Research Article

Approximation of a Common Element of the Fixed Point Sets of Multivalued Strictly Pseudocontractive-Type Mappings and the Set of Solutions of an Equilibrium Problem in Hilbert Spaces

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Received 8 December 2015; Revised 14 February 2016; Accepted 17 February 2016

1. Introduction

Let $X$ be a nonempty set and let $T : X \rightarrow X$ be a map. A point $x \in X$ is called a fixed point of $T$ if $x = Tx$. If $T : X \rightarrow 2^X$ is a multivalued map then $x$ is a fixed point of $T$ if $x \in Tx$. If $Tx = \{x\}$ then $x$ is called a strict fixed point of $T$. The set $F(T) = \{x \in D(T) : x \in Tx\}$ (resp., $F(T) = \{x \in D(T) : x = Tx\}$) is called the fixed point set of multivalued (resp., single-valued) map $T$, while the set $F_s(T) = \{x \in D(T) : Tx = \{x\}\}$ is called the strict fixed point set of $T$.

Let $X$ be a normed space. A subset $K$ of $X$ is called proximinal if for each $x \in X$ there exists $k \in K$ such that

$$
\|x - k\| = \inf \{\|x - y\| : y \in K\} = d(x, K).
$$

(1)

It is known that every closed convex subset of a uniformly convex Banach space is proximinal. We will denote the family of all nonempty closed and bounded subsets of $X$ by $CB(X)$, the family of all nonempty subsets of $X$ by $2^X$, and the family of all proximinal subsets of $X$ by $P(X)$, for a nonempty set $X$.

Let $H$ denote the Hausdorff metric induced by the metric $d$ on $X$; that is, for every $A, B \in CB(X),$

$$
H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.
$$

(2)

Let $X$ be a normed space. Let $T : D(T) \subseteq X \rightarrow 2^X$ be a multivalued mapping on $X$. A multivalued mapping $T : D(T) \subseteq X \rightarrow 2^X$ is called $L$-Lipschitzian if there exists $L \geq 0$ such that for all $x, y \in D(T)$

$$
H(Tx, Ty) \leq L\|x - y\|.
$$

(3)

In (3) if $L \in [0, 1)$ $T$ is said to be a contraction while $T$ is nonexpansive if $L = 1$. $T$ is called quasi-nonexpansive if $F(T) = \{x \in D(T) : x \in Tx\} \neq \emptyset$ and for all $p \in F(T),$

$$
H(Tx,Tp) \leq \|x - p\|.
$$

(4)

Clearly every nonexpansive mapping with nonempty fixed point set is quasi-nonexpansive. $T$ is said to be $k$-strictly pseudocontractive-type of Isiogugu [1] if there exists $k \in (0, 1)$ such that, given any pair $x, y \in D(T)$ and $u \in Tx$, there exists $v \in Ty$ satisfying $\|u - v\| \leq H(Tx, Ty)$ and

$$
H^2(Tx, Ty) \leq \|x - y\|^2 + k\|x - u - (y - v)\|^2.
$$

(5)

If $k = 1$ in (5), $T$ is said to be pseudocontractive-type, while $T$ is nonexpansive-type if $k = 0$. Every multivalued nonexpansive mapping $T : D(T) \subseteq X \rightarrow P(X)$ is nonexpansive-type. In a real Hilbert space $H, T : D(T) \subseteq


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\( H \rightarrow \text{CB}(H) \) is said to be \( k \)-strictly pseudocontractive of Chidume et al. [2] if there exists \( k \in (0,1) \) such that for all \( x, y \in D(T) \)

\[
H^2(Tx,Ty) \leq \|x - y\|^2 + k \|x - u - (y - v)\|^2, \\
\forall u \in Tx, \ v \in Ty.
\]

(6)

