Research Article

A Degree Theory for Compact Perturbations of Monotone Type Operators and Application to Nonlinear Parabolic Problem

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Abstract and Applied Analysis

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Let $X$ be a real locally uniformly convex reflexive Banach space with locally uniformly convex dual space $X^*$. Let $T : X \ni D(T) \to 2^{X^*}$ be maximal monotone, $S : X \to 2^{X^*}$ be bounded and of type $(S_u)$, and $C : D(C) \to X^*$ be compact with $D(T) \subseteq D(C)$ such that $C$ lies in $\mathcal{F}(\mathcal{F}^*)$ (i.e., there exist $\sigma \geq 0$ and $\tau \geq 0$ such that $\|Cx\| \leq \sigma \|x\| + \tau$ for all $x \in D(C)$). A new topological degree theory is developed for operators of the type $T + S + C$. The theory is essential because no degree theory and/or existence result is available to address solvability of operator inclusions involving operators of the type $T + S + C$, where $C$ is not defined everywhere. Consequently, new existence theorems are provided. The existence theorem due to Asfaw and Kartsatos is improved. The theory is applied to prove existence of weak solution(s) for a nonlinear parabolic problem in appropriate Sobolev spaces.

1. Introduction: Preliminaries

In what follows, the norm of the spaces $X$ and $X^*$ will be denoted by $\| \cdot \|$. For $x \in X$ and $x^* \in X^*$, the pairing $\langle x^*, x \rangle$ denotes the value $x^*(x)$. Let $X$ and $Y$ be real Banach spaces. For an operator $T : X \to 2^Y$, we define the domain $D(T)$ of $T$ by $D(T) = \{ x \in X : Tx \neq \emptyset \}$, and the range $R(T)$ of $T$ by $R(T) = \bigcup_{x \in D(T)} Tx$. We also use the symbol $G(T)$ for the graph of $T$: $G(T) = \{(x, x^*) : x \in D(T), x^* \in Tx\}$.

An operator $T : X \ni D(T) \to Y$ is “demicontinuous” if it is continuous from the strong topology of $D(T)$ to the weak topology of $Y$. It is “compact” if it is strongly continuous and maps bounded subsets of $D(T)$ to relatively compact subsets of $Y$. An operator $T : X \ni D(T) \to 2^X$ is “bounded” if it maps each bounded subset of $D(T)$ into a bounded subset of $Y$. It is “finitely continuous” if it is upper semicontinuous from each finite dimensional subspace $F$ of $X$ to the weak topology of $Y$. Let $\phi : [0, \infty) \to (-\infty, \infty)$ be a continuous and strictly increasing function such that $\phi(t) \to \infty$ as $t \to \infty$. The mapping $J_\phi : X \ni 2^{X^*}$ defined by

$$J_\phi(x) = \{ x^* \in X^* : \langle x^*, x \rangle = \phi(\|x\|^2 \|x^*\|), \quad \|x^*\| = \phi(\|x\|) \}$$

is called the “duality mapping” associated with $\phi$. As a consequence of the Hahn-Banach theorem, it is well-known that $I_\phi(x) \neq \emptyset$ for all $x \in X$. Since $X$ and $X^*$ are locally uniformly convex, $I_\phi$ is single valued, bounded, monotone, and bicontinuous. The following definitions are needed throughout the paper.

Definition 1. An operator $T : X \ni D(T) \to 2^{X^*}$ is said to be

(i) “monotone” if for every $x \in D(T)$, $y \in D(T)$, $u^* \in Tx$, and $v^* \in Ty$, we have $\langle u^* - v^*, x - y \rangle \geq 0$;

(ii) “maximal monotone” if $T$ is monotone and $R(T + \lambda I) = X^*$ for every $\lambda > 0$; that is, $T$ is maximal monotone if and only if $T$ is monotone and $\langle u^* - u^*_0, x - x_0 \rangle \geq 0$ for every $(x, u^*) \in G(T)$ implies $x_0 \in D(T)$ and $u^*_0 \in Tx_0$;

(iii) “coercive” if either $D(T)$ is bounded or there exists a function $\psi : [0, \infty) \to (-\infty, \infty)$ such that $\psi(t) \to \infty$ as $t \to \infty$ and $\langle y^*, x \rangle \geq \psi(\|x\|)\|x\|$ for all $x \in D(T)$ and $y^* \in Tx$;

