Research Article

On Estimates of Deviation of Functions from Matrix Operators of Their Fourier Series by Some Expressions with r -Differences of the Entries

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We generalize the results of Krasniqi 2012 and Wei and Yu 2012 to the case of r -differences.

1. Introduction

Let $X = L^p$ or $X = C$, where L^p ($1 \leq p \leq \infty$) $[C]$ is the class of all 2π -periodic real-valued functions, integrable in the Lebesgue sense with p th power when $p \geq 1$ and essentially bounded when $p = \infty$ [continuous], over $Q = [-\pi, \pi]$ with the norm

$$\|f\|_{L^p} := \|f(\cdot)\|_{L^p} = \begin{cases} \left(\int_Q |f(t)|^p dt \right)^{1/p} & \text{when } 1 \leq p < \infty, \\ \text{ess sup}_{t \in Q} |f(t)| & \text{when } p = \infty, \end{cases} \quad (1)$$

$$\|f\|_C := \|f(\cdot)\|_C = \sup_{t \in Q} |f(t)|$$

and consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_{\nu}(f) \cos \nu x + b_{\nu}(f) \sin \nu x) \quad (2)$$

with the partial sums $S_k f$.

Let $A := (a_{n,k})$ be an infinite matrix of real numbers such that

$$a_{n,k} \geq 0$$

$$\text{when } k, n = 0, 1, 2, \dots, \lim_{n \rightarrow \infty} a_{n,k} = 0, \sum_{k=0}^{\infty} a_{n,k} = 1. \quad (3)$$

The A -transformation of $(S_k f)$ be given by

$$T_{n,A} f(x) := \sum_{k=0}^{\infty} a_{n,k} S_k f(x) \quad (n = 0, 1, 2, \dots). \quad (4)$$

In this paper, we study the upper bounds of $\|T_{n,A} f - f\|_X$ by the second modulus of continuity of f in the space X defined by the formula

$$\omega_2(f, \delta)_X := \sup_{0 \leq |t| \leq \delta} \|\varphi_x(t)\|_X, \quad (5)$$

where

$$\varphi_x(t) := f(x+t) + f(x-t) - 2f(x). \quad (6)$$

The deviation $T_{n,A} f - f$ with lower triangular infinite matrix such that

$$\begin{aligned} a_{n,k} &\geq 0 & \text{when } k = 0, 1, 2, \dots, n, \\ a_{n,k} &= 0 & \text{when } k > n, \end{aligned} \quad (7)$$

$$\sum_{k=0}^n a_{n,k} = 1, \quad \text{where } n = 0, 1, 2, \dots, \quad (8)$$

was estimated in the sup norm $\|\cdot\|_C$ by Krasniqi [1, Theorem 3.4, p. 290] (see also [2]) as follows.

Theorem 1. Let $f \in C$ and $(a_{n,k})$ satisfy (8). Then,

$$\begin{aligned} \|T_{n,A}f - f\|_C &= O\left(\omega\left(f, \frac{\pi}{n}\right)_C\right. \\ &+ \sum_{k=1}^n \frac{\omega(f, \pi/k)_C}{k} \sum_{l=0}^{k+1} a_{n,l} \\ &\left. + \sum_{k=1}^n \omega\left(f, \frac{\pi}{k}\right)_C \sum_{l=k}^n |a_{n,l} - a_{n,l+2}|\right), \end{aligned} \quad (9)$$

where $\omega(f, \cdot)_C$ denotes the modulus of continuity of f and if $\omega(f, \cdot)_C$ is such that

$$\int_u^\pi t^{-2} \omega(f, t)_C dt = O(H(u)) \quad \text{when } u \in [0, \pi], \quad (10)$$

where $H(u) \geq 0$, we have

$$\begin{aligned} \|T_{n,A}f - f\|_C &= O\left(\omega\left(f, \frac{\pi}{n}\right)_C + H\left(\frac{\pi}{n}\right) \sum_{k=0}^n |a_{n,k} - a_{n,k+2}|\right). \end{aligned} \quad (11)$$

Additionally, if H satisfies the condition

$$\int_0^u H(t) dt = O(uH(u)) \quad \text{when } u \in [0, \pi], \quad (12)$$

then

$$\begin{aligned} \|T_{n,A}f - f\|_C &= O\left(H\left(\sum_{k=0}^n |a_{n,k} - a_{n,k+2}|\right) \sum_{k=0}^n |a_{n,k} - a_{n,k+2}|\right), \\ \|T_{n,A}f - f\|_C &= O\left(H\left(\frac{\pi}{n}\right) \sum_{k=0}^n |a_{n,k} - a_{n,k+2}|\right). \end{aligned} \quad (13)$$

In our theorems we will consider the r -differences $a_{n,k} - a_{n,k+r}$ instead of $a_{n,k} - a_{n,k+2}$ considered above. We will formulate the general relation for $r \in \mathbb{N}$ like it formulated only for $r = 2$ in [1, Theorem 3.4, p. 290].

2. Statement of the Results

Let us consider a function ω of modulus of continuity type on the interval $[0, 2\pi]$, that is, a nondecreasing continuous function having the following properties: $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$.

Suppose that ω satisfies the condition

$$\int_u^\pi t^{-2} \omega(t) dt = O(H(u)) \quad \text{when } u \in [0, \pi], \quad (14)$$

where $H(u) \geq 0$ is such that

$$\int_0^u H(t) dt = O(uH(u)) \quad \text{when } u \in [0, \pi]. \quad (15)$$

Let

$$X_\omega = \{f \in X : \omega_2(f, \delta)_X = O(\omega(\delta)) \text{ when } \delta \in [0, 2\pi]\}. \quad (16)$$

Our main results on the degrees of approximation are the following.

