v_\varepsilon\},
\]
\[
\varphi(v_\varepsilon) \leq \varphi(u), \text{ and } d(v_\varepsilon, u) \leq \lambda.
\]

**Definition 1.** Let \( X \) be a real normed space, \( \beta \) be a bornology on \( X \), and let \( \varphi : X \to \mathbb{R} \) be \( \beta \)-differentiable; \( c \in \mathbb{R} \) \( \Rightarrow K_c(\varphi) = \{ x \in X : \varphi(x) = c, \nabla \beta \varphi(x) = 0 \}. \) (11)

Explanations. Let us introduce the definition of the metric gradient in order to provide other observations relative to this central notion for this statement. In a real normed space \( X \), consider the Gâteaux differentiable functional \( f : X \to \mathbb{R} \). The \textbf{metric gradient} of \( f \) is the multiple-valued mapping \( Vf : X \to \mathcal{P}(X) \), \( Vf(x) = i^{-1} J_f(\mathcal{P}(x)) \), where \( i : X^* \to \mathcal{P}(X^*) \) is the duality mapping on \( X^* \) corresponding to the identity and \( i \) the canonical injection of \( X \) into \( X^{**} \). Let \( f = i(x) = x^* \), \( (x^*, x^*) = (x^*, x) \), \( \forall x^* \in X^* \). Consequently, for any \( x \in X \), \( Vf(x) = \{ y \in X : i(y)y \in J_f(x) \} = \{ y \in X : i(y), f'(x)(y) = \| f'(x)(y) \| \leq \| f'(x)(x) \| \}. \) If \( X \) is reflexive, for any \( x \in X \), \( Vf(x) \) is nonempty. \( X^* \) being strictly convex, \( i_f \) is single-valued. So, if \( X \) is reflexive and strictly convex, then \( Vf : X \to X \), \( Vf(x) = i^{-1} J_f(x) \), and the following equalities hold:
\[
\langle f'(u)(\nu), f'(v)(\nu) \rangle = \| f'(u)(\nu) \|^2, \quad \| f'(v)(\nu) \|^2.
\]

Remark 3. This result is reported in [14] as Lemma 9 in the frame of the Hilbert spaces having the function \( \varphi \) of \( C^1 \)-class Fréchet, but condition (5) is more complicated due to another condition imposed to the set \( F \).

**Proposition 2.** Let \( X \) be a real reflexive strictly convex Banach space, let \( \varphi : X \to \mathbb{R} \) be lower semicontinuous and Gâteaux differentiable and let \( F \) be a closed subset of \( X \) such that for every \( u \) from \( F \) with the metric gradient \( \nabla \varphi(u) \neq 0 \), for sufficiently small \( r > 0 \),
\[
\left( u - \delta \frac{\nabla \varphi(u)}{\| \nabla \varphi(u) \|} \right) \in F, \quad \forall \delta \in (0, r].
\]

Then, if \( \varphi \) is lower bounded on \( F \), for every \( (v_n)_{n \geq 1} \) a minimizing sequence for \( \varphi \) on \( F \), there exists a sequence \( (u_n)_{n \geq 1} \) in \( F \) such that
\[
\| \varphi'(u_n) \| \leq \sqrt{r}, \quad \| \varphi'(v_n) \| \leq \sqrt{r}, \quad \forall n,
\]
(5)
\[
\varphi(u_n) \leq \varphi(v_n), \quad \forall n, \quad \lim_{n \to \infty} \| u_n - v_n \| = 0,
\]
(7)
where \( \varepsilon_n > 0 \) and \( \varepsilon_n \to 0 \).

Remark 4. The Gâteaux derivative from the above statement can be replaced by any \( \beta \)-derivative and the result remains the same. In the case of the Fréchet derivative, it must remove the condition "\( \varphi \) lower semicontinuous" from the statement.
Proposition 5. Let $X$ be a real reflexive strictly convex Banach space and $\varphi : X \to \mathbb{R}$ lower semicontinuous and Gâteaux differentiable and $F$ is a nonempty convex closed subset such that $(I - \nabla \varphi)(F) \subset F$, $I$ the identity map. If $\varphi$ is lower bounded on $F$, then for every $(\nu_n)_{n \geq 1}$ a minimizing sequence for $\varphi$ on $F$, there is a sequence $(u_n)_{n \geq 1}$ in $F$ such that
\[
\varphi(u_n) \leq \varphi(\nu_n) \quad \forall n,
\]
and
\[
\lim_{n \to \infty} \|u_n - \nu_n\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|\varphi'_w(u_n)\| = 0.
\]
Moreover, if $\varphi$ satisfies (PS)$_{C_\varepsilon}$, where $c = \inf \varphi(F)$, then $F \cap K_c(\varphi) \neq \emptyset$.

Proof. Applying Proposition 2, (4) is satisfied indeed: if $u \in F$ and $\varphi'_w(u) \neq 0$, then, $F$ being convex,
\[
u = \frac{1}{\|\varphi'(w)\|} \left( u - \frac{\varphi'(w)}{\|\varphi'(w)\|} \|\varphi(w)\| \right) + \frac{\delta}{\|\varphi'(w)\|} (I - \nabla \varphi)(w) \in F.
\]

Let $(u_n)_{n \geq 1}$ be the sequence given by the statement. $c \leq \varphi(u_n) \leq \varphi(\nu_n) \forall n$, hence $\varphi(u_n) \to c$, $\|\varphi'(u_n)\| \to 0$, and consequently $(u_n)_{n \geq 1}$ has a convergent subsequence $(u_{n_k})_{k \geq 1}$, $u_{n_k} \to u_0 \in F$. This implies $\|\varphi'(u_{n_k})\| \to 0$, and $u_0$ is a global critical point of $\varphi$ contained in $F$.