(iv) “weakly coercive” if either $D(T)$ is bounded or $\|Tx\| \to \infty$ as $\|x\| \to \infty$, where for each $x \in D(T)$, $[Tx] = \inf\{\|v^*\| : v^* \in Tx\}$. 
It is important to note here that the class of weakly coercive operators includes the classes of coercive operators. For a maximal monotone operator \( T: X \supseteq D(T) \rightarrow 2^{X^*} \), we know that \( R(T + \lambda I) = X^* \) for all \( \lambda > 0 \) and \( (T + \lambda I)^{-1} : X^* \rightarrow D(T) \) is single valued and demicontinuous. In addition, the operator \( T_t: X \rightarrow X^*, t \in (0, \infty) \), defined by \( T_t x = (T^{-1} + tI)^{-1} x \), is the "Yosida approximant" of \( T \). It is bounded, continuous, and maximal monotone with domain \( X \) such that \( T_t x \rightarrow T^0_0 x \) as \( t \rightarrow 0^+ \), for every \( x \in D(T) \), where \( \|T^0_0 x\| = \inf \{\|y^*\| : y^* \in Tx \} \). Furthermore, the operator \( I_T: X \rightarrow D(T) \), defined by \( I_T x = x - I^{T-1} T(x) \), is called the "Yosida resolvent" of \( T \). It is continuous, \( T_T x \in T(I_T x) \) for every \( x \in X \), and \( \lim_{t \to 0^+} I_T x = x \) for all \( x \in \text{co} D(T) \), where \( \text{co} D(T) \) is the convex hull of the set \( D(T) \). Furthermore, for each \( x \in D(T) \), \( \|T_t x\| \leq |T x| \) for all \( t > 0 \). Browder and Hess [1] introduced the following definitions. The original definition of single valued pseudomonotone operator is due to Brézis [2].

**Definition 2.** An operator \( T: X \supseteq D(T) \rightarrow 2^{X^*} \) is said to be

(a) "pseudomonotone" if the following conditions are satisfied:

(i) For every \( x \in D(T) \), \( Tx \) is nonempty, closed, convex, and bounded subset of \( X^* \);

(ii) \( T \) is finitely continuous; that is, for every \( x_0 \in D(T) \cap F \) and every weak neighborhood \( V \) of \( Tx_0 \) in \( X^* \), there exists a neighborhood \( U \) of \( x_0 \) in \( F \) such that \( TU \subseteq V \);

(iii) for each sequence \( \{x_n\} \subset D(T) \) with \( y^*_n \in Tx_n \) such that \( x_n \rightarrow x_0 \) in \( D(T) \) and

\[
\lim_{n \to \infty} \langle y^*_n, x_n - x_0 \rangle \leq 0,
\]

we have that, for every \( x \in D(T) \), there exists \( y^*(x) \in Tx_0 \) such that

\[
\langle y^*(x), x_0 - x \rangle \leq \liminf_{n \to \infty} \langle y^*_n, x_n - x \rangle;
\]

in particular, letting \( x_0 \) in place of \( x \) in the above inequality, the pseudomonotonicity of \( T \) implies

\[
\liminf_{n \to \infty} \langle y^*_n, x_n - x_0 \rangle \geq 0;
\]

(b) "of type \((S_\lambda)\)" if (i) and (ii) of (a) hold and for each sequence \( \{x_n\} \) in \( D(T) \) such that \( x_n \rightarrow x_0 \) in \( X \) as \( n \to \infty \) and every \( y^*_n \in Sx_n \) with

\[
\lim_{n \to \infty} \langle y^*_n, x_n - x_0 \rangle \leq 0,
\]

we have \( x_n \rightarrow x_0 \in D(T) \) and there exists a subsequence of \( \{y^*_n\} \), denoted again by \( \{y^*_n\} \), such that \( y^*_n \rightarrow y^*_0 \in Tx_0 \) as \( n \to \infty \);

(c) "of type \((S)\)" if (i) and (ii) of (a) hold and for any sequence \( x_n \in D(T) \), \( y^*_n \in Tx_n \) such that \( y^*_n \rightarrow y^*_0 \) as \( n \rightarrow \infty \), it follows that there exists a subsequence of \( \{x_n\} \), denoted again by \( \{x_n\} \), such that \( x_n \rightarrow x_0 \) as \( n \rightarrow \infty \).

It is not difficult to see that the class of operators of type \((S)\) includes the classes of operators of type \((S_\lambda)\). Furthermore, it holds that \( T + C \) is of type \((S)\) provided that \( T \) is of type \((S)\) and \( C \) is compact. The main goals of this paper are

(i) to develop suitable degree theory for operators of the type \( T + S + C \), where \( T: X \supseteq D(T) \rightarrow 2^{X^*} \) is maximal monotone, \( S: X \mapsto 2^{X^*} \) is bounded of type \((S_\lambda)\), and \( C: D(C) \rightarrow X^* \) is compact with \( D(T) \subseteq D(C) \) and sublinear; that is, there exist \( r \geq 0 \) and \( \sigma \geq 0 \) such that \( \|Cx\| \leq r \|x\| + \sigma \) for all \( x \in D(C) \). The existing degree theories for operators of the type \( T + S \) cannot be used to treat inclusions involving operators of the type \( T + S + C \) because the compact operator is not everywhere defined. For recent degree theories for multivalued bounded \((S_\lambda)\) or bounded pseudomonotone perturbations of arbitrary maximal monotone operators, the reader is referred to the papers by Asfaw and Kartasatos [3], Asfaw [4], Adhikari and Kartasatos [5], and the references therein. In these theories, the maximal monotone operator is arbitrary and \((S_\lambda)\) and/or pseudomonotone operator is everywhere defined. The original degree mapping due to Browder [6] is for operators of the type \( T + f \), where \( f \) is single valued bounded operator of type \((S_\lambda)\) defined from the closure of a nonempty, bounded, and open subset \( G \) of \( X \). Hu and Papageorgiou [7] generalized Browder's theory for multivalued compact perturbation of \( T + f \), where the compact operator is defined on \( G \). All these theories do not include the case where \( C \) is not defined on \( G \), in particular, when \( D(C) \) contains \( D(T) \).