Theorem 2. If $f \in X_\omega$, where ω satisfies condition (14) such that (15) holds and $r \in \mathbb{N}$, then

$$\begin{aligned} \|T_{n,A}f - f\|_X &= O\left(H\left(\frac{\pi}{n+1}\right) \left(\frac{\pi}{n+1} + \sum_{k=0}^\infty |a_{n,k} - a_{n,k+r}|\right)\right). \end{aligned} \quad (17)$$

Additionally, if a matrix A is such that

$$\left[\sum_{l=0}^n \sum_{k=l}^{r+l-1} a_{n,k}\right]^{-1} = O(1), \quad (18)$$

then

$$\|T_{n,A}f - f\|_X = O\left(H\left(\frac{\pi}{n+1}\right) \sum_{k=0}^\infty |a_{n,k} - a_{n,k+r}|\right). \quad (19)$$

Theorem 3. If $f \in X_\omega$, where ω satisfies condition (14) such that (15) holds and $r \in \mathbb{N}$, then

$$\begin{aligned} \|T_{n,A}f - f\|_X &= O\left(H\left(\sum_{k=0}^\infty |a_{n,k} - a_{n,k+r}|\right) \sum_{k=0}^\infty |a_{n,k} - a_{n,k+r}|\right). \end{aligned} \quad (20)$$

Theorem 4. Let $f \in X_\omega$, where ω satisfies condition (14) such that (15) holds and $r \in \mathbb{N}$. If a matrix A is such that

$$\sum_{k=0}^\infty (k+1) a_{n,k} = O(n+1), \quad (21)$$

then

$$\begin{aligned} \|T_{n,A}f - f\|_X &= O\left(\omega\left(\frac{\pi}{n+1}\right) + H\left(\frac{\pi}{n+1}\right) \sum_{k=0}^\infty |a_{n,k} - a_{n,k+r}|\right). \end{aligned} \quad (22)$$

Theorem 5. If $f \in X$ and a matrix A is such that (21) holds and $r \in \mathbb{N}$, then

$$\begin{aligned} \|T_{n,A}f - f\|_X &= O\left(\omega_2\left(f, \frac{\pi}{n+1}\right)_X\right. \\ &+ \sum_{\mu=1}^n \frac{\omega_2(f, \pi/\mu)_X}{\mu} \sum_{k=0}^{\mu+1} a_{n,k} \\ &\left. + \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \sum_{k=\mu}^\infty |a_{n,k} - a_{n,k+r}|\right). \end{aligned} \quad (23)$$

Corollary 6. For a lower triangular infinite matrix A conditions (18) and (21) hold always and therefore for a lower triangular infinite matrix such that conditions (7) and (8) hold one can obtain the results from Theorems 2, 4, and 5 without the assumptions (18) and (21), where the mentioned results of Krasniqi follow as the special cases with $r = 2$. Moreover, one can consider the essentially wider class of sequences than in the mentioned paper with the same degrees of approximation (see, e.g., [3, Theorem 2]).

Corollary 7. From Theorem 5 it follows that if $f \in X_\omega$ where ω satisfies condition (14) such that (15) holds and a matrix A is such that (21) and

$$\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+r}| \leq K \sum_{k=m/c}^{\infty} \frac{a_{n,k}}{k}, \quad (24)$$

with some $c > 1$ and $r \in \mathbb{N}$, hold, then

$$\begin{aligned} & \|T_{n,A}f - f\|_X \\ &= O\left(\frac{H(\pi/(n+1))}{n+1} + \sum_{k=1}^n a_{n,k} \frac{H(\pi/(k+1))}{k+1}\right). \end{aligned} \quad (25)$$

Remark 8. The class of sequences defined by condition (24) was introduced by the second author in [4]. The similar classes were considered by Dyachenko and Tikhonov in [5] with $r = 1$ (see also [6]).

Remark 9. We note that instead of (14) and (15) one can use Bari-Stechkin conditions

$$\begin{aligned} & \int_0^u t^{-1} \omega(t) dt = O(\omega(u)), \\ & \int_u^{2\pi} t^{-2} \omega(t) dt = O(\omega(u)) \end{aligned} \quad (26)$$

when $u \in [0, 2\pi]$.

Then all results are true for $t^{-1}\omega(t)$ instead of $H(y)$ and Lemmas 12 and 14 are not necessary.

3. Auxiliary Results

We begin this section by some notations from [7]. Let for $r = 1, 2, \dots$

$$\begin{aligned} D_{k,r}(t) &= \frac{\sin((2k+r)t/2)}{2 \sin(rt/2)}, \\ \bar{D}_{k,r}(t) &= \frac{\cos((2k+r)t/2)}{2 \sin(rt/2)}. \end{aligned} \quad (27)$$

It is clear by [8] that

$$\begin{aligned} S_k f(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_{k,1}(t) dt, \\ T_{n,A} f(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{k=0}^n a_{n,k} D_{k,1}(t) dt. \end{aligned} \quad (28)$$

Hence

$$T_{n,A} f(x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \varphi_x(t) \sum_{k=0}^n a_{n,k} D_{k,1}(t) dt. \quad (29)$$

Next, we present the known estimates.