2.2. Weak Solutions. Open set of $C^1$ class in $R^N$. Use the notations (the norm is that Euclidean from $R^{N+1}$): $R^N = \{x = (x', x_N) : x_N > 0\}$, $Q = \{x = (x', x_N) : |x'| < 1, |x_N| < 1\}$, $Q_s = Q \cap R^{N+1}$, $Q_w = \{x = (x', x_N) : |x'| < 1, x_N = 0\}$. Let $\Omega$ be an open nonempty set in $R^{N}$, $\Omega \neq R^N$, and $\partial \Omega$ its boundary. By definition, $\Omega$ is of $C^1$ class if $\forall x$ from $\partial \Omega \exists U$ a neighbourhood of $x$ in $R^N$ and $f : Q \to U$ bijective such that $f \in C^1(\bar{Q})$, $f^{-1} \in C^1(\bar{U})$, $f(Q_s) = U \cap \Omega$, and $f(\Omega_0) = U \cap \partial \Omega$. Weak solution. Let $\Omega$ be an open bounded nonempty set in $R^{N}$, $N > 1$, $f : \Omega \times R^N \to R$, and $u_0 \in W_0^{1,p}(\Omega)$. Consider the problems
\[
-\Delta_p u = f(x,u), \quad x \in \Omega \quad (*)
\]
\[
-\Delta_p' u = f(x,u), \quad x \in \Omega \quad (**) \quad u = 0 \quad \text{on} \partial \Omega.
\]
Actually $u = u_0$ on $\partial \Omega$ means $u \mid \partial \Omega = u_0$, where $\gamma : u \to u \mid \partial \Omega$ is the trace operator, a continuous linear mapping from $W^{1,p}(\Omega)$ in $L^p(\partial \Omega)$, $p \in [1, +\infty)$. We have $\gamma^{-1}(0) = W_0^{1,p}(\Omega) = C_c^\infty(\bar{\Omega}) \cap W^{1,p}(\Omega)$ and $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega}) \Rightarrow \gamma(u) = u \mid \partial \Omega$. If $\Omega$ is bounded and $1/p + 1/q = 1$, then $\gamma(1/p)^{-1}(\Omega)$ is a $W_0^{1,p}$ class of $\nabla$ with $N$ continuous; moreover, $\gamma(p)^{-1} \subset L^p(\Omega)$ with $N$ continuous (see, e.g., [2]), where $F(x,s) = \int_0^s f(x,t) dt$, and $\Phi : L^p(\Omega) \to R$, $\Phi(\Omega) = \int_0^\infty f(x,u(x)) dx$ is of Fréchet $C^1$ class and $\Phi' = N_f [14]$, so it is also Gâteaux differentiable.

Remark 6. Here $\nabla u$ is the weak gradient, and it is equal to $(\partial w/\partial x_1, \ldots, \partial w/\partial x_N)$, $\partial w/\partial x_i$: the weak derivatives; $|\nabla u|^p = \int_\Omega |\nabla u(x)|^p dx$. If $\Omega = W_0^{1,p}(\Omega)$ is endowed in the first case $(\ast)$ with the norm $\|\cdot\|_p$, that is, $\|u\|_{p,\Omega} = \|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)}^p + \sum_{i=1}^N \|\partial u/\partial x_i\|_{L^p(\Omega)}$, which is equivalent to the norm $u \to (\|u\|_{p,\Omega}^p + \sum_{i=1}^N \|\partial u/\partial x_i\|_{L^p(\Omega)}^p)^{1/p}$. For the second case $(\ast\ast)$, equip the same vector space with the norm $u \to \|u\|_{L^p(\Omega)}^p = (\sum_{i=1}^N \|\partial u/\partial x_i\|_{L^p(\Omega)}^p)^{1/p}$, which is equivalent to $u \to \|u\|_{p,\Omega} = \sum_{i=1}^N \|\partial u/\partial x_i\|_{L^p(\Omega)}$. Nemytskii Operator. Let be $R^N$, $N \geq 1$, $\mu$ the Lebesgue measure in $R^N$, $\Omega$ open nonempty Lebesgue measurable. $U(\Omega) = \{u : \Omega \to \mathbb{R} | u \text{ Lebesgue measurable}\}$. By definition $f : \Omega \times R \to R$ is a Carathéodory function if $f(\cdot, s)$ is Lebesgue measurable $\forall s \in R$ and $f(x, \cdot)$ is continuous $\forall x \in \Omega \setminus A$, $\mu(A) = 0$. In this case, for every $u$ from $U(\Omega)$ one may consider the function $N_f : U(\Omega) \to U(\Omega)$, $N_f u : N_f u(x) = f(x,u(x))$, $\text{Nemytskii operator}$. Suppose $\mu(\Omega) < +\infty$. Then $u_0(x) \xrightarrow{\mu_{x \in \Omega}} u_0(x) \Rightarrow N_f u_0(x) \xrightarrow{\mu_{x \in \Omega}} N_f u_0(x)$. Assume that $f$ satisfies the growth condition: $|f(x,s)| \leq c|s|^{p-1} + b(x)$, $\forall x \in \Omega \setminus A$ with $\mu(A) = 0$, $\forall s \in R$, where $c \geq 0$, $p > 1$, and $b \in L^q(\Omega)$, $q \in [1, +\infty)$.

Then $N_f(L^{p-1}\Omega) \subset L^q(\Omega)$; $N_f$ is continuous $(q < +\infty)$ and bounded on $L^{p-1}\Omega$. If $\Omega$ is bounded and $1/p + 1/q = 1$, then $N_f(L^p(\Omega)) \subset L^q(\Omega)$ with $N_f$ continuous; moreover, $N_f(L^p(\Omega)) \subset L^q(\Omega)$ with $N_f$ continuous (see, e.g., [2]), where $F(x,s) = \int_0^s f(x,t) dt$, and $\Phi : L^p(\Omega) \to R$, $\Phi(u) = \int_\Omega F(x,u(x)) dx$ is of Fréchet $C^1$ class and $\Phi' = N_f [14]$, so it is also Gâteaux differentiable.

Theorem 7. Let $\Omega$ be an open bounded nonempty set in $R^N$ and $f : \Omega \times R \to R$ a Carathéodory function with the growth condition:
\[
|f(x,s)| \leq c|s|^{p-1} + b(x),
\]
where $c > 0$, $2 \leq p \leq 2N/(N-2)$ when $N \geq 3$, and $2 \leq p < +\infty$ when $N = 1, 2$, and where $b \in L^q(\Omega)$, $1/p + 1/q = 1$.  