In view of those, our work in developing a degree theory for operators of the type \( T + S + C \), where \( C \) is a compact operator with \( D(T) \subseteq D(C) \), is essential. It is worth mentioning that the theory associated with (i) is a generalization of the previous degree theories for bounded \((S_\lambda)\) perturbations of maximal monotone operators due to Browder [6], Kobayashi and Otani [8], Hu and Papageorgiou [7], Asfaw and Kartasatos [3], and the references therein. The most general degree theory currently available which is due to Asfaw [9] is for pseudomonotone perturbations of the sum of two maximal monotone operators with one of the maximal monotone operators which is of type \( I_{pq}^0 \);

(ii) to derive existence theorem(s) in order to establish solvability of operator inclusion problems involving operators of the type \( T + S + C \). Consequently, the theory developed in (i) is applied to prove existence of solution for the inclusion problem \( f^* \in (T + S + C)(D(T) \cap B_R(0)) \) provided that there exists \( R = R(f^*) > 0 \) such that

\[
\langle v^* + \omega^* + Cx - f^*, x \rangle > 0
\]

for all \( x \in D(T) \cap \partial B_R(0), v^* \in Tx \), and \( \omega^* \in Sx \); that is, \( R(T + S + C) = X^* \) provided that \( T + S + C \) is coercive. The result is a generalization of the existence result due to Asfaw and Kartasatos [3, Theorem 17] for
Throughout the paper, we shall use the following definition of a homotopy of class \((S_t)\).

**Definition 3.** Let \(t \in [0, 1]\) and \(S^t : X \supset D(S^t) \to 2^{X^*}\). The family \([S^t]_{t \in [0, 1]}\) is said to be a "homotopy of type \((S_t)\)" if the following are true:

(i) For each \(t \in [0, 1]\), \(x \in D(S^t)\), \(S^t x\) is a nonempty, closed, convex, and bounded subset of \(X^*\).

(ii) For each \(t \in [0, 1]\), \(S^t\) is finitely continuous.

(iii) Let \(\{t_n\} \subset [0, 1], x_n \in D(S^{t_n})\) be such that \(t_n \to t_0\) and \(x_n \to x_0 \in X\). Let \(f_n \in S^{t_n} x_n\) be such that
\[
\limsup_{n \to \infty} \langle f_n, x_n - x_0 \rangle \leq 0.
\]
Then \(x_n \to x_0 \in D(S^{t_0})\) and there exists a subsequence of \(\{f_n\}\), denoted again by \(\{f_n\}\), such that \(f_n \to f \in S^{t_0} x_0\) as \(n \to \infty\).

The following lemma is due to Ibrahimou and Kartosatos [10].

**Lemma 4.** Let \(T : X \supset D(T) \to 2^{X^*}\) be maximal monotone and \(G \subset X\) be bounded. Let \(0 < s_1 \leq s_2\), \(0 < t_1 < t_2\). Let \(T^s = ST\). Then there exists a constant \(K_1 > 0\), independent of \(t\) and \(s\), such that \(||T^s u|| \leq K_1\) for all \(u \in G\), \(s \in [s_1, s_2]\), and \(t \in [t_1, t_2]\).

For basic definitions and further properties of mappings of monotone type, the reader is referred to Barbu [11], Pascali and Sburlan [12], Browder and Hess [1], and Zeidler [13].

The content of the following important lemma is due to Brezis et al. [14].

**Lemma 5.** Let \(B\) be a maximal monotone set in \(X \times X^*\). If \((u_n, u_n^*) \in B\) such that \(u_n \to u\) in \(X\), \(u_n^* \to u^*\) in \(X^*\), and
\[
\limsup_{n \to \infty} \langle u_n^* - u^* , u_n - u\rangle \leq 0,
\]
then \((u, u^*) \in B\) and \((u_n^*, u_n) \to (u^*, u)\) as \(n \to \infty\).

Browder [6] introduced the concept of a pseudomonotone homotopy as given below.

**Definition 6.** Let \([T^t]_{t \in [0, 1]}\) be a family of maximal monotone operators from \(X\) to \(2^{X^*}\) such that \(0 \in T^t(0), t \in [0, 1]\). Then \([T^t]_{t \in [0, 1]}\) is called a "pseudomonotone homotopy" if it satisfies the following equivalent conditions:

(i) Suppose that \(t_n \to t_0 \in [0, 1]\) and \((x_n, y_n) \in G(T^{t_n})\) are such that \(x_n \to x_0 \in X\), \(y_n \to y_0 \in X^*\) and
\[
\limsup_{n \to \infty} \langle y_n, x_n \rangle \leq \langle y_0, x_0 \rangle.
\]

Then \((x_0, y_0) \in G(T^{t_0})\) and \(\lim_{n \to \infty} \langle y_n, x_n \rangle = \langle y_0, x_0 \rangle\).