Lemma 10 (see [8]). If $0 < |t| \leq \pi$ then

$$\begin{aligned} |D_{k,1}(t)| &\leq \frac{\pi}{2|t|}, \\ |\bar{D}_{k,1}(t)| &\leq \frac{\pi}{2|t|} \end{aligned} \quad (30)$$

and, for any real t , we have

$$|D_{k,1}(t)| \leq k + \frac{1}{2}. \quad (31)$$

Lemma 11 (see [7]). Let $r \in \mathbb{N}$, $l \in \mathbb{Z}$, and $a := (a_n) \subset \mathbb{C}$. If $x \neq 2l\pi/r$ then for every $m \geq n$

$$\begin{aligned} \sum_{k=n}^m a_k \sin kx &= - \sum_{k=n}^m (a_k - a_{k+r}) \bar{D}_{k,r}(t) \\ &\quad + \sum_{k=m+1}^{m+r} a_k \bar{D}_{k,-r}(t) \\ &\quad - \sum_{k=n}^{n+r-1} a_k \bar{D}_{k,-r}(t), \\ \sum_{k=n}^m a_k \cos kx &= \sum_{k=n}^m (a_k - a_{k+r}) D_{k,r}(t) \\ &\quad - \sum_{k=m+1}^{m+r} a_k D_{k,-r}(t) \\ &\quad + \sum_{k=n}^{n+r-1} a_k D_{k,-r}(t). \end{aligned} \quad (32)$$

Lemma 12 (see [9]). If (14) and (15) hold then

$$\int_0^u t^{-1} \omega(t) dt = O(uH(u)) \quad \text{when } u \in [0, 2\pi]. \quad (33)$$

We additionally proved two slight changed estimates.

Lemma 13. If (14) and (15) hold, then, for $c \geq 1$ and $\beta > \alpha > 0$,

$$\begin{aligned} \int_{\alpha}^{\beta} t^{-1} \omega(t) dt &= O((\beta - \alpha) H(c(\beta - \alpha))) \\ &\quad \text{when } (\beta - \alpha) \in [0, 2\pi]. \end{aligned} \quad (34)$$

Proof. By substitution of $t = u + \alpha$ and monotonicity of $\omega(u)/u$ (see Lemma 15) we obtain, for $c \geq 1$,

$$\begin{aligned} \int_{\alpha}^{\beta} t^{-1} \omega(t) dt &= \int_0^{\beta-\alpha} \frac{\omega(u+\alpha)}{u+\alpha} du \leq 2 \int_0^{\beta-\alpha} \frac{\omega(u)}{u} du \\ &\leq 2 \int_0^{c(\beta-\alpha)} \frac{\omega(u)}{u} du \end{aligned} \quad (35)$$

and the desired result follows from the Lemma 12. $\square$

Lemma 14. If (14) and (15) hold, then, for $b \geq 1$,

$$\int_u^\pi t^{-2} \omega(t) dt = O(H(bu)) \quad \text{when } u \in [0, \pi]. \quad (36)$$

Proof. Using Lemma 13 with $c = 1$, $\alpha = 0$ and (14) our result follows

$$\begin{aligned} \int_u^\pi t^{-2} \omega(t) dt &= \left(\int_u^{bu} + \int_{bu}^\pi \right) t^{-2} \omega(t) dt \\ &\leq \frac{1}{u} \int_u^{bu} \frac{\omega(t)}{t} dt + O(H(bu)) \\ &\leq \frac{1}{u} \int_0^{bu} \frac{\omega(t)}{t} dt + O(H(bu)) \\ &= \frac{1}{u} O(buH(bu)) + O(H(bu)) \\ &= O((b+1)H(bu)), \end{aligned} \quad (37)$$

for $b \geq 1$. $\square$

Finally, we present very useful property of the modulus of continuity.

Lemma 15 (see [8]). A function ω of modulus of continuity type on the interval $[0, 2\pi]$ satisfies the following condition:

$$\delta_2^{-1} \omega(\delta_2) \leq 2\delta_1^{-1} \omega(\delta_1) \quad \text{for } \delta_2 \geq \delta_1 > 0. \quad (38)$$

4. Proofs of the Results

Proof of Theorem 2. It is clear that for an odd r

$$\begin{aligned} T_{n,A} f(x) - f(x) &= \frac{1}{\pi} \sum_{m=0}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r + \pi/r} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) dt \\ &\quad + \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r + \pi/r}^{2(m+1)\pi/r} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) dt \\ &= I_1(x) + I_2(x) \end{aligned} \quad (39)$$

and for an even r

$$\begin{aligned} T_{n,A} f(x) - f(x) &= \frac{1}{\pi} \\ &\quad \cdot \sum_{m=0}^{[r/2]-1} \left(\int_{2m\pi/r}^{2m\pi/r + \pi/r} + \int_{2m\pi/r + \pi/r}^{2(m+1)\pi/r} \right) \varphi_x(t) \\ &\quad \cdot \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) dt = I_1'(x) + I_2(x). \end{aligned} \quad (40)$$

Then,

$$\|T_{n,A} f - f\|_X \leq \|I_1\|_X + \|I_1'\|_X + \|I_2\|_X. \quad (41)$$

Since, by Lemma 11,

$$\begin{aligned} \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) &= \sum_{k=0}^{\infty} a_{n,k} \frac{\sin((2k+1)t/2)}{2 \sin(t/2)} \\ &= \frac{1}{2 \sin(t/2)} \left(\sum_{k=0}^{\infty} a_{n,k} \sin kt \cos(t/2) \right. \\ &\quad \left. + \sum_{k=0}^{\infty} a_{n,k} \cos kt \sin(t/2) \right) \\ &= \frac{\cos(t/2)}{2 \sin(t/2)} \left(- \sum_{k=0}^{\infty} (a_{n,k} - a_{n,k+r}) \bar{D}_{k,r}(t) \right. \\ &\quad \left. - \sum_{k=0}^{r-1} a_{n,k} \bar{D}_{k,-r}(t) \right) + \frac{1}{2} \left(\sum_{k=0}^{\infty} (a_{n,k} - a_{n,k+r}) D_{k,r}(t) \right. \\ &\quad \left. + \sum_{k=0}^{r-1} a_{n,k} D_{k,-r}(t) \right), \end{aligned} \quad (42)$$

then

$$\begin{aligned} \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| &\leq \frac{1}{2 |\sin(t/2) \sin(rt/2)|} \left(\sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \right. \\ &\quad \left. + \sum_{k=0}^{r-1} |a_{n,k}| \right) \leq \frac{1}{|\sin(t/2) \sin(rt/2)|} \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}|. \end{aligned} \quad (43)$$