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Then the energy functional \( \varphi : W^{1, p}_0(\Omega) \to \mathbb{R} \),
\[
\varphi(u) = \frac{1}{p} \|u\|_{L^p}^p - \int_{\Omega} F(x, u(x)) \, dx,
\]
for the problem \((*)\)
\[
\varphi(u) = \frac{1}{p} \|u\|_{L^p}^p - \int_{\Omega} F(x, u(x)) \, dx
\]
for the problem \((**)*\), respectively

where \( F(x, s) \) is Gâteaux differentiable on \( W^{1, p}_0(\Omega) \setminus \{0\} \) and
\[
\varphi'(u)(v) = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx - \int_{\Omega} f(x, u(x)) \, v \, dx
\]
\[
\varphi'_w(u)(v) = \sum_{i=1}^N \int_{\Omega} \frac{\partial u}{\partial x_i} |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx - \int_{\Omega} f(x, u(x)) \, v \, dx = 0
\]
\[
\forall u, v \in W^{1, p}_0(\Omega) \text{ respectively}
\]

Proof. One may consider \( \varphi \), in both cases, as the sum of two other functions. The second of these functions being Gâteaux differentiable (see the above consideration), it is sufficient to remark that also the maps \( u \to (1/p) \|u\|_{L^p}^p \) and \( u \to (1/p) \|u\|_{L^p}^p \) are Gâteaux differentiable on \( W^{1, p}_0(\Omega) \setminus \{0\} \) [2, 19] and then \( \varphi \) is Gâteaux differentiable on \( W^{1, p}_0(\Omega) \setminus \{0\} \).

Corollary 8. Let \( \Omega \) and \( f \) be as in Theorem 7. Then the weak solutions of \((*)\) and \((**)*\), respectively, are precisely the critical points of the functional \( \varphi : W^{1, p}_0(\Omega) \to \mathbb{R} \):
\[
\varphi(u) = \frac{1}{p} \|u\|_{L^p}^p - \int_{\Omega} F(x, u(x)) \, dx
\]
\[
F(x, s) = \int_0^s f(x, t) \, dt
\]
\[
\varphi'(u)(v) = 0 \quad \forall v \in W^{1, p}_0(\Omega) \text{ respectively}
\]

Proof. Indeed, if \( \bar{u} \) is a weak solution of \((*)\) and \((**)*\), respectively, then \( \varphi'_w(\bar{u}) \equiv 0 \forall v \in W^{1, p}_0(\Omega) \) (\ref{25b}) and (\ref{26b}), resp., Theorem 7); hence \( \varphi'_w(\bar{u}) = 0 \). The inverse assertion is obvious.

Weak Subsolutions and Weak Supersolutions of \((*)\) and \((**)*\). Let \( \Omega \) be an open bounded set of \( C^1 \) class in \( \mathbb{R}^N, N \geq 3 \), let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function, and let \( \bar{u} \in W^{1, p}_0(\Omega) \). \( \bar{u} \) is a weak subsolution and a weak supersolution, respectively, of \((*)\) or \((**)*\) if
\[
\bar{u} \leq 0 \quad \text{on } \partial \Omega \quad \text{respectively } \bar{u} \geq 0 \quad \text{on } \partial \Omega, \quad \text{and}
\]
\[
\int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla v \, dx \leq \int_{\Omega} f(x, \bar{u}(x)) \, v \, dx \tag{24a}
\]
\[
\forall v \in W^{1, p}_0(\Omega), \quad v \geq 0,
\]
respectively,
\[
\int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u} \cdot \nabla v \, dx \geq \int_{\Omega} f(x, \bar{u}(x)) \, v \, dx \tag{25b}
\]
\[
\forall v \in W^{1, p}_0(\Omega), \quad v \geq 0,
\]
or
\[
\sum_{i=1}^N \int_{\Omega} |\partial \bar{u}/\partial x_i|^{p-2} \partial \bar{u}/\partial x_i \, dx \leq \int_{\Omega} f(x, \bar{u}(x)) \, v \, dx \tag{26a}
\]
\[
\forall v \in W^{1, p}_0(\Omega), \quad v \geq 0,
\]
respectively,
\[
\sum_{i=1}^N \int_{\Omega} |\partial \bar{u}/\partial x_i|^{p-2} \partial \bar{u}/\partial x_i \, dx \geq \int_{\Omega} f(x, \bar{u}(x)) \, v \, dx \tag{26b}
\]
\[
\forall v \in W^{1, p}_0(\Omega), \quad v \geq 0.
\]

Proposition 9. Let \( \Omega \) be an open bounded set of \( C^1 \) class in \( \mathbb{R}^N, N \geq 3 \) and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) a Carathéodory function and \( u_1, u_2 \) from \( W^{1, p}_0(\Omega) \) bounded weak subsolution and weak supersolution of \((*)\), respectively, with \( u_1(x) \leq u_2(x) \) a.e. on \( \Omega \). Suppose that \( f \) verifies \( \ell \) and there is \( p > 0 \) such that the function \( g : g(x, s) = f(x, s) + ps \) is strictly increasing in \( s \) on \([\inf u_1(\Omega), \sup u_2(\Omega)]\). Then there is a weak solution \( u \) of \((*)\) in \( W^{1, p}_0(\Omega) \) with the property
\[
u_1(x) \leq u(x) \leq u_2(x) \quad \text{a.e. on } \Omega.
\]