(ii) The mapping \(\phi : X^* \times [0, 1] \to X\) defined by
\[
\phi(w, t) = (T^t + f)^{-1}(w)
\]
is continuous.

(iii) For each \(w \in X^*\), the mapping \(\phi_w : [0, 1] \to X\) defined by
\[
\phi_w(t) = (T^t + f)^{-1}(w)
\]
is continuous.

(iv) For any \((x, y) \in G(T^{s_0})\) and any sequence \(t_n \to t_0\), there exists a sequence \((x_n, y_n) \in G(T^{s_0})\) such that \(x_n \to x\) and \(y_n \to y\) as \(n \to \infty\).

For a maximal monotone operator \(T : X \supset D(T) \to 2^{X^*}\), Kobayashi and Otani [8] proved that the family \([T^t]_{t \in [0, 1]}\) is a pseudomonotone homotopy of maximal monotone operators if and only if \(T\) is densely defined. It is worth mentioning that the proof of this fact does not require the hypothesis \(0 \in T(0)\). It is essential herein to mention that the original degree theory for single-value \((S_t)\) perturbations of maximal monotone operators is due to Browder [6]. For a generalization of Browder's degree for multivalued compact perturbations of \(T + f\), where \(T : X \supset D(T) \to 2^{X^*}\) is maximal monotone and \(f : \overline{G} \to X^*\) is bounded demicontinuous of type \((S_t)\), the reader is referred to the paper due to Hu and Papageorgiou [7]. For existence results for compact perturbation of maximal monotone operators, the reader is referred to the paper due to Kartosatos [15]. For a relevant degree mapping for single multivalued operator of type \((S_t)\), we cite the paper of Zhang and Chen [16]. Recent developments on degree theories for perturbations of the sum of two maximal monotone operators can be found in the papers due to Adhikari and Kartosatos [5] and Asfaw [4].

In Section 2 we construct a degree mapping for operators of the type \(T + S + C\), where \(T : X \supset D(T) \to 2^{X^*}\) is maximal monotone, \(S : X \to 2^{X^*}\) is bounded and of type \((S_t)\) or bounded pseudomonotone, and \(C : D(C) \to X^*\) is compact with \(D(T) \subseteq D(C)\) and satisfies a sublinearity condition. The existence of solutions for operator inclusion problems of the type \(Tu + Su + Cu \ni f^*\) is included in Section 3. In Section 4, the theory is applied to establish existence of weak solution(s) for a nonlinear parabolic problem in appropriate Sobolev spaces.

**2. Degree Theory for \(T + S + C\) with \(D(T) \subseteq D(C)\)**

2.1. Degree Theory for \(T + S + C\) with \(S\) Bounded and of Type \((S_t)\). The goal of this section is to develop a degree theory for operators of the type \(T + S + C\), where \(T : X \supset D(T) \to 2^{X^*}\) is maximal monotone, \(S : X \to 2^{X^*}\) is bounded and of type \((S_t)\), and \(C : D(C) \to X^*\) is compact with \(D(T) \subseteq D(C)\).

Throughout the paper, we assume that \(C\) belongs to \(L^{\sigma}_0\) (i.e., there exist \(\sigma \geq 0\) and \(\tau \geq 0\) such that \(||Cx|| \leq \tau ||x|| + \sigma\) for all \(x \in D(C)\)). To this end, we start by proving the following useful lemma.
Lemma 7. Let $G$ be a nonempty, bounded, and open subset of $X$. Let $T : X \ni D(T) \to 2^{X^*}$ be maximal monotone, $S : X \to 2^{X^*}$ be bounded and of type $(S_\infty)$, and $C : X \ni D(C) \to X^*$ be compact with $D(C) \subseteq D(T)$ such that $C$ belongs to class $\Gamma^*_0$. Assume, further, that $f^* \notin (T + S + C)(D(T) \cap \partial G)$. Then there exists $\varepsilon_0 > 0$ such that $d(T_\varepsilon + S + Cl_{\varepsilon}G, f^*)$ is well-defined and independent of $\varepsilon \in (0, \varepsilon_0]$.

Proof. In the first step, we claim that there exists $\varepsilon_0 > 0$ such that $d(T_\varepsilon + S + Cl_{\varepsilon}G, f^*)$ is well-defined for all $\varepsilon \in (0, \varepsilon_0]$. Suppose that this is false; that is, there exist $\varepsilon_n \downarrow 0^+$, $x_n \in \partial G$, and $w_n^* \in Sx_n$ such that

$$v_n^* + w_n^* + Cl_{\varepsilon_n}(x_n) = f^* \quad \forall n,$$

where $v_n^* = T_{\varepsilon_n}x_n$. By the definitions of $T_{\varepsilon_n}$ and $I_{\varepsilon_n}$, we have

$$J_{\varepsilon_n}x_n = x_n - \varepsilon_n J^{-1}(v_n^*) \in D(T),$$

$$v_n^* \in T(I_{\varepsilon_n}x_n) \quad \forall n.$$  

Since $\{x_n\}$ and $S$ are bounded, it follows that $\{w_n^*\}$ is bounded. Since $C$ belongs to $\Gamma^*_0$, we get that