Hence, by Lemma 10,

$$\begin{aligned} \|I_1\|_X &\leq \frac{1}{\pi} \sum_{m=0}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r + \pi/r} \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt = \frac{1}{\pi} \\ &\quad \cdot \sum_{m=0}^{[r/2]} \left(\int_{2m\pi/r}^{2m\pi/r + \pi/r} + \int_{2m\pi/r + \pi/r}^{2(m+1)\pi/r} \right) \|\varphi_x(t)\|_X \\ &\quad \cdot \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \leq \frac{1}{2} \sum_{m=0}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r + \pi/r} \frac{O(\omega(t))}{t} dt \\ &\quad + \frac{1}{\pi} \\ &\quad \cdot \sum_{m=0}^{[r/2]} \int_{2m\pi/r + \pi/r}^{2(m+1)\pi/r} \frac{O(\omega(t))}{|\sin(t/2) \sin(rt/2)|} \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| dt. \end{aligned} \quad (44)$$

Using Lemmas 13 and 14 with $c = b = r$ and the estimates $|\sin(t/2)| \geq |t|/\pi$ for $t \in [0, \pi]$ and $|\sin(rt/2)| \geq rt/\pi - 2m$ for $t \in [2m\pi/r, 2m\pi/r + \pi/r]$, where $m \in \{0, \dots, [r/2]\}$, we obtain

$$\begin{aligned} \|I_1\|_X &\leq O(1) \left(\left[\frac{r}{2} \right] + 1 \right) \frac{\pi}{r(n+1)} H\left(\frac{\pi}{n+1} \right) \\ &\quad + \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \end{aligned}$$

$$\begin{aligned}
& \cdot \sum_{m=0}^{[r/2]} \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r} \frac{O(\omega(t))}{t(rt/\pi-2m)} dt = O(1) \frac{\pi}{n+1} \\
& \cdot H\left(\frac{\pi}{n+1}\right) + \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \\
& \cdot \sum_{m=0}^{[r/2]} \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r} \frac{O(\omega(t))}{(rt/\pi)(t-2m\pi/r)} dt = O(1) \\
& \cdot \frac{\pi}{n+1} H\left(\frac{\pi}{n+1}\right) + \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \\
& \cdot \sum_{m=0}^{[r/2]} \int_{\pi/r(n+1)}^{\pi/r} \frac{O(\omega(t+2m\pi/r))}{(rt/\pi)(t+2m\pi/r)} dt \leq O(1) \\
& \cdot \left[\frac{\pi}{n+1} H\left(\frac{\pi}{n+1}\right) + \left(\left[\frac{r}{2}\right] + 1\right) \frac{2\pi}{r} \right. \\
& \cdot \left. \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \int_{\pi/r(n+1)}^{\pi/r} \frac{\omega(t)}{t^2} dt \right] = O(1) \\
& \cdot \left[\frac{\pi}{n+1} H\left(\frac{\pi}{n+1}\right) \right. \\
& \cdot \left. + \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| H\left(\frac{\pi}{n+1}\right) \right].
\end{aligned} \tag{45}$$

Analogously

$$\begin{aligned}
\|I_1'\|_X &= O(1) \left[\frac{r}{2} \right] \left[\frac{\pi}{r(n+1)} H\left(\frac{\pi}{n+1}\right) \right. \\
& \cdot \left. + \frac{2\pi}{r} \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| H\left(\frac{\pi}{n+1}\right) \right].
\end{aligned} \tag{46}$$

Similarly, by Lemmas 10, 13, and 14 with $c = b = r$ and the estimates $|\sin(t/2)| \geq |t|/\pi$ for $t \in [0, \pi]$ and $|\sin(rt/2)| \geq 2(m+1) - rt/\pi$ for $t \in [2(m+1)\pi/r - \pi/r, 2(m+1)\pi/r - \pi/r(n+1)]$, where $m \in \{0, \dots, [r/2] - 1\}$, we get

$$\begin{aligned}
\|I_2\|_X &\leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r} \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \\
&= \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \left(\int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} + \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \right) \\
&\cdot \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \leq \frac{1}{2} \\
&\cdot \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{O(\omega(t))}{t} dt + \frac{1}{\pi} \sum_{k=0}^{\infty} |a_{n,k}
\end{aligned}$$

$$\begin{aligned}
& - a_{n,k+r}| \\
& \cdot \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} \frac{O(\omega(t))}{|\sin(t/2) \sin(rt/2)|} dt \leq \frac{1}{2} \\
& \cdot \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{O(\omega(t))}{t} dt + \sum_{k=0}^{\infty} |a_{n,k} \\
& - a_{n,k+r}| \\
& \cdot \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} \frac{O(\omega(t))}{(rt/\pi)[2(m+1)\pi/r-t]} dt \\
& = \frac{1}{2} \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{O(\omega(t))}{t} dt + \sum_{k=0}^{\infty} |a_{n,k} \\
& - a_{n,k+r}| \sum_{m=0}^{[r/2]-1} \int_{\pi/r(n+1)}^{\pi/r} \frac{O(\omega(-t+2(m+1)\pi/r))}{(r/\pi)t(-t+2(m+1)\pi/r)} dt \\
& \leq \frac{1}{2} \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{O(\omega(t))}{t} dt + \sum_{k=0}^{\infty} |a_{n,k} \\
& - a_{n,k+r}| \left[\frac{r}{2} \right] \frac{2\pi}{r} \int_{\pi/r(n+1)}^{\pi/r} \frac{O(\omega(t))}{t^2} dt.
\end{aligned} \tag{47}$$

Thus

$$\begin{aligned}
\|I_2\|_X &= O(1) \left[\frac{\pi}{n+1} H\left(\frac{\pi}{n+1}\right) \right. \\
& \cdot \left. + \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| H\left(\frac{\pi}{n+1}\right) \right].
\end{aligned} \tag{48}$$

Collecting these estimates we obtain the first result.