If \( k = 1 \), \( T \) is said to be pseudocontractive. It is easy to see that every \( k \)-strictly pseudocontractive mapping \( T : D(T) \subseteq H \rightarrow P(H) \) is \( k \)-strictly pseudocontractive-type mappings. Second, establish the convexity property for a strict fixed point set of a multivalued \( k \)-strictly pseudocontractive-type mapping. Let \( F : C \times C \rightarrow \mathbb{R} \) be a bifunction, where \( \mathbb{R} \) is the set of real numbers. The equilibrium problem for \( F : C \times C \rightarrow \mathbb{R} \) is to find \( x \in C \) such that

\[
F(x, y) \geq 0 \quad \forall y \in C.
\]

The set of solutions of (7) is denoted by EP(\( F \)). Several algorithms were introduced by authors for approximating solutions of equilibrium problems for a bifunction (or finite family of bifunctions) (see, e.g., [3] and references therein). Given a mapping \( A : C \rightarrow H \) and \( x \in C \), let \( z \in EP(F) \) if and only if \( \langle Az, y - z \rangle \geq 0 \) for all \( y \in C \); that is, \( z \) is a solution of the variational inequality problem of finding \( z \in VIP(A, C) \). Numerous problems in physics, optimization, and economics are reduced to the problem of finding the solutions of (7) (see, e.g., [4–6] and the references therein).

The purpose of this work is to first establish closed and convexity property for a strict fixed point set of a multivalued strictly pseudocontractive-type mappings. Second, establish with a strict fixed point set condition a strong convergence of a hybrid algorithm to a common element of the fixed point sets of two multivalued strictly pseudocontractive-type mappings and the set of solutions of an equilibrium problem in Hilbert spaces. The obtained results extend, complement, and improve the results on equilibrium problems as well as multivalued and single-valued mappings in the contemporary literature.

2. Preliminaries

In the sequel, we will need the following definitions and lemmas.

Definition 1. Let \( T : X \rightarrow 2^X \) be a multivalued mapping; for each \( x \in X \), \( P_T(x) \) is defined by

\[
P_T(x) = \{ y \in Tx : \|x - y\| = d(x, Tx) \}.
\]

(8)

For solving the equilibrium problems for a bifunction \( F : C \times C \rightarrow \mathbb{R} \), let us assume that \( F \) satisfies the following conditions:

(A1) \( F(x, x) = 0 \) for all \( x \in C \).

(A2) \( F \) is monotone; that is, \( F(x, y) + F(y, x) \leq 0 \), for all \( x, y \in C \).

(A3) For each \( x, y, z \in C \), \( \lim_{t \downarrow 0} F(tx+(1-t)x, y) \leq F(x, y) \).

(A4) For each \( x \in C \), \( y \rightarrow F(x, y) \) is convex and lower semicontinuous.

Lemma 2 (see [4]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( F : C \times C \rightarrow \mathbb{R} \) be a bifunction satisfying (A1)–(A4). Let \( r > 0 \) and \( x \in H \). Then, there exists \( z \in C \) such that

\[
F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.
\]

(9)

Lemma 3 (see [6]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Assume that \( F : C \times C \rightarrow \mathbb{R} \) satisfying (A1)–(A4). Let \( r > 0 \) and \( x \in H \). Define \( T_r : H \rightarrow 2^C \) by

\[
T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \right\},
\]

(10)

\[ \forall y \in C. \]

Then the following hold:

(1) \( T_r \) is single valued.

(2) \( T_r \) is firmly nonexpansive; that is, for any \( x, y \in H \), \( \|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle \).

(3) \( F(T_r) = EP(\mathcal{F}) \).

(4) \( EP(F) \) is closed and convex.