$$\|v_n^*\| \leq \|f^* - w_n^*\| + \|Cl_{\varepsilon_n}(x_n)\| \leq \kappa_0 + \tau \|J_{\varepsilon_n}x_n\| + \sigma$$

$$= \kappa_0 + \tau \|x_n - \varepsilon_n J^{-1}(v_n^*)\| + \sigma$$

$$\leq \kappa_0 + \tau \|x_n\| + \tau \varepsilon_n \|v_n^*\| + \sigma$$

for all $n$, where $\kappa_0$ is an upper bound for $\|f^* - w_n^*\|$. This yields the estimate

$$\left(1 - \varepsilon_n\right) \|v_n^*\| \leq \kappa_0 + \tau \|x_n\| + \sigma$$

for all $n$. Since $\varepsilon_n \downarrow 0^+$ and $\{x_n\}$ is bounded, it follows that $\{v_n^*\}$ and $\{J_{\varepsilon_n}x_n\}$ are bounded. The compactness of $C$ implies the boundedness of $\{Cl_{\varepsilon_n}(x_n)\}$. Now, assume without loss of generality that $x_n \to x_0$, $v_n^* \to v_0^*$, and $w_n^* \to w_0^*$ as $n \to \infty$. Since $C$ is compact, we may assume, by passing into a subsequence if necessary, that $Cl_{\varepsilon_n}x_n \to g_0^*$ as $n \to \infty$. The maximality of $T$ along with Lemma 5 gives

$$\liminf_{n \to \infty} \langle v_n^*, x_n - x_0 \rangle \geq 0;$$

that is, we obtain from (12) that

$$\limsup_{n \to \infty} \langle w_n^*, x_n - x_0 \rangle$$

$$= \limsup_{n \to \infty} \left( - \langle v_n^* + Cl_{\varepsilon_n}(x_n) - f^*, x_n - x_0 \rangle \right)$$

$$= - \liminf_{n \to \infty} \left( \langle v_n^* + Cl_{\varepsilon_n}(x_n) - f^*, x_n - x_0 \rangle \right)$$

$$\leq - \liminf_{n \to \infty} \langle v_n^*, x_n - x_0 \rangle$$

$$- \liminf_{n \to \infty} \langle Cl_{\varepsilon_n}(x_n) - f^*, x_n - x_0 \rangle$$

$$= - \liminf_{n \to \infty} \langle v_n^*, x_n - x_0 \rangle \leq 0.$$  

Since $S$ is of type $(S_\infty)$, we conclude that $x_n \to x_0 \in \partial G$ as $n \to \infty$ and $\overline{w_n}^* \in Sx_0$. Consequently, using (12) we arrive at

$$\limsup_{n \to \infty} \langle v_n^*, x_n - x_0 \rangle \leq 0.$$  

The maximality of $T$ along with Lemma 5 yields $x_0 \in D(T) \cap \partial G$ and $v_0^* \in T_{\varepsilon_0}x_0$ and $\langle v_n^*, x_0 \rangle \to \langle v_0^*, x_0 \rangle$ as $n \to \infty$. Since $C$ is compact and $I_{\varepsilon_n}x_n = x_n - \varepsilon_n J^{-1}(v_n^*) \to x_0 \in D(T) \subseteq D(C)$ as $n \to \infty$, it follows that $Cl_{\varepsilon_n}x_n \to Cx_0 = g_0^*$ as $n \to \infty$. Letting $n \to \infty$ in (12), we get $f^* \in (T + S + C)(D(T) \cap \partial G)$. However, this is impossible. Thus, there exists $\varepsilon_0 > 0$ such that $d(T_{\varepsilon_0} + S + Cl_{\varepsilon_0}G, f^*)$ is well-defined for all $\varepsilon \in (0, \varepsilon_0]$.

Next, we shall prove that $d(T_{\varepsilon_0} + S + Cl_{\varepsilon_0}G, f^*)$ is independent of $\varepsilon \in (0, \varepsilon_0]$. Let $\epsilon_i \in (0, \varepsilon_0]$ ($i = 1, 2$) be such that $0 < \epsilon_1 < \epsilon_2 \leq \epsilon_0$, $q(t) = \epsilon_1 + (1 - t)\epsilon_2$, $t \in [0, 1]$. We consider the homotopy operator

$$H(t, x) = T_{q(t)}x + Sx + C_{q(t)}x, \quad (t, x) \in [0, 1] \times \overline{G}.$$  