Applying condition (18) we have

$$\begin{aligned}
& \left[(n+1) \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \right]^{-1} \\
& = \left[\sum_{l=0}^n \sum_{k=l}^{\infty} |a_{n,k} - a_{n,k+r}| \right]^{-1} \\
& \leq \left[\sum_{l=0}^n \sum_{k=l}^{\infty} |a_{n,k} - a_{n,k+r}| \right]^{-1} \\
& \leq \left[\sum_{l=0}^n \left| \sum_{k=l}^{\infty} (a_{n,k} - a_{n,k+r}) \right| \right]^{-1} \leq \left[\sum_{l=0}^n \sum_{k=l}^{r+l-1} a_{n,k} \right]^{-1} \\
& = O(1)
\end{aligned} \tag{49}$$

and the second result also follows. $\square$

Proof of Theorem 3. Analogously, as in the proof of Theorem 2, we consider an odd r and an even r . Then,

$$T_{n,A}f(x) - f(x) = I_1(x) + I_2(x) \tag{50}$$

or

$$T_{n,A}f(x) - f(x) = I_1'(x) + I_2(x), \quad (51)$$

respectively, and

$$\|T_{n,A}f - f\|_X \leq \|I_1\|_X + \|I_1'\|_X + \|I_2\|_X. \quad (52)$$

Since $A_{n,r} = \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \leq 2$

$$\begin{aligned} \|I_1\|_X &\leq \frac{1}{\pi} \sum_{m=0}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r+\pi/r} \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \\ &= \frac{1}{\pi} \sum_{m=0}^{[r/2]} \left(\int_{2m\pi/r}^{2m\pi/r+(1/r)A_{n,r}} + \int_{2m\pi/r+\pi/r}^{2m\pi/r+(1/r)A_{n,r}} \right) \\ &\quad \cdot \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt, \\ \|I_2\|_X &\leq \frac{1}{\pi} \\ &\quad \cdot \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r} \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \leq \frac{1}{\pi} \\ &\quad \cdot \sum_{m=0}^{[r/2]-1} \left(\int_{2(m+1)\pi/r-(1/r)A_{n,r}}^{2(m+1)\pi/r-(1/r)A_{n,r}} + \int_{2(m+1)\pi/r-(1/r)A_{n,r}}^{2(m+1)\pi/r} \right) \\ &\quad \cdot \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt. \end{aligned} \quad (53)$$

Therefore, in the terms I_1 , I_1' , and I_2 we can estimate analogously as in the proof of Theorem 2 and thus we obtain the desired estimate. $\square$

Proof of Theorem 4. Similarly, as in the proof of Theorem 2, we consider an odd r and an even r . Then,

$$\begin{aligned} T_{n,A}f(x) - f(x) &= \frac{1}{\pi} \int_0^{\pi/r(n+1)} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) dt \\ &\quad + \frac{1}{\pi} \int_{\pi/r(n+1)}^{\pi/r} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) dt \end{aligned}$$

$$\begin{aligned} &+ \frac{1}{\pi} \sum_{m=1}^{[r/2]} \int_{2m\pi/r}^{2m\pi/r+\pi/r} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) dt \\ &+ \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r} \varphi_x(t) \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) dt \\ &= J_1(x) + J_2(x) + I_1''(x) + I_2(x) \end{aligned} \quad (54)$$

or

$$\begin{aligned} T_{n,A}f(x) - f(x) &= J_1(x) + J_2(x) \\ &+ \frac{1}{\pi} \left(\sum_{m=1}^{[r/2]-1} \int_{2m\pi/r}^{2m\pi/r+\pi/r} + \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r} \right) \varphi_x(t) \\ &\quad \cdot \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) dt = J_1(x) + J_2(x) + I_1'''(x) \\ &\quad + I_2(x), \end{aligned} \quad (55)$$

respectively. Therefore,

$$\|T_{n,A}f - f\|_X \leq \|J_1\|_X + \|J_2\|_X + \|I_1''\|_X + \|I_1'''\|_X + \|I_2\|_X. \quad (56)$$

By Lemma 10 and (18),

$$\begin{aligned} \|J_1\|_X &\leq \frac{1}{\pi} \int_0^{\pi/r(n+1)} \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \\ &\leq \frac{1}{\pi} \sum_{k=0}^{\infty} (k+1) a_{n,k} \int_0^{\pi/r(n+1)} \omega(t) dt \\ &= O(1)(n+1) \int_0^{\pi/r(n+1)} \omega(t) dt \\ &\leq O(1) \frac{\pi}{r} \omega\left(\frac{\pi}{r(n+1)}\right) = O\left(\omega\left(\frac{\pi}{n+1}\right)\right). \end{aligned} \quad (57)$$