Lemma 4 (see [7]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( F : C \times C \rightarrow \mathbb{R} \) be a bifunction satisfying (A1)–(A4). Let \( r > 0 \) and \( x \in H \). Then for all \( x \in H \) and \( p \in F(T_r x) \)

\[
\|p - T_r x\|^2 + \|T_r x - x\|^2 \leq \|p - x\|^2.
\]

(11)

Lemma 5. Let \( H \) be a real Hilbert space, and let \( C \) be a nonempty closed convex subset of \( H \). Let \( P_C \) be the convex projection onto \( C \). Then, convex projection is characterized by the following relations:

(i) \( x^* = P_C(x) \Leftrightarrow (x - x^*, y - x^*) \leq 0, \forall y \in C \).

(ii) \( \|x - P_C x\|^2 \leq \|x - y\|^2 - \|y - P_C y\|^2 \).

(iii) \( \|x - P_C y\|^2 \leq \|x - y\|^2 - \|P_C y\|^2 \).

3. Main Results

Proposition 6. Let \( K \) be a nonempty subset of a real Hilbert space \( H \). And let \( T : K \rightarrow P(K) \) be a \( k \)-strictly pseudocontractive-type mapping such that \( F(T) \) is nonempty. Then \( F(T) \) is closed and convex.

Proof. Let \( \{x_n\}_{n=1}^{\infty} \subseteq F(T) \) such that \( \{x_n\}_{n=1}^{\infty} \) converges to \( x \in K \). We show that \( x \in F(T) \). Let \( u \in Tx \) be arbitrary:

\[
\|x - u\| \leq \|x - x_{n}\| + \|x_{n} - u\| \leq \|x - x_{n}\| + H(Tx_{n}, Tx)
\]

\[
\leq \|x - x_{n}\| + \|x - x_{n}\| + \sqrt{r} \|x - u\|.
\]

(12)
Taking limits as \( n \to \infty \), we have that \( \|x - u\| \leq \sqrt{k} \|x - u\| \). Hence, \( x = u \in T x \). Since \( u \) was arbitrary, we have that \( T x = \{x\} \).

We now prove that \( F(T) \) is convex. Let \( p_1, p_2 \in F(T) \) and \( z = \alpha p_1 + (1 - \alpha)p_2 \) and then \( z - p_1 = (1 - \alpha)(p_2 - p_1) \) and \( z - p_2 = \alpha(p_1 - p_2) \):

\[
d^2(z, Tz) \leq \|z - u\|^2, \quad \forall u \in Tz
\]

\[
= \|\alpha p_1 + (1 - \alpha)p_2 - u\|^2
\]

\[
= \alpha \|p_1 - u\|^2 + (1 - \alpha) \|p_2 - u\|^2
\]

\[
- \alpha (1 - \alpha) \|p_2 - p_1\|^2.
\]

Now, \( k \)-strictly pseudocontractive-type condition on \( T \) and a strict fixed point condition on \( p_1 \) and \( p_2 \) imply that, for all \( u \in Tz, \|z - p_1\| \leq H(Tz, Tp_1) \) and \( H^2(Tz, Tp_1) \leq \|z - p_1\|^2 + k\|z - u\|^2 \). \( \|u - p_2\| \leq H(Tz, Tp_2) \) and \( H^2(Tz, Tp_2) \leq \|z - p_2\|^2 + k\|z - u\|^2 \). It then follows that

\[
d^2(z, Tz) \leq \|u - z\|^2
\]

\[
= \alpha \|p_1 - u\|^2 + (1 - \alpha) \|p_2 - u\|^2
\]

\[
- \alpha (1 - \alpha) \|p_2 - p_1\|^2
\]

\[
\leq \alpha H^2(Tz, Tp_1) + (1 - \alpha) H^2(Tz, Tp_2)
\]

\[
- \alpha (1 - \alpha) \|p_1 - p_2\|^2
\]

\[
\leq \alpha \left[ \|z - p_1\|^2 + k\|z - u\|^2 \right]
\]

\[
+ (1 - \alpha) \left[ \|z - p_2\|^2 + k\|z - u\|^2 \right]
\]

\[
- \alpha (1 - \alpha) \|p_1 - p_2\|^2.
\]