Proof. Take the equivalent norm on \( X = W^{1, p}_0(\Omega) \):
\[
\|u\| = \left( \rho \|u\|_{L^p(\Omega)}^p + \sum_{i=1}^N \|\partial u/\partial x_i\|_{L^p(\Omega)}^p \right)^{1/p}
\]
Consider the functional \( \varphi : W^{1, p}_0(\Omega) \to \mathbb{R} \):
\[
\varphi(u) = \frac{1}{p} \|u\|^p - \int_{\Omega} G(x, u(x)) \, dx,
\]
\[
G(x, s) = \int_0^s g(x, t) \, dt.
\]
\( \varphi \) is Gâteaux differentiable and its critical points are the weak solutions of \((*)\) (see Corollary 8). \( \varphi \) is also lower bounded,
the norm on $L^p(\Omega)$ actually being of Fréchet $C^1$ class (see, e.g., [20] or [21]). Use Proposition 5, $(X,\|\cdot\|)$ being a reflexive strictly convex Banach space (see, e.g., [2]). Let be
\[\mathcal{F} = \left\{ u \in W_0^{1,p}(\Omega) : u_1(x) \leq u(x) \right\} \quad \text{and} \quad u_2(x) \text{ a.e. on } \Omega.\]
$\mathcal{F}$ is closed convex. We also get
\[\mathcal{F} = \left\{ u \in W_0^{1,p}(\Omega) : u_1(x) \leq u(x) \right\},\]
Here $V\varphi$ denotes the metric gradient of $\varphi$. Since $(X,\|\cdot\|)$ is reflexive and strictly convex (see, e.g., [2]), thus $V\varphi$ is univaluated and it has the above described properties. Indeed, let $u$ be in $\mathcal{F}$ and $v = (I - \varphi)(u)$. We should prove that $v \in \mathcal{F}$.
\[v = u - V\varphi(u) \in W_0^{1,p}(\Omega) \quad \text{and} \quad u_1(x) \leq v(x) \leq u_2(x).\]
Since the definition relation of the subsolution for $u_1$ actually means $\varphi'_1(u_1)(u) \leq 0 \quad \forall u \in W_0^{1,p}(\Omega)$ with $u(x) \geq 0$ almost everywhere (a.e.) on $\Omega$ and that of supersolution for $u_2$ is $\varphi'_2(u_2)(u) \geq 0 \quad \forall u \in W_0^{1,p}(\Omega)$ verifying $u(x) \geq 0$ a.e. on $\Omega$, we will prove that $v(x) - u_1(x) \geq 0$ a.e. on $\Omega$ and $u_2(x) - v(x) \leq 0$ a.e. on $\Omega$ using the Gâteaux derivatives of $\varphi$ in $u_1$ and $u_2$, respectively.
\[\varphi'_1(u_1)(v - u) + \varphi'_2(u_2)(v - u) \geq 0 \quad \forall u \in W_0^{1,p}(\Omega) \quad \text{and} \quad \varphi'_1(u_1)(\nabla \varphi(u)) = \varphi'_2(u_2)(\nabla \varphi(u)).\]
Also $\varphi'_1(u_1)(v - u) = \varphi'_2(u_1)(v - u)$ is a linear map and some properties of the metric gradient) of $\varphi$ being actually continuous (for this see, for instance, [2]). Until now, applying Proposition 5, for every $(\nu_n)_{n \geq 1}$ a minimizing sequence for $\varphi$ on $\mathcal{F}$, there is a sequence $(\nu_n)_{n \geq 1}$ in $\mathcal{F}$ such that $\varphi(\nu_n) \leq \varphi(\nu_n)$, $\lim_{n \to \infty} \|u_n - \nu_n\| = 0$, $\lim_{n \to \infty} \varphi(\nu_n) = c$ since $c = \inf \varphi(\mathcal{F})$, we have $\lim_{n \to \infty} \varphi(\nu_n) = 0$ already; also the last proof from (P9) is condition is verified. Finish the proof applying once again Proposition 5.

Example 10. Consider the problem $(\Omega$ open bounded of $C^1$ class in $\mathbb{R}^N, N \geq 3$)
\[-\Delta_p u = \alpha(x) u |u|^{p-2},\]
where $p = 2N/(N - 2)$; $\alpha$ is continuous with $1 \leq \alpha(x) \leq a < +\infty$ on $\Omega$. Then $u_1 = 1$ is a weak subsolution, $u_2 = M$, $M > 1$ sufficiently big, is a weak supersolution. $|f(x, s)| \leq \alpha|s|^{p-1}$ (condition (17)), and $s \to \alpha(x)|s|^{p-2} + s$ is increasing in $s$ on $[1, M]$; consequently, according to Proposition 11, (32) has a weak solution $\pi$ with $1 \leq \pi(x) \leq M$ a.e. on $\Omega$.

Proposition 11. Let $\Omega$ be an open bounded of $C^1$ class set in $\mathbb{R}^N, N \geq 3$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function and $u_1, u_2$ from $W_0^{1,p}(\Omega)$ bounded weak subsolution and weak supersolution, respectively, of $(\star \star)$ with $u_1(x) \leq u_2(x)$ a.e. on $\Omega$. Suppose that $f$ verifies (17) and there is $r > 0$ such that the function $g : g(x, s) = f(x, s) + ps$ is strictly increasing in $s$ on $[\inf u_1(\Omega), \sup u_2(\Omega)]$. Then there is a weak solution $\bar{u}$ of $(\star \star)$ in $W_0^{1,p}(\Omega)$ with the property
\[u_1(x) \leq \bar{u}(x) \leq u_2(x) \quad \text{a.e. on } \Omega.\]

Proof. Follow step by step the above proof for Proposition 9 considering the reflexive strictly convex Banach space $X = W_0^{1,p}(\Omega)$ endowed with the norm $u \to \|u\| = (\sum_{i=1}^N \|\partial u/\partial x_i\|_{L^p(\Omega)})^{1/p}$ or the equivalent norm $u \to \|u\|_{1,p} = \sum_{i=1}^N \|\partial u/\partial x_i\|_{L^1(\Omega)}$, which both are also equivalent to the other two norms used at Proposition 9. The function $\varphi$ is from (19) having the weak derivative given in Theorem 7. Using similar calculus, obtain similar conclusion.

Example 12. Consider the problem $(\Omega$ open bounded of $C^1$ class in $\mathbb{R}^N, N \geq 3$)
\[-\Delta_p u = \alpha(x) u |u|^{p-2},\]
where $p = 2N/(N - 2)$ and $\alpha$ is continuous with $1 \leq \alpha(x) \leq a < +\infty$ on $\Omega$. Then $u_1 = 1$ is a weak subsolution, $u_2 = M$, $M > 1$ sufficiently big, is a weak supersolution, $|f(x, s)| \leq \alpha|s|^{p-1}$ (condition (17)), and $s \to \alpha(x)|s|^{p-2} + s$ is increasing in $s$ on $[1, M]$; consequently, according to Proposition 11, (34) has a weak solution $\pi$ with $1 \leq \pi(x) \leq M$ a.e. on $\Omega$.

3. Critical Points for Nondifferentiable Functionals

The sense of the title actually is "not compulsory differentiable." Start this section with the following.