Further, by the same lemmas and conditions as above and Lemma 15, we obtain, with

$$\kappa = \begin{cases} 1 & \text{when } r \text{ is even,} \\ 0 & \text{when } r \text{ is odd,} \end{cases} \quad (58)$$

that

$$\begin{aligned} \|I_1''\|_X + \|J_2\|_X + \|I_1'''\|_X &\leq \frac{1}{\pi} \left(\sum_{m=1}^{[r/2]-\kappa} \int_{2m\pi/r}^{2m\pi/r+\pi/r} + \int_{\pi/r(n+1)}^{\pi/r} \right) \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \\ &= \frac{1}{\pi} \left(\sum_{m=1}^{[r/2]-\kappa} \int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} + \sum_{m=0}^{[r/2]-\kappa} \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r} \right) \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\pi} \sum_{m=1}^{[r/2]-\kappa} \int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} \frac{O(\omega(t))}{2|\sin(t/2)|} dt \\
&\quad + \frac{1}{\pi} \sum_{m=0}^{[r/2]-\kappa} \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r} \frac{O(\omega(t))}{|\sin(t/2)\sin(rt/2)|} \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| dt \\
&\leq \frac{1}{2} \sum_{m=1}^{[r/2]-\kappa} \int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} \frac{O(\omega(t))}{t} dt + \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \sum_{m=0}^{[r/2]-\kappa} \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r} \frac{O(\omega(t))}{t(rt/\pi - 2m)} dt \\
&\leq \sum_{m=1}^{[r/2]-\kappa} \frac{O(\omega(2m\pi/r))}{2m\pi/r} \int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} dt \\
&\quad + \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \sum_{m=0}^{[r/2]-\kappa} \int_{\pi/r(n+1)}^{\pi/r} \frac{O(\omega(t+2m\pi/r))}{(rt/\pi)(t+2m\pi/r)} dt \\
&\leq 2 \sum_{m=1}^{[r/2]-\kappa} \frac{O(\omega(2\pi/r))}{2\pi/r} \frac{\pi}{r(n+1)} + \frac{2\pi}{r} \left(\left[\frac{r}{2} \right] + 1 \right) \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \int_{\pi/r(n+1)}^{\pi/r} \frac{O(\omega(t))}{t^2} dt \\
&= O(1) \left[\omega\left(\frac{\pi}{n+1}\right) + \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| H\left(\frac{\pi}{n+1}\right) \right], \\
\|I_2\|_X &\leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \left(\int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} + \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \right) \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \\
&\leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} \frac{O(\omega(t))}{|\sin(t/2)\sin(rt/2)|} \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| dt \\
&\quad + \frac{1}{2} \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{O(\omega(t))}{t} dt \\
&\leq \sum_{m=0}^{[r/2]-1} \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} \frac{O(\omega(t))}{(rt/\pi)[2(m+1)\pi/r-t]} dt \\
&\quad + \sum_{m=0}^{[r/2]-1} \frac{O(\omega(2(m+1)\pi/r-\pi/r(n+1)))}{2(m+1)\pi/r-\pi/r(n+1)} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} dt \\
&\leq \left[\frac{r}{2} \right] \left[\frac{2\pi}{r} \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| \int_{\pi/r(n+1)}^{\pi/r} \frac{O(\omega(t))}{t^2} dt + 2 \frac{O(\omega(\pi/r))}{\pi/r} \frac{\pi}{r(n+1)} \right] \\
&= O(1) \left[\sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| H\left(\frac{\pi}{n+1}\right) + \omega\left(\frac{\pi}{n+1}\right) \right].
\end{aligned} \tag{59}$$

Thus our proof is complete. $\square$

Proof of Theorem 5. Let as above

$$\begin{aligned}
&\|T_{n,A}f - f\|_X \\
&\leq \|J_1\|_X + \|J_2\|_X + \|I_1''\|_X + \|I_1'''\|_X + \|I_2\|_X,
\end{aligned}$$

$$\begin{aligned}
\|J_1\|_X &\leq \frac{1}{\pi} \int_0^{\pi/r(n+1)} \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \\
&\leq \frac{1}{\pi} \sum_{k=0}^{\infty} (k+1) a_{n,k} \int_0^{\pi/r(n+1)} \omega_2(f, t)_X dt \\
&= O(n+1) \int_0^{\pi/r(n+1)} \omega_2(f, t)_X dt
\end{aligned}$$

$$\begin{aligned}
&\leq O(1) \frac{\pi}{r} \omega_2 \left(f, \frac{\pi}{r(n+1)} \right)_X \\
&= O(1) \omega_2 \left(f, \frac{\pi}{n+1} \right)_X.
\end{aligned} \tag{60}$$

Further, taking $\tau_m^1 = [\pi/(rt - 2m\pi)]$ and $\tau = [\pi/rt]$, using Lemma 15, we obtain, with

$$\kappa = \begin{cases} 1 & \text{when } r \text{ is even,} \\ 0 & \text{when } r \text{ is odd,} \end{cases} \tag{61}$$

that

$$\begin{aligned}
&\|I_1''\|_X + \|J_2\|_X + \|I_1'''\|_X \leq \frac{1}{\pi} \left(\sum_{m=1}^{[r/2]-\kappa} \int_{2m\pi/r}^{2m\pi/r+\pi/r} \right. \\
&\quad \left. + \int_{\pi/r(n+1)}^{\pi/r} \right) \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \\
&= \frac{1}{\pi} \left(\sum_{m=1}^{[r/2]-\kappa} \int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} \right. \\
&\quad \left. + \sum_{m=0}^{[r/2]-\kappa} \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r} \right) \|\varphi_x(t)\|_X \\
&\quad \cdot \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \leq \frac{1}{\pi} \\
&\quad \cdot \sum_{m=1}^{[r/2]-\kappa} \int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} \frac{\omega_2(f, t)_X}{2 |\sin(t/2)|} \sum_{k=0}^{\infty} a_{n,k} dt + \frac{1}{\pi} \\
&\quad \cdot \sum_{m=0}^{[r/2]-\kappa} \int_{2m\pi/r+\pi/r(n+1)}^{2m\pi/r+\pi/r} \left(\frac{\omega_2(f, t)_X}{2 |\sin(t/2)|} \sum_{k=0}^{\tau_m^1} a_{n,k} \right.
\end{aligned}$$