We will show that the family $\{H(t, \cdot)\}_{t \in [0, 1]}$ is a homotopy of class $(S_\infty)$ such that $0 \notin H(t, \partial G)$ for all $t \in [0, 1]$. To this end, let $x_0 \in \overline{G}$, $w_0^* \in Sx_0$, $t_n \in [0, 1]$, $f_n^* = T_{q(t_n)}x_0 + w_0^* + Cl_{q(t_n)}x_n$, $x_n \to x_0$, and $t_n \to t_0$ as $n \to \infty$ be such that

$$\limsup_{n \to \infty} \langle f_n^*, x_n - x_0 \rangle \leq 0.$$  

Since $\{x_n\}$ and $S$ are bounded, it follows that $\{w_n^*\}$ is also bounded. Since $q(t_n) \to q(t_0) = q(t_0) > 0$ as $n \to \infty$, we use the continuity of $(0, \infty) \times X \ni (t, x) \to T_tx$ to $T_tx$ is continuous, we conclude that

$$\limsup_{n \to \infty} \langle w_n^*, x_n - x_0 \rangle \leq 0.$$  

By the compactness of $C$, we may assume without loss of generality that $Cl_{q(t_n)}x_n \to g_0^*$ as $n \to \infty$. Since $q(t_n) \to q(t_0) = q(t_0) > 0$ as $n \to \infty$, we use the continuity of $(0, \infty) \times X \ni (t, x) \to T_tx$ is continuous, we conclude that

$$\limsup_{n \to \infty} \langle v_n^*, x_n - x_0 \rangle \leq 0;$$  

that is, $\{H(t, \cdot)\}_{t \in [0, 1]}$ is a homotopy of class $(S_\infty)$ such that $0 \notin H(t, \partial G)$ for all $t \in [0, 1]$. Therefore, $d(H(t, \cdot), G, f^*)$ is independent of $t \in [0, 1]$; that is, $d(T_{\epsilon_1} + S + Cl_{\epsilon_1}G, f^*) = d(T_{\epsilon_2} + S + Cl_{\epsilon_2}G, f^*)$. Since $\epsilon_1$ and $\epsilon_2$ are arbitrary in $(0, \epsilon_0]$, we conclude that $d(T_{\epsilon_1} + S + Cl_{\epsilon_1}G, f^*)$ is well-defined and independent of $\epsilon \in (0, \epsilon_0]$. This completes the proof. \qed
Based on Lemma 7, the associated degree mapping is defined as follows.

**Definition 8.** Let $G$ be a nonempty, bounded, and open subset of $X$, $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone, $S : X \rightarrow 2^{X^*}$ be bounded and of type $(S_i)$, and $C : D(C) \rightarrow X^*$ be compact with $D(T) \subseteq D(C)$ and belonging to the class $I^*_m$. Assume, further, that $f^* \notin (T + S + C)(D(T) \cap \partial G)$. Then the degree mapping $d$ for $T + S + C$ at $f^* \in X^*$ with respect to $G$ is defined by

$$
d(T + S + C, G, f^*) = \lim_{\varepsilon \downarrow 0} d_s(T_\varepsilon + S + C_I, G, f^*),
$$

(24)

where $d_s$ is the degree mapping for multivalued bounded operators of type $(S_i)$ from [16].

### 2.2. Basic Properties of the Degree

**Theorem 9.** Let $G$ be a nonempty, bounded, and open subset of $X$. Let $T : X \supset D(T) \rightarrow 2^{X^*}$ be maximal monotone, $S : X \rightarrow 2^{X^*}$ be bounded and of type $(S_i)$, and $C : D(C) \rightarrow X^*$ be compact with $D(T) \subseteq D(C)$ such that $C$ belongs to $I^*_m$. Then the following properties hold:

(i) **(Normalization)** $d(J, G, 0) = 1$ if $0 \in G$ and $d(J, G, 0) = 0$ if $0 \notin \overline{G}$.

(ii) **(Existence)** if $f^* \notin (T + S + C)(D(T) \cap \partial G)$ and $d(T + S + C, G, f^*) = 0$, then $f^* \in (T + S + C)(D(T) \cap G)$.

(iii) **(Decomposition)** let $G_1$ and $G_2$ be nonempty, disjoint, and open subsets of $G$ such that $f^* \notin (T + S + C)(D(T) \cap (\overline{G} \setminus (G_1 \cup G_2)))$. Then

$$
d(T + S + C, G, f^*) = d(T + S + C, G_1, f^*) + d(T + S + C, G_2, f^*).$$

(25)

(iv) **(Translation invariance)** let $f^* \notin (T + S + C)(D(T) \cap \partial G)$. Then we have

$$
d(T + S + C - f^*, G, 0) = d(T + S + C, G, f^*).$$

(26)

Proof. The proof of (i) follows by setting $T = \emptyset$ and $C = \emptyset$. To prove (ii), assume that $f^* \notin (T + S + C)(D(T) \cap \partial G)$ and $d(T + S + C, G, f^*) = 0$. By the definition of $d$, there exists $\varepsilon_0 > 0$ such that $d(T_\varepsilon + S + C_I, G, f^*) \neq 0$ for all $\varepsilon \in (0, \varepsilon_0]$; that is, for each $\varepsilon_n \downarrow 0^+$ there exist $x_n \in D(T) \cap G$ and $w_n \in Sx_n$ such that

$$
v_n^* + w_n^* + C_I x_n = f^*, \quad v_n^* = T_\varepsilon x_n \forall n.
$$

(27)

Since $S$ is bounded, it follows that $\{w_n^*\}$ is bounded. By using $\Gamma_d^*$ condition on $C$ along with the arguments used in the proofs of Lemma 7, it is easy to see that $\{v_n^*\}$ and $\{I_{C_I} x_n\}$ are bounded. Assume without loss of generality that $x_n \to x_0$, $w_n \to w_0$, $v_n^* \to v_0^*$, and $C_I x_n \to g_n^*$ as $n \to \infty$. By the maximality of $T$, the $(S_i)$ condition on $S$, and the arguments used in the proof of Lemma 7, we conclude that $x_0 \in D(T) \cap G$, $v_0^* \in T x_0$, and $w_0^* \in S x_0$ such that $v_0^* + w_0^* + C x_0 = f^*$. This shows that $f^* \in (T + S + C)(D(T) \cap G)$.