Definition 13. $x_0$ is a critical point (in the sense of Clarke subdervative) for the real function $f$ if $0 \in \partial f(x_0)$. In this case $f(x_0)$ is a critical value (in the sense of Clarke subdervative) for $f$.

To clarify this notion, the Clarke subderivative should be introduced. Let $X$ be a real normed space, $E \subset X, f : E \to \mathbb{R}, x_0 \in E$, and $v \in X$. We set $f^0(x_0; v) = \lim_{t \to 0, t \neq 0} ((f(x + tv) - f(x))/t)$, $f^0(x_0; v)$ is by definition Clarke derivative (or the generaliized directional derivative) of the function $f$ at $x_0$ in the direction $v$. The functional $\xi$ from $X^*$ is by definition Clarke subderivative (or generaliized gradient) of $f$ in $x_0$ if $f^0(x_0; v) \geq \xi(v) \forall v \in X$. The set of these generalized gradients is designated by $\partial f(x_0)$.

Here it is a generalization at $p$-Laplacian and $p$-pseudo-Laplacian of an application from [16] of this concept.

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$ with the smooth boundary $\partial \Omega$ (topological boundary). Consider the nonlinear boundary value problems $(\star \star \star)$ and $(\star \star \star \star)$ where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function with subcritical growth; that is,
\[|f(x, s)| \leq a + b|s|^{\sigma} \quad \forall s \in \mathbb{R}, x \in \Omega \text{ a.e.},\]
where $a, b > 0, 0 \leq \sigma < (N + 2)/(N - 2)$ for $N > 2$ and $\sigma \in [0, +\infty)$ for $N = 1$ or $N = 2$.\]
Set as in [15]
\[
\bar{f}(x, t) = \lim_{s \to t} f(x, s),
\]
\[
\underline{f}(x, t) = \lim_{s \to t} f(x, s).
\]

Suppose
\[
\bar{f} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}
\]
are measurable with respect to \(x\). (II)

We emphasize that (II) is verified in the following two cases:

(i) \(f\) is independent of \(x\).

(ii) \(f\) is Baire measurable and \(s \to f(x, s)\) is decreasing \(\forall x \in \Omega\), in which case we have
\[
\begin{align*}
\bar{f}(x, t) &= \max \{ f(x, t+), f(x, t-) \}, \\
\underline{f}(x, t) &= \min \{ f(x, t+), f(x, t-) \}.
\end{align*}
\]

Definition 14. \(u\) from \(W^{1,p}_0(\Omega)\), \(p > 1\) is solution of \((*)\) and \((***)\), respectively, if \(u = 0\) on \(\partial \Omega\) in the sense of trace and
\[
-\Delta_p u(x) \in \left[ \underline{f}(x, u(x)), \bar{f}(x, u(x)) \right] \quad \text{in } \Omega \text{ a.e.}
\]
respectively
\[
-\Delta_p u(x) \in \left[ \underline{f}(x, u(x)), \bar{f}(x, u(x)) \right] \quad \text{in } \Omega \text{ a.e.}
\]

Define \(W^{1,p}(\Gamma)\) with \(p \in (1, +\infty)\), \(\Gamma\) regular differential manifold, e.g., \(\Gamma = \partial \Omega\), \(\Omega\) open of \(C^1\) class with \(\partial \Omega\) bounded. In this situation, there exists a unique linear continuous operator \(\gamma : W^{1,p}(\Omega) \rightarrow W^{1-1/p,p}(\partial \Omega)\), the trace, such that \(\gamma\) is surjective and \(u \in W^{1,p}(\Omega) \cap C(\Omega) \Rightarrow \gamma(u) = u \mid \partial \Omega\). This gives a sense for \(u \mid \partial \Omega\), for any \(u \in W^{1,p}(\Omega)\). Moreover, \(\gamma^{-1}(0) = W^{1,p}_0(\Omega)\).

Let \(X = W^{1,p}_0(\Omega)\), but in the first case \((*)\), the norm endowing \(X\) is \(\| u \|_{1,p} = \| u \|_{W^{1,p}(\Omega)} + \sum_{i=1}^{N} \| \partial u_i / \partial x_i \|_{L^p(\Omega)}\), which is equivalent to the norm \(u \rightarrow \sum_{i=1}^{N} \| \partial u_i / \partial x_i \|_{L^p(\Omega)}\), for the second case \((***)\), equip the same set \(X\) with \(\| u \|_{1,p} = \sum_{i=1}^{N} \| \partial u_i / \partial x_i \|_{L^p(\Omega)}\).

Associate to \((*)\) the locally Lipschitz functional \(\Phi : X \rightarrow \mathbb{R}\):
\[
\Phi(u) = \frac{1}{p} \| u \|_{1,p}^p - \int_{\Omega} F(x, u) \, dx, \quad u \in X,
\]
and associate to \((***)\)
\[
\Phi(u) = \frac{1}{p} \| u \|_{1,p}^p - \int_{\Omega} F(x, u) \, dx, \quad u \in X,
\]
where \(F(x, s) = \int_0^s f(x, t) \, dt\). Set
\[
Q(u) = \frac{1}{p} \| u \|_{1,p}^p, \quad u \in X,
\]
\[
\Psi_1(u) = \int_{\Omega} F(x, u) \, dx, \quad u \in X,
\]
respectively \(Q(u) = \frac{1}{p} \| u \|_{1,p}^p, \quad u \in X,
\]
\[
\Psi_1(u) = \int_{\Omega} F(x, u) \, dx, \quad u \in X,
\]
\(F\), a map defined on \(\Omega \times \mathbb{R}\), taking values in \(\mathbb{R}\), is locally Lipschitz (use (I)). The functional \(\Psi : L^{p+1}(\Omega) \rightarrow \mathbb{R}\), \(\Psi(u) = \int_{\Omega} F(x, u) \, dx\), is also locally Lipschitz (again (I)). Using Sobolev embedding \(X \subset L^{p+1}(\Omega)\), we obtain that \(\Psi_1 = \Psi \mid X\) is locally Lipschitz on \(X\), which implies \(F\) locally Lipschitz on \(X\), and consequently, according to a local extremum result for Lipschitz functions (if \(x_0\) is a point of local extremum for \(f\), then \(0 \in \partial f(x_0)\), the critical points of \(\Phi\) for Clarke subderivative can be taken into account. One may state the following.