$$\begin{aligned}
&\quad \left. + \frac{\omega_2(f, t)_X}{|\sin(t/2) \sin(rt/2)|} \sum_{k=\tau_m^1}^{\infty} |a_{n,k} - a_{n,k+r}| \right) dt \\
&\leq \frac{1}{2} \sum_{m=1}^{[r/2]-\kappa} \int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} \frac{\omega_2(f, t)_X}{t} dt + \frac{1}{2} \left(\left[\frac{r}{2} \right] \right. \\
&\quad \left. + 1 \right) \int_{\pi/r(n+1)}^{\pi/r} \frac{\omega_2(f, t)_X}{t} \sum_{k=0}^{\tau} a_{n,k} dt + \frac{2\pi}{r} \left(\left[\frac{r}{2} \right] + 1 \right) \\
&\quad \cdot \int_{\pi/r(n+1)}^{\pi/r} \frac{\omega_2(f, t)_X}{t^2} \sum_{k=\tau}^{\infty} |a_{n,k} - a_{n,k+r}| dt \\
&\leq \sum_{m=1}^{[r/2]-\kappa} \frac{\omega_2(f, 2m\pi/r)_X}{2m\pi/r} \int_{2m\pi/r}^{2m\pi/r+\pi/r(n+1)} dt \\
&\quad + \frac{1}{2} \left(\left[\frac{r}{2} \right] + 1 \right) \sum_{\mu=1}^n \int_{\mu}^{\mu+1} \frac{\omega_2(f, \pi/rt)_X}{\pi/rt} \sum_{k=0}^{[t]} a_{n,k} \frac{\pi dt}{rt^2} \\
&\quad + \frac{2\pi}{r} \left(\left[\frac{r}{2} \right] + 1 \right) \\
&\quad \cdot \sum_{\mu=1}^n \int_{\mu}^{\mu+1} \frac{\omega_2(f, \pi/rt)_X}{(\pi/rt)^2} \sum_{k=[t]}^{\infty} |a_{n,k} - a_{n,k+r}| \frac{\pi dt}{rt^2} \\
&\leq O(1) \omega_2 \left(f, \frac{\pi}{n+1} \right)_X + O(1) \\
&\quad \cdot \sum_{\mu=1}^n \frac{\omega_2(f, \pi/\mu)_X}{\mu} \sum_{k=0}^{\mu+1} a_{n,k} + O(1) \sum_{\mu=1}^n \omega_2 \left(f, \frac{\pi}{\mu} \right)_X \\
&\quad \cdot \sum_{k=\mu}^{\infty} |a_{n,k} - a_{n,k+r}|.
\end{aligned} \tag{62}$$

Next, taking $\tau_m^2 = [\pi/(-rt + 2(m+1)\pi)]$, we obtain

$$\begin{aligned}
&\|I_2\|_X \leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi/r+\pi/r}^{2(m+1)\pi/r} \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \\
&\leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \left(\int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} + \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \right) \|\varphi_x(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} D_{k,1}(t) \right| dt \\
&\leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r}^{2(m+1)\pi/r-\pi/r(n+1)} \left(\frac{\omega_2(f, t)_X}{2 |\sin(t/2)|} \sum_{k=0}^{\tau_m^2} a_{n,k} + \frac{\omega_2(f, t)_X}{|\sin(t/2) \sin(rt/2)|} \sum_{k=\tau_m^2}^{\infty} |a_{n,k} - a_{n,k+r}| \right) dt \\
&\quad + \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{\omega_2(f, t)_X}{2 |\sin(t/2)|} \sum_{k=0}^{\infty} a_{n,k} dt \\
&\leq \frac{1}{2} \sum_{m=0}^{[r/2]-1} \int_{\pi/r(n+1)}^{\pi/r} \frac{\omega_2(-t + 2(m+1)\pi/r)_X}{-t + 2(m+1)\pi/r} \sum_{k=0}^{\tau} a_{n,k} dt
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{m=0}^{[r/2]-1} \int_{\pi/r(n+1)}^{\pi/r} \frac{\omega_2(-t+2(m+1)\pi/r)_X}{(r/\pi)t(-t+2(m+1)\pi/r)} \sum_{k=\tau}^{\infty} |a_{n,k} - a_{n,k+r}| dt + \frac{1}{2} \sum_{m=0}^{[r/2]-1} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} \frac{\omega_2(t)_X}{t} dt \\
 & \leq \left[\frac{r}{2} \right] \int_{\pi/r(n+1)}^{\pi/r} \frac{\omega_2(f,t)_X}{t} \sum_{k=0}^{\tau} a_{n,k} + \frac{2\pi}{r} \left[\frac{r}{2} \right] \int_{\pi/r(n+1)}^{\pi/r} \frac{\omega_2(f,t)_X}{t^2} \sum_{k=\tau}^{\infty} |a_{n,k} - a_{n,k+r}| dt \\
 & + \sum_{m=0}^{[r/2]-1} \frac{\omega_2(f, 2(m+1)\pi/r - \pi/r(n+1))_X}{2(m+1)\pi/r - \pi/r(n+1)} \int_{2(m+1)\pi/r-\pi/r(n+1)}^{2(m+1)\pi/r} dt \\
 & \leq O(1) \sum_{\mu=1}^n \frac{\omega_2(f, \pi/\mu)_X}{\mu} \sum_{k=0}^{\mu+1} a_{n,k} + O(1) \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \sum_{k=\mu}^{\infty} |a_{n,k} - a_{n,k+r}| + O(1) \omega_2\left(f, \frac{\pi}{n+1}\right)_X.
 \end{aligned}$$