In particular, for each \( u \in P_z \),

\[
d^2(z, Tz) \leq \alpha \left[ \|z - p_1\|^2 + k\alpha d^2(z, Tz) \right]
\]

\[
+ (1 - \alpha) \left[ \|z - p_2\|^2 + k\alpha d^2(z, Tz) \right]
\]

\[
- \alpha (1 - \alpha) \|p_1 - p_2\|^2
\]

\[
= \|\alpha p_1 + (1 - \alpha)p_2 - z\|^2 + k\alpha d^2(z, Tz)
\]

\[
d^2(z, Tz).
\]

Hence, \( d(z, Tz) = 0 \). Since \( Tz \) is proximinal, there exists \( w \in Tz \) such that \( \|w - z\| = 0 \); consequently, \( z \in Tz \). Also, if \( v \in Tz \), then

\[
\|v - z\|^2 = \|v - \alpha p_1 + (1 - \alpha)p_2\|^2
\]

\[
\leq \alpha \left[ \|z - p_1\|^2 + k\|w - v\|^2 \right]
\]

\[
+ (1 - \alpha) \left[ \|z - p_2\|^2 + k\|w - v\|^2 \right]
\]

\[
- \alpha (1 - \alpha) \|p_1 - p_2\|^2 = k\|w - v\|^2
\]

which shows that \( z = v \). Thus, \( Tz = \{z\} \).}

We now prove a strong convergence of multivalued version of the hybrid algorithm considered in [8] to a common element of the set of fixed points of two \( k \)-strictly pseudocontractive-type mappings and the set of solutions of an equilibrium problem in Hilbert spaces. As a corollary, we obtain a hybrid algorithm for finding common elements of the set of fixed points of two multivalued strictly pseudocontractive mappings of [2] and the set of solutions of an equilibrium problem, with a strict fixed point set condition.

Theorem 7. Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \), let \( f : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)–(A4), and let \( S, T : C \to P(C) \) be two strictly pseudocontractive-type mappings with contractive coefficients \( \lambda_1 \) and \( \lambda_2 \), respectively, such that \( F = F(S) \cap F(T) \neq \emptyset \).

Let \( \{x_n\} \) be a sequence generated from an arbitrary \( x_0 \in C \) as follows:

\[
x_0 \in H,
\]

\[
C_1 = C,
\]

\[
x_1 = P_{S x_0} y_n = \alpha_n x_n + (1 - \alpha_n) \left[ \beta_n v_n + (1 - \beta_n) z_n \right],
\]

\[
u_n \in C \text{ such that}
\]

\[
F(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - y_n) \geq 0, \quad \forall y \in C,
\]

\[
C_{n+1} = \left\{ z \in C_n : \|z - u_n\|^2 \leq \|z - x_n\|^2 \right\},
\]

\[
x_{n+1} = P_{C_{n+1}} x_0,
\]

where \( v_n \in T x_n \) and \( z_n \in S x_n \). \( \{\alpha_n\}_{n=1}^{\infty} \) and \( \{\beta_n\}_{n=1}^{\infty} \) are sequences in \( [0, 1] \) satisfying

(i) \( \alpha_n \geq \max\{\lambda_1, \lambda_2\} \),

(ii) \( \lim \inf_{n \to \infty} (1 - \alpha_n)(1 - \beta_n)(\alpha_n - \lambda_1) > 0 \) and \( \lim \inf_{n \to \infty} (1 - \alpha_n)(\alpha_n - \lambda_2) \beta_n > 0 \),

(iii) \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \).

Then \( \{x_n\} \) converges strongly to \( p \in P_x x_0 \).