Proposition 15. Suppose (I) and (II) are satisfied. Then \(\Psi\) is locally Lipschitz on \(L^{p+1}(\Omega)\) and

\(\Psi_1\) is locally Lipschitz on \(W^{1,p}_0(\Omega)\).

(i) \(\partial \Psi(u) \subseteq \{ f(x, u(x)), \bar{f}(x, u(x)) \}\) in \(\Omega\) a.e.,

(ii) if \(\Psi_1 = \Psi\mid X\), where \(X = W^{1,p}_0(\Omega)\) endowed with the norm \(\| u \|_{1,p}\) for the problem \((*)\) and \(\| u \|_{1,p}\) for the problem \((***)\), respectively, then
\[
\partial \Psi_1(u) \subseteq \partial \Psi(u) \quad \forall u \in X.
\]

Proof. The proof for (i) can be found in [15], Theorem 2.1, which remained here the same, while the problem was solved for the Laplacian with \(X = H^1_0(\Omega)\) only. In order to prove (ii), use Theorem 2.2 from [15] observing for both cases (\(X\) endowed with each one of those two norms) that \(X\) is reflexive and dense in \(L^{p+1}(\Omega)\) as can be seen, for instance, summarized in [2].

Proposition 16. If (I) and (II) are verified, every critical point of \(\Phi\) is solution for \((*)\) and \((***)\), respectively.

Proof.

Problem \((*)\). Let \(u_0\) be a critical point for \(\Phi\). We have
\[
0 \in \partial \Phi(u_0) \subset \partial Q(u_0) + \partial (-\Psi_1)(u_0),
\]
since \(\Phi = Q - \Psi_1\), and apply some rules of subdifferential calculus concerning finite sums. \(\partial Q(u_0) = \{ Q'(u_0) \}\), where \(Q'(u_0)(v) = \int_{\Omega} |\nabla u_0|^{p-2} \cdot \nabla v \, dx\) [2].

Using (44) and a specific property of a function \(f\) Lipschitz around \(x_0 (f^0(x_0) v) = \sup_{0 \in \partial f(x_0)} \xi(v)\), \(v \in X\), \(f^0\) the Clarke derivative of \(f\), we find
\[
0 \leq \int_{\Omega} |\nabla u_0|^{p-2} \cdot \nabla v \, dx + (-\Psi_1)^0(u_0; v).
\]
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But \((-\Psi^1)_0(u^0,v) = \Psi^0(u^0; -v)\) (a property of the Clarke derivative, see [1]) and thus
\[
\int_{\Omega} |\nabla u_0|^{p-2} \cdot \nabla (-v) \, dx \leq \Psi^0(u_0; -v) \quad \forall v \in X; \quad (46)
\]
that is,
\[
\mu_0(v) = \int_{\Omega} |\nabla u_0|^{p-2} \cdot \nabla v \, dx \leq \Psi^0(u_0, v) \quad \forall v \in X, \quad (47)
\]
\[
\mu_0 = -\Delta_p u_0 \in \partial \Psi_1(u_0) \quad \text{and, using Proposition 15,} \quad -\Delta_p u_0 \in \partial \Psi(u_0).
\]

Problem (***). Let \(u_0\) be a critical point for \(\Phi\). We have
\[
0 \in \partial \Phi(u_0) \subset \partial Q(u_0) + \partial (-\Psi_1)(u_0), \quad (49)
\]
since \(\Phi = Q - \Psi_1\) and apply some rules of subdifferential calculus concerning finite sums. \(\partial Q(u_0) = \{Q'(u_0)\}\), where \(Q'(u_0)(v) = \sum_{i=1}^{N} \int_{\Omega} \frac{\partial u_0}{\partial x_i} \frac{\partial u_0}{\partial x_i} ||\nabla v||_2^2 \, dx = \langle Q'(u_0), v \rangle\) [2].

Using (49) and a mentioned property of a function \(f\) Lipschitz around \(x_0\), we find
\[
0 \leq \sum_{i=1}^{N} \int_{\Omega} \frac{\partial u_0}{\partial x_i} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx + (-\Psi^0(0,u^0;v)). \quad (50)
\]
But \((-\Psi^1)_0(u^0,v) = \Psi^0(u^0; -v)\) (a property of the Clarke derivative, see [1]) and thus
\[
\sum_{i=1}^{N} \int_{\Omega} \frac{\partial u_0}{\partial x_i} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx \leq \Psi^0(u_0; -v) \quad \forall v \in X; \quad (51)
\]
that is,
\[
\mu_0(v) = \sum_{i=1}^{N} \int_{\Omega} \frac{\partial u_0}{\partial x_i} |\nabla u_0|^{p-2} \frac{\partial u_0}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx \leq \Psi^0(u_0, v) \quad \forall v \in X, \quad (52)
\]
\[
\mu_0 = -\Delta_p u_0 \in \partial \Psi_1(u_0) \quad \text{and, using Proposition 15,} \quad -\Delta_p u_0 \in \partial \Psi(u_0).
\]

4. An Application of Ghoussoub-Maurey Linear Principle to \(p\)-Laplacian and to \(p\)-Pseudo-Laplacian

Start with the statement of this generalized perturbed variational principle.

**Theorem 17** (Ghoussoub-Maurey linear principle). Let \(X\) be reflexive separable space and \(\varphi : X \rightarrow (-\infty, +\infty]\) lower semicontinuous and proper:

(I) If \(\varphi\) is bounded from below on the closed bounded nonempty subset \(C\), the set
\[
\{\xi \in X^* : \varphi + \xi \text{ strongly exposes } C \text{ from below}\}
\]
is of \(G_0\) type and everywhere dense.

(II) If, for any \(\xi\) from \(X^*\), \(\varphi + \xi\) is bounded from below, the set
\[
\{\xi \in X^* : \varphi + \xi \text{ strongly exposes } X \text{ from below}\}
\]
is of \(G_0\) type and everywhere dense.