Next we prove (iii). Suppose the hypotheses in (iii) hold. By the definition of $d$, we see that $d(T + S + C, G, f^*) = d(T_\varepsilon + S + C_I, G, f^*)$ for all sufficiently small $\varepsilon > 0$. Since $T_\varepsilon + S + C_I$ is bounded and of type $(S_i)$, the decomposition property of the degree mapping for multivalued $(S_i)$ operators implies

$$
d(T + S + C, G, f^*) = \lim_{\varepsilon \downarrow 0^+} d(T_\varepsilon + S + C_I, G, f^*)
$$

$$
= \lim_{\varepsilon \downarrow 0^+} d(T_\varepsilon + S + C_I, G_1, f^*) + \lim_{\varepsilon \downarrow 0^+} d(T_\varepsilon + S + C_I, G_2, f^*)
$$

(28)

$$
= d(T + S + C, G_1, f^*) + d(T + S + C, G_2, f^*);
$$

that is, (iii) holds.

(iv) Suppose that $f^* \notin (T + S + C)(D(T) \cap \partial G)$; that is, $0 \notin (T + S + C - f^*)(D(T) \cap \partial G)$. This implies that $d(T + S + C - f^*, G, 0)$ is well-defined. Since $d(T + S + C - f^*, G, 0) = d(T_\varepsilon + S + C_I - f^*, G, 0)$, by the translation property of the degree mapping for multivalued bounded operators of type $(S_i)$, we see that $d(T_\varepsilon + S + C_I - f^*, G, 0) = d(T_\varepsilon + S + C_I, G, f^*)$. Thus,

$$
d(T + S + C - f^*, G, 0)
$$

$$
= \lim_{\varepsilon \downarrow 0^+} d(T_\varepsilon + S + C_I - f^*, G, 0)
$$

(29)

$$
= \lim_{\varepsilon \downarrow 0^+} d(T_\varepsilon + S + C_I, G, f^*)
$$

$$
= d(T + S + C, G, f^*).
$$

(v) Suppose that $0 \notin M(t, D(T) \cap \partial G)$ for all $t \in [0, 1]$, where $M(t, x) = T x + t(S_1 x + C x) + (1 - t)S_2 x$, $(t, x) \in [0, 1] \times (D(T) \cap C)$. For every $\varepsilon > 0$, we consider

$$
M_\varepsilon(t, x) = T x + t(S_1 x + C I x) + (1 - t)S_2 x,
$$

(30)

$(t, x) \in [0, 1] \times \overline{C}$. 

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We shall show that there exists \( \varepsilon_0 > 0 \) such that \( d(M_t(t, \cdot), G, 0) \) is well-defined and independent of all \((t, \varepsilon) \in [0, 1] \times (0, \varepsilon_0] \). To do this, we assume that there exist \( \varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0] \) such that

\[
\begin{align*}
\phi(t, \varepsilon_1) &< 1, \\
\phi(t, \varepsilon_2) &> 1,
\end{align*}
\]

since \( \phi(t, \cdot) \) is bounded monotonically and of type \( (S_2) \).

To show that \( d(M_t(t, \cdot), G, 0) \) is constant for all \( t \in [0, 1] \) and \( \varepsilon \in (0, \varepsilon_0], \) with \( \varepsilon_0 \) as in the proof of (ii), we let \( 0 < \varepsilon_1 < \varepsilon_2 \leq \varepsilon_0 \) and consider the homotopy operator

\[
\bar{M}(t, x) = T_{q(t)} x + \theta \left( S_1 x + C_{J_{q(t)}} x \right) + (1 - \theta) S_2 x,
\]

where

\[
q(t) = t\varepsilon_1 + (1 - t)\varepsilon_2, \quad (t, x) \in [0, 1] \times G.
\]

Since for each \( t \in [0, 1] \) \( T_{q(t)} \) is monotone, \( S_1 \) and \( S_2 \) are bounded and of type \( (S_2) \) and \( C_{J_{q(t)}} \) is compact, it follows that \( M_t(t, \cdot) \) is bounded demicontinuous and of type \( (S_2) \). It is not hard to verify that \( 0 \notin \bar{M}(t, \partial G) \) for all \( t \in [0, 1] \).