Proof. Observe that \( C_n \) is closed and convex for all \( n \geq 1 \); therefore \( P_{C_{n+1}} x_0 \) is well defined and note that \( u_n = T_{r_n} y_n \). Next we show that \( F \subset C_n \) for all \( n \geq 1 \). \( F \subset C_1 \subset C \) is obvious. Suppose \( F \subset C_k \), set \( u_n = \beta_n v_n + (1 - \beta_n) z_n \), and then using Lemma 3, for all \( q \in F \), we have

\[
\|q - u_k\|^2 = \|q - T_{r_k} y_k\|^2 \leq \|q - y_k\|^2
\]

\[
= \|q - [\alpha_k x_k + (1 - \alpha_k) (1 - \beta_k) z_k] + (1 - \beta_k) z_k\|^2
\]

\[
= \|q - [\alpha_k x_k + (1 - \alpha_k) u_k]\|^2
\]

\[
= \alpha_k \|x_k - q\|^2 + (1 - \alpha_k) \|u_k - q\|^2
\]

\[
- \alpha_k (1 - \alpha_k) \|x_k - u_k\|^2.
\]
Also,
\[
\|w_k - q\|^2 = \|\beta_k v_k + (1 - \beta_k) z_k - q\|^2
\]
\[
= \beta_k \|v_k - q\|^2 + (1 - \beta_k) \|z_k - q\|^2
- \beta_k (1 - \beta_k) \|v_k - z_k\|^2. \tag{19}
\]

Using (19) we obtain from (18) that
\[
\|q - u_k\|^2
\leq \alpha_k \|x_k - q\|^2 + (1 - \alpha_k) \beta_k \|v_k - q\|^2
+ (1 - \alpha_k) (1 - \beta_k) \|z_k - q\|^2
- \alpha_k (1 - \alpha_k) \|x_k - u_k\|^2
\leq \alpha_k \|x_k - q\|^2 + (1 - \alpha_k) \beta_k H^2 (Tx_k, Tq)
+ (1 - \alpha_k) (1 - \beta_k) H^2 (Sx_k, Tq)
- \alpha_k (1 - \alpha_k) \|x_k - u_k\|^2. \tag{20}
\]

Using (21) we obtain from (20) that
\[
\|q - u_k\|^2
\leq \alpha_k \|x_k - q\|^2 + (1 - \alpha_k) \beta_k \|v_k - q\|^2
+ (1 - \alpha_k) (1 - \beta_k) \|z_k - q\|^2
- \alpha_k (1 - \alpha_k) \|x_k - u_k\|^2
\leq \alpha_k \|x_k - q\|^2 + (1 - \alpha_k) \beta_k \|v_k - q\|^2
+ (1 - \alpha_k) (1 - \beta_k) \|z_k - q\|^2
- \alpha_k (1 - \alpha_k) \|x_k - u_k\|^2. \tag{22}
\]

This shows that \(q \in C_{k+1}\). It then follows that \(F \subseteq C_n\) for all \(n \geq 1\). From \(x_n = P_{C_n} x_0\) we have from Lemma 5(i) that
\[
\langle x_n - y, x_0 - x_n \rangle \geq 0, \quad \forall y \in C_n. \tag{23}
\]

Since \(F \subseteq C_n\) for all \(n \geq 1\), we have
\[
\langle x_n - q, x_0 - x_n \rangle \geq 0, \quad \forall q \in F. \tag{24}
\]

Using Lemma 5(ii) we obtain
\[
\|x_n - x_0\|^2 = \|P_{C_n} x_0 - x_0\|^2 \leq \|x_0 - q\|^2 - \|q - x_n\|^2 \tag{25}
\]
for each \(q \in F \subseteq C_n\) and for all \(n \geq 1\). Consequently the sequence \(\{x_n\}\) is bounded, and so are \(\{z_n\}\) and \(\{v_n\}\). Furthermore, since \(x_n = P_{C_n} x_0\) and \(x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subseteq C_n\), then from definition of \(P_{C_n}\) we have \(\|x_n - x_0\| \leq \|x_{n+1} - x_0\|\) for all \(n \geq 1\). Therefore the sequence \(\|x_n - x_0\|\) is nondecreasing. It then follows that \(\lim_{n \to \infty} \|x_n - x_0\|\) exists. From the construction of \(C_n\), we have that \(C_n \subseteq C_n\) and \(x_n = P_{C_n} x_0 \in C_n\) for any integer \(m \geq n\). It also follows from Lemma 5(iii) that
\[
\|x_n - x_0\|^2 = \|x_n - P_{C_n} x_0\|^2
\leq \|x_n - x_0\|^2 - \|P_{C_n} x_0 - x_0\|^2 \tag{26}
\]