(63)

Thus the result follows. $\square$

Proof of Corollary 7. Theorem 5 implies that

$$\begin{aligned}
 \|T_{n,A}f - f\|_X & = O\left(\omega_2\left(f, \frac{\pi}{n+1}\right)_X\right. \\
 & + \sum_{\mu=1}^n \frac{\omega_2(f, \pi/\mu)_X}{\mu} \sum_{k=0}^{\mu+1} a_{n,k} \\
 & \left. + \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \sum_{k=\mu}^{\infty} |a_{n,k} - a_{n,k+r}|\right). \quad (64)
 \end{aligned}$$

Since in (24)

$$\begin{aligned}
 & \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \sum_{k=\mu}^{\infty} |a_{n,k} - a_{n,k+r}| = O(1) \\
 & \cdot \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \left(\sum_{k=\mu/c}^{\mu-1} \frac{a_{n,k}}{k} + \sum_{k=\mu}^n \frac{a_{n,k}}{k} + \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \right) \\
 & \leq O(1) \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \left(\sum_{k=\mu/c}^{\mu-1} \frac{a_{n,k}}{k} \right) + O(1) \\
 & \cdot \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \left(\sum_{k=\mu}^n \frac{a_{n,k}}{k} + \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \right) \leq O(1) \\
 & \cdot c \sum_{\mu=1}^n \frac{\omega_2(f, \pi/\mu)_X}{\mu} \sum_{k=0}^{\mu+1} a_{n,k} + O(1) \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \\
 & \cdot \left(\sum_{k=\mu}^n \frac{a_{n,k}}{k} + \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \right)
 \end{aligned} \quad (65)$$

one has

$$\|T_{n,A}f - f\|_X = O(1) \omega_2\left(f, \frac{\pi}{n+1}\right)_X + O(1)(1+c)$$

$$\begin{aligned}
 & \cdot \sum_{\mu=1}^n \frac{\omega_2(f, \pi/\mu)_X}{\mu} \sum_{k=0}^{\mu+1} a_{n,k} + O(1) \\
 & \cdot \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \sum_{k=\mu}^n \frac{a_{n,k}}{k} + O(1) \\
 & \cdot \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \leq O(1) \omega_2\left(f, \frac{\pi}{n+1}\right)_X \\
 & + O(1)(1+c) \left\{ \sum_{\mu=1}^n \frac{\omega_2(f, \pi/\mu)_X}{\mu} \sum_{k=0}^{\mu-1} a_{n,k} \right. \\
 & \left. + 2 \sum_{\mu=1}^n \frac{\omega_2(f, \pi/\mu)_X}{\mu+1} a_{n,\mu+1} + \sum_{\mu=1}^n \frac{\omega_2(f, \pi/\mu)_X}{\mu} a_{n,\mu} \right\} \\
 & + O(1) \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \sum_{k=\mu}^n \frac{a_{n,k}}{k} + O(1) \\
 & \cdot \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \leq O(1) \omega_2\left(f, \frac{\pi}{n+1}\right)_X \\
 & + O(1)(1+c) \sum_{\mu=0}^n \frac{\omega_2(f, \pi/(\mu+1))_X}{\mu+1} \sum_{k=0}^{\mu} a_{n,k} \\
 & + O(1)[3(1+c)+1] \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X \sum_{k=\mu}^n \frac{a_{n,k}}{k} \\
 & + O(1) \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \sum_{\mu=1}^n \omega_2\left(f, \frac{\pi}{\mu}\right)_X = O(1) \\
 & \cdot \omega_2\left(f, \frac{\pi}{n+1}\right)_X + O(1) \\
 & \cdot \sum_{k=0}^n a_{n,k} \sum_{\mu=k}^n \frac{\omega_2(f, \pi/(\mu+1))_X}{\mu+1} + O(1) \\
 & \cdot \sum_{k=1}^n \frac{a_{n,k}}{k} \sum_{\mu=1}^k \omega_2\left(f, \frac{\pi}{\mu}\right)_X + O(1)
 \end{aligned}$$

$$\sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \sum_{\mu=1}^n \omega_2 \left(f, \frac{\pi}{\mu} \right)_X, \quad (66)$$

If (14) and (15) hold, then

$$\begin{aligned} \omega_2 \left(f, \frac{\pi}{n+1} \right)_X &\leq \frac{1}{n} \sum_{\mu=1}^n \omega_2 \left(f, \frac{\pi}{\mu} \right)_X \\ &\leq O(1) 4\pi \frac{H(\pi/(n+1))}{n+1}, \\ \sum_{\mu=k}^n \frac{\omega_2(f, \pi/(\mu+1))_X}{\mu+1} &\leq 8 \int_{\pi/(n+2)}^{\pi/(k+1)} \frac{\omega_2(f, t)_X}{t} dt \\ &\leq O(1) 8\pi \frac{H(\pi/(k+1))}{k+1} \end{aligned} \quad (67)$$

and therefore

$$\begin{aligned} \|T_{n,A}f - f\|_X &= O\left(\frac{H(\pi/(n+1))}{n+1}\right) \\ &\quad + O\left(\sum_{k=1}^n a_{n,k} \frac{H(\pi/(k+1))}{k+1}\right) \\ &\quad + O\left(H\left(\frac{\pi}{n+1}\right) \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k}\right). \end{aligned} \quad (68)$$

Since

$$\sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k} \leq \frac{1}{n+1} \sum_{k=n+1}^{\infty} a_{n,k} = O\left(\frac{1}{n+1}\right) \quad (69)$$

the result follows (cf. [6]). $\square$

Competing Interests

The authors declare that they have no competing interests.

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