To clarify the involved notions, let \(X\) be a real normed space, \(f : X \rightarrow (-\infty, +\infty]\), \(C\) nonempty subset of \(X\), and \(x_0 \in C\). \(f\) strongly exposes \(C\) from below in \(x_0\), when \(f(x_0) = \inf \{f(x) < +\infty \text{ and } x \in C\} \geq 1, f(x_0) = f(x_0) \rightarrow x_0 \rightarrow x_0\). \(f\) strongly exposes \(C\) from above in \(x_0\) with a similar definition. Remark that, taking \(C = X\) in the given definition, we fall on the definition of strongly minimum point. And also, a set of \(G_0\) type means a set which is a countable intersection of open sets. A set of \(F_0\) type means a set which is a countable union of closed sets.

We imply this theorem in two generalizations of a minimization problem of the form [22]
\[
C_f = \min \left\{ \int_{\Omega} \left[ \frac{1}{p} |u|^p + \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^p \right] - f(u) \right\} \quad dx : \quad u \in W_0^{1,p}(\Omega), ||u||_{L^p} = 1 \right\},
\]
\[
C_f = \min \left\{ \int_{\Omega} \left[ \frac{1}{p} \sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^p \right] - f(u) \right\} \quad dx : \quad u \in W_0^{1,p}(\Omega), ||u||_{L^p} = 1 \right\},
\]
where \(\Omega\) is open set of \(C^1\) class in \(\mathbb{R}^N, N \geq 3, f \in W^{-1,p}(\Omega) (= W_0^{1,p}(\Omega)^*)\), \(1/p + 1/p' = 1, p^* = 2N/(N - 2)\), the critical exponent for the Sobolev embedding (for the necessary explanations, here and in the following, see [1, 1, §4, last section]).

Let \(\Omega\) be an open bounded set of \(C^1\) class in \(\mathbb{R}^N, N \geq 3\). Consider the problems (*) and (**), where \(f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) is a Carathéodory function with the growth condition.
\[ f(x,s) \leq c|s|^{p-1} + b(x), \]
\[ c > 0, \ 2 \leq p \leq \frac{2N}{N-2}, \ b \in L^{p'}(\Omega), \ \frac{1}{p} + \frac{1}{p'} = 1. \] (58)

The functionals \( \varphi : W^{1,p}_0(\Omega) \rightarrow \mathbb{R} \),

\[ \varphi (u) = \int_{\Omega} \left[ \frac{1}{p} |u|^p + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \right] dx, \] (59)

\[ \varphi (u) = \int_{\Omega} \left( \frac{1}{p} \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \right)^p - F(x,u(x)) \right) dx, \] (60)

with \( F(x,s) = \int_{0}^{s} f(x,t) dt \), are of \( C^1 \)-class Fréchet and their critical points are the weak solutions of the problems \((*)\) and \((***)\), respectively.

**Problem \((*)\).** Let \( \lambda_1 \) be the first eigenvalue of \(-\Delta_p \) in \( W^{1,p}_0(\Omega) \) with homogeneous boundary condition. We have (see [2, 62])

\[ \lambda_1 = \inf \left\{ \|u\|_{1,p}^p : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\} \] (61)

(The Rayleigh-Ritz quotient).

And now give an answer for (56). Use the norm \( \| \cdot \|_p \) on \( W^{1,p}_0(\Omega) \) (see above). Denote the dual of \( (W^{1,p}_0(\Omega), \| \cdot \|_p) \) by \( W^{-1,p'}(\Omega) \), where \( p' \) is the conjugate of \( p \) (i.e., \( 1/p + 1/p' = 1 \)).

**Proposition 18.** Under the above assumptions and in addition the growth condition

\[ F(x,s) \leq c_1 s^p + \alpha(x)s, \] (62)

with \( 0 < c_1 < \lambda_1, \alpha \in L^{p'}(\Omega) \) for some \( 2 \leq q \leq 2N/(N-2) \) and \( f(x,s) = -f(x,s), \forall x \in \Omega \), the following assertions hold:

(i) The set of functions \( h \) from \( W^{-1,p'}(\Omega) \), having the property that the functional \( \varphi_h : W^{1,p}_0(\Omega) \rightarrow \mathbb{R} \),

\[ \varphi_h (u) = \frac{1}{p} \|u\|_p^p - \int_{\Omega} \left( F(x,u(x)) + h(u(x)) \right) dx \] (63)

has in only one point an attained minimum includes a \( G_\delta \) set everywhere dense.

(ii) The set of functions \( h \) from \( W^{-1,p'}(\Omega) \), having the property

the problem \( \begin{align*}
-\Delta_p u &= f(x,u) + h(u) & \text{in } \Omega \\
\ u &= 0 & \text{on } \partial\Omega 
\end{align*} \] (64)

has solutions, includes a \( G_\delta \) set everywhere dense.

(iii) Moreover, if \( s \rightarrow f(x,s) \) is increasing, then the set of functions \( h \) from \( W^{-1,p'}(\Omega) \), having the property

the problem \( \begin{align*}
-\Delta_p u &= f(x,u) + h(u) & \text{in } \Omega \\
\ u &= 0 & \text{on } \partial\Omega 
\end{align*} \) (65)

has a unique solution, includes a \( G_\delta \) set everywhere dense.

**Remark 19.** This is a generalization to \( p \)-Laplacian and at \( W^{1,p}_0(\Omega) \) of Theorem 2.13 from [14].

**Proof.** It is sufficient to justify (i). Consider, for each \( h \) from \( W^{-1,p'}(\Omega) \), the functional \( \xi_h \) from \( W^{-1,p'}(\Omega) \):

\[ \xi_h (u) = -\int_{\Omega} h(u(x)) dx. \] (66)

One may see that \( \varphi_h = \varphi + \xi_h \) (see (59)). Consequently, according to Ghoussoub-Maurey linear principle, (II), if we show that \( \varphi_h \) is bounded from below for any \( h \) from \( W^{-1,p'}(\Omega) \), (i) is proven. But, taking into account the Sobolev embedding and (62), we have \( \forall u \in W^{1,p}_0(\Omega) \):

\[ (\varphi + \xi_h)(u) = \frac{1}{p} \|u\|_{1,p}^p - \int_{\Omega} F(x,u(x)) dx \]

\[ -\int_{\Omega} h(u(x)) dx \]

\[ \geq \frac{1}{p} \|u\|_{1,p}^p - c_1 \int_{\Omega} |u(x)|^p dx \]

\[ -\int_{\Omega} \alpha(x) u(x) dx - \int_{\Omega} h(u(x)) dx \] (67)

\[ \geq \frac{1}{p} \|u\|_{1,p}^p - \frac{c_1}{\lambda_1} \|u\|_{p'} - \|\alpha\|_{q'} \|u\|_q \]

\[ -\|h\|_{W^{-1,p'}} \|u\|_{1,p} \]

\[ \geq \frac{1}{p} \left( 1 - \frac{c_1}{\lambda_1} \right) \|u\|_{p'}^p - r \|u\|_{1,p} \]

\[ = \|u\|_{1,p} \left( \frac{1}{p} \left( 1 - \frac{c_1}{\lambda_1} \right) \|u\|_{p'}^p - r \right), \]