Letting \(m, n \to \infty\) in (26), we have \(\|x_m - x_n\| \to 0\). Hence \(\{x_n\}\) is a Cauchy sequence. Since \(H\) is Hilbert and \(C\) is closed and convex, we can assume that \(x_n \to p \in C\) as \(n \to \infty\); that is, \(\lim_{n \to \infty} \|x_n - p\| = 0\). We now show that \(p \in F(S)\). In particular when \(m = n + 1\) in (26) we obtain
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{27}
\]
Also, since \( x_{n+1} \in C_{n+1} \), we obtain
\[
\|x_{n+1} - u_n\| \leq \|x_{n+1} - x_n\|. \tag{28}
\]
It then follows from (27) that
\[
\lim_{n \to \infty} \|x_{n+1} - u_n\| = 0. \tag{29}
\]
Combining (27) and (29) we obtain
\[
\lim_{n \to \infty} \|x_n - u_n\| = 0. \tag{30}
\]
It follows from \( \lim_{n \to \infty} \|x_n - p\| = 0 \) and (30) that
\[
\lim_{n \to \infty} \|u_n - p\| = 0. \tag{31}
\]
Setting \( n = k \) in (22) we have
\[
\|u_n - q\|^2 \leq \|x_n - q\|^2 - \beta_n (1 - \alpha_n) (\alpha_n - \lambda_2) \|x_n - v_n\|^2 \tag{32}
\]
\[
- (1 - \alpha_n) (1 - \beta_n) (\alpha_n - \lambda_1) \|x_n - z_n\|^2
- (1 - \alpha_n)^2 (1 - \beta_n) \|v_n - z_n\|^2.
\]
Observe that
\[
\|q - x_n\|^2 - \|q - u_n\|^2 = \|x_n\|^2 - \|u_n\|^2 - 2 \langle q, x_n - u_n \rangle \leq \|x_n - u_n\| (\|x_n\| + \|u_n\|) + \|q\| \|x_n - u_n\|. \tag{33}
\]
It then follows from (30) that
\[
\lim_{n \to \infty} \|q - x_n\| - \|q - u_n\| = 0. \tag{34}
\]
Using \( \lim_{n \to \infty} (1 - \alpha_n) (1 - \beta_n) (\alpha_n - \lambda_1) > 0 \) and \( \lim_{n \to \infty} (1 - \alpha_n) (\alpha_n - \lambda_2) \beta_n > 0 \) we obtain from (32) that \( \lim_{n \to \infty} \|x_n - v_n\| = 0 \) and \( \lim_{n \to \infty} \|x_n - z_n\| = 0 \). Hence \( p \in F(S) \cap F(T) \). It remains to show that \( p \) is in EP(f). Now from (32)
\[
\|q - y_n\| \leq \|q - x_n\|. \tag{35}
\]
Also, using \( u_n = T_{r_n} y_n \), Lemma 4, and (35) we have
\[
\|u_n - y_n\|^2 = \|T_{r_n} y_n - y_n\|^2 \leq \|q - y_n\|^2 - \|q - T_{r_n} y_n\|^2 \leq \|q - x_n\|^2 - \|q - T_{r_n} y_n\|^2 \leq \|q - x_n\|^2 - \|q - u_n\|^2. \tag{36}
\]
It then follows from (34) and (36) that
\[
\lim_{n \to \infty} \|u_n - y_n\| = 0. \tag{37}
\]
Consequently, we obtain from (31) and (37) that
\[
\lim_{n \to \infty} \|y_n - p\| = 0. \tag{38}
\]
From the assumption that \( r_n \geq a > 0 \)
\[
\lim_{n \to \infty} \|u_n - y_n\| = 0. \tag{39}
\]
Since \( u_n = T_{r_n} y_n \) implies
\[
f(\langle u_n, y \rangle + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle) \geq 0, \tag{40}
\]
we have from (A2) that
\[
\frac{\|u_n - y_n\|^2}{r_n} \geq \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq f(\langle u_n, y \rangle) \tag{41}
\]
\[
\geq f(y, u_n), \forall y \in C.
\]
By taking limit as \( n \to \infty \) of the above inequality and from (A4), (31), and (38) we have \( f(\langle y, p \rangle) \leq 0 \), for all \( y \in C \). Let \( t \in (0, 1) \) and for all \( y \in C \), since \( p \in C \), we have that \( y_t = ty + (1 - t)p \in C \). Hence \( f(\langle y_t, p \rangle) \leq 0 \). It follows from (A1) that
\[
0 = f(\langle y_t, y_t \rangle) = tf(\langle y_t, y \rangle) + (1 - t)f(\langle y_t, p \rangle) \leq tf(\langle y_t, y \rangle); \tag{42}
\]
that is, \( f(\langle y_t, y \rangle) \geq 0 \). Letting \( t \downarrow 0 \), from (A3) we obtain \( f(\langle p, y \rangle) \geq 0 \) for all \( y \in C \) so that \( p \in \text{EP}(f) \). Hence \( y \in F \).