\( r \in \mathbb{R} \), and hence the conclusion since \( 1 - c_1/\lambda_1 > 0 \). To prove some of these inequalities,

\[ \int_{\Omega} F(x,u(x)) dx \leq \int_{\Omega} \left( \|u(x)\|_p^p + \alpha(x) u(x) \right) dx \]

\[ = \frac{1}{p} \|u\|_{1,p}^p + \int_{\Omega} \alpha(x) u(x) dx \]

\[ \leq \frac{1}{p \lambda_1} \|u\|_{1,p}^p + \|\alpha\|_{q'} \|u\|_q \]

\[ \leq \frac{1}{p \lambda_1} \|u\|_{1,p}^p + K \|u\|_{1,p} \] (68)
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(see \( q \) and properties of Sobolev spaces, e.g., [2]) and
\[
\int_{\Omega} h(u(x)) \, dx = \langle h, u \rangle \leq \|h\|_{W^{1-\theta'}_{0}} \|u\|_{1,\theta'} \quad \text{(the norm of the linear continuous map).}
\]

Problem (**). Let \( \lambda_1 \) be the first eigenvalue of \(-\Delta_\theta^p\) in \( W^{1,p}_0(\Omega) \) with homogeneous boundary condition. We have (see [2, 7.2])
\[
\lambda_1 = \inf \left\{ \frac{\int_{\Omega} F(x, u(x)) \, dx}{\|u\|_{1,\theta'}^p} : u \in W^{1,p}_0(\Omega) \setminus \{0\} \right\}
\]
(69) (the Rayleigh - Ritz quotient).

And now give an answer for (57). Use the norm \( \mathbf{1} \cdot \mathbf{1}_p \) on \( W^{1,p}_0(\Omega) \) (see above). Denote also the dual of \((W^{1,p}_0(\Omega), \mathbf{1} \cdot \mathbf{1}_p)\) by \( W^{-1,p'}(\Omega) \), where \( p' \) is the conjugate of \( p \) (i.e., \( 1/p + 1/p' = 1 \)).

Proposition 20. Under the above assumptions and in addition the growth condition
\[
F(x, s) \leq c_1 s^p + \alpha(x) s, \quad (70)
\]
with \( 0 < c_1 < \lambda_1 \), \( \alpha \in L^\theta(\Omega) \) for some \( 2 \leq q \leq 2N/(N-2) \) and \( f(x, -s) = -f(x, s), \forall x \) from \( \Omega \), the following assertions hold.

(i) The set of functions \( h \) from \( W^{-1,p'}(\Omega) \), having the property that the functional \( \varphi_h : W^{1,p}_0(\Omega) \to \mathbb{R} \)
\[
\varphi_h(u) = \frac{1}{p} \mathbf{1} \! \cdot \! \mathbf{1}_p - \int_{\Omega} (F(x, u(x)) + h(u(x))) \, dx
\]
(71) has in only one point an attained minimum includes a \( G_\Theta \) set everywhere dense.

(ii) The set of functions \( h \) from \( W^{-1,p'}(\Omega) \), having the property
the problem
\[
\begin{cases}
-\Delta_\theta^p u = f(x, u(x)) + h(u(x)) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
(72) has solutions,

includes a \( G_\Theta \) set everywhere dense.

(iii) Moreover, if \( f(x, s) \) is increasing, then the set of functions \( h \) from \( W^{-1,p'}(\Omega) \), having the property
the problem
\[
\begin{cases}
-\Delta_\theta^p u = f(x, u(x)) + h(u(x)) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]
(73) has a unique solution,

includes a \( G_\Theta \) set everywhere dense.

Remark 21. This is a generalization to \( p \)-pseudo-Laplacian and at \( W^{1,p}_0(\Omega) \) of Theorem 2.13 from [14].

Proof. The proof and the afferent calculus follow step by step those for Proposition 18. One replaces there the norm \( \| \cdot \|_p \) on \( W^{1,p}_0(\Omega) \) by the norm \( \mathbf{1} \cdot \mathbf{1}_p \) and the inequalities and the considerations remain the same.

5. Conclusions

Three ways to obtain and/or characterize weak solutions for two problems of mathematical physics equations involving Dirichlet problems for the \( p \)-Laplacian and the \( p \)-pseudo-Laplacian are developed.

The first sequence of results used the Ekeland variational principle to obtain theoretical propositions which generalize two statements given by Ghoussoub in which the author replaced the real Hilbert space by real reflexive uniformly convex Banach space, and the Fréchet \( C^1 \)-class of the goal function by the conditions imposed to this to be lower semi-continuous and Gâteaux differentiable. It is also worthwhile to underline that the Gâteaux differentiability can be replaced by the property of \( p \)-differentiability, \( p \) being any bornology. These theoretical statements have been used to characterize weak solutions for the \( p \)-Laplacian and for the \( p \)-pseudo-Laplacian. Some adequate examples were also given.

The second succession of statements establishes results for nondifferentiable functionals using Clarke gradient and other specific notions until their insertion in characterization of weak solutions for Dirichlet problems with the \( p \)-Laplacian and the \( p \)-pseudo-Laplacian, respectively, in \( W^{1,p}_0(\Omega) \).

The last sequence of assertions starts from Ghoussoub-Maurey linear principle which is used in order to solve some minimization problems. Generalizations of a minimization principle for the Laplacian given by Brezis and Nirenberg have been obtained in conjunction with characterization of weak solutions of Dirichlet problems for the \( p \)-Laplacian and for the \( p \)-pseudo-Laplacian.

Competing Interests

The author declares that they have no competing interests.

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