Finally we show that \( P = P_{tx_0} \). By taking the limits as \( n \to \infty \) in (23) we have
\[
\langle y - x_0 - p \rangle \geq 0, \forall y \in F. \tag{43}
\]
It then follows from Lemma 5(i) that \( p = P_{tx_0} \). This completes the proof.

**Corollary 8.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \), let \( f : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)–(A4), and let \( S, T : C \to P(C) \) be two strictly pseudocontraction mappings with contractive coefficients \( \lambda_1 \) and \( \lambda_2 \), respectively, such that \( F = F(T) \cap F(S) \cap \text{EP}(f) \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated from an arbitrary \( x_0 \in C \) as follows:

\[
x_0 \in H,
C_1 = C,
x_1 = P_{Cx_0},
y_n = \alpha_n x_n + (1 - \alpha_n) [\beta_n v_n + (1 - \beta_n) z_n],
u_n \in C \text{ such that}
\]
\[
F(\langle u_n, y \rangle) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \forall y \in C,
C_{n+1} = \{z \in C_n : \|z - u_n\|^2 \leq \|z - x_n\|^2\},
x_{n+1} = P_{C_{n+1}} x_0.
\]
where \( v_n \in P_{T}x_n \) and \( z_n \in P_{S}x_n \). \( \{\alpha_n\}^\infty_{n=1} \) and \( \{\beta_n\}^\infty_{n=1} \) are sequences in \([0,1]\) satisfying

(i) \( \alpha_n \geq \max\{\lambda_1, \lambda_2\} \),

(ii) \( \liminf_{n \to \infty} (1 - \alpha_n)(1 - \beta_n)(\alpha_n - \lambda_1) > 0 \) and \( \liminf_{n \to \infty} (1 - \alpha_n)(\alpha_n - \lambda_2)\beta_n > 0 \),

(iii) \( \{r_n\} \subset [a, \infty) \) for some \( a > 0 \).

Then \( \{x_n\} \) converges strongly to \( p \in P_{T}x_0 \).

Proof. The proof follows easily from Theorem 7. \( \square \)

Competing Interests

The author declares that there are no competing interests.

References