In the arguments used in the proof of Lemma 7, we shall show that \( \bar{M}(t, \cdot) \in [0, 1] \) is a homotopy of class \( (S_2) \). To this end, let \( x_n \in G \) and \( t_n \in [0, 1] \) be such that \( x_n \rightarrow x \) and \( t_n \rightarrow t \in [0, 1] \) as \( n \rightarrow \infty \), \( u_n^* \in S_1 x_n \) and \( u_n^* \in S_2 x_n \) be such that

\[
\limsup_{n \rightarrow \infty} \left( T_{q(t_n)} x_n + t_n \left( u_n^* + C_{J_{q(t_n)}} x_n \right) \right) + (1 - t_n) u_n^* \rightarrow x_0 \leq 0.
\]

Since \( T_{q(t_0)} \) is monotone with domain \( X \), it follows that

\[
\limsup_{n \rightarrow \infty} \left( T_{q(t_n)} x_0 + t_n \left( u_n^* + C_{J_{q(t_n)}} x_0 \right) \right) + (1 - t_n) u_n^* \rightarrow x_0 \leq 0;
\]

that is,

\[
\limsup_{n \rightarrow \infty} \left( t_n \left( u_n^* + C_{J_{q(t_n)}} x_n \right) + (1 - t_n) u_n^* \right) \rightarrow x_0 \leq 0.
\]
the continuity of \( (0, \infty) \times X \ni (t, x) \mapsto T_t x \), it follows that \( T_{g(t, x)} x_n \to T_{g(t, x)} x_0 \) as \( n \to \infty \). From the continuity of \( J \) and \( C \), we obtain that
\[
J_{g(t, x)} x_n = x_n - q(t_n) J^{-1}(T_{g(t, x)} x_n) \to x_n - q(t_0) J^{-1}(T_{g(t, x)} x_0) = y_0 \in D(T) \subset D(C)
\]
as \( n \to \infty \) and \( CJ_{g(t, x)} x_n \to C y_0 = g_0^* \) as \( n \to \infty \). Thus, we arrive at
\[
T_{g(t, x)} x_n = \frac{1}{q(t_n)} J(x_n - J_{g(t, x)} x_n)
\]
for all \( n \), where \( v_n = T_{g(t, x)} x_n \in T_n T(y_n) \) and \( y_n = J_{g(t, x)} x_n \in D(T) \).

By using the \( \Gamma^* \) condition on \( C \), we arrive at
\[
\|v_n\| \leq t_n \|w_n^*\| + (1 - t_n) \|u_n^*\| + t_n \|C y_n\|
\]
for all \( n \), where \( \kappa_1 \) is an upper bound for \( \{\|w_n\|, \|u_n^*\|\} \). This gives the boundedness of \( \{v_n\} \) and \( \{y_n\} \). Since \( y_n \to y_0 \) as \( n \to \infty \), \( y_n - x_0 \) is bounded, it follows that \( y_n \to 0 \) as \( n \to \infty \). Assume without loss of generality that \( C y_n \to g_0^* \) as \( n \to \infty \). Since \( y_n - x_0 \to 0 \) as \( n \to \infty \), the quasimonotonicity of \( S_1 \) and \( S_2 \) implies
\[
\limsup_{n \to \infty} \langle v_n^*, y_n - x_0 \rangle
\]
for all \( n \), where \( v_n^* = T_{g(t, x)} x_n \in T_n T(y_n) \) and \( y_n = J_{g(t, x)} x_n \in D(T) \).

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Since \( \overline{D(T)} = X \), the result of Kobayashi and Otani [8] says that the family \( \{T(t)\}_{t \in [0,1]} \) is a pseudomonotone homotopy of maximal monotone operators. By (i) of Definition 6, we conclude that \( x_0 \in D(T_{g(t)} = t_0 T) \), \( v_0^* = t_0 T x_0 \), and \( \langle v_0^*, y_0 \rangle \to \langle \overline{v_0^*}, x_0 \rangle \) as \( n \to \infty \). Applying analogous arguments to those of the proof of (iv) along with the \( (S_2) \) condition on \( S_1 \) and \( S_2 \), one easily verify that \( x_n \to x_0 \in D(T) \cap dG, C y_n \to C x_0, w_n^* \to w_0^* \in S_1 x_0 \), and \( u_n^* \to u_0^* \in S_2 x_0 \), so that \( v_n^* + t_n w_n^* + (1 - t_n) u_n^* \) is bounded, it follows that \( y_n \to 0 \) as \( n \to \infty \). By the hypotheses. In conclusion, we have proved that \( d(T_{g(t)} + tS_1 + (1 - t)S_2 + tCJ_1, G, 0) \) is well-defined for all \( t \in [0, 1] \) and sufficiently small \( \gamma > 0 \).

Finally, we shall show that \( d(T_{g(t)} + tS_1 + (1 - t)S_2 + tCJ_1, G, 0) \) is independent of \( t \in [0, 1] \) and \( \gamma \in (0, \varepsilon_0) \). To this end, let \( 0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0, q(t) = t \varepsilon_1 + (1 - t) \varepsilon_2, 0 < t_1 < t_2 \leq 1 \), and \( y_1 = T t_1 + (1 - t) t_2 \in [0, 1] \). To complete the proof, we consider the homotopy operator
\[
N_1(t, x) = T_{q(g(t))}^\gamma x + tS_1 x + (1 - t) S_2 x + tCJ_1^{(g(t))},
\]
for all \( t \in [0, 1] \times \overline{G} \).

It is sufficient to show that \( \{N_1(t, \cdot)\}_{t \in [a, 1]} \) is a homotopy of class \((S_1)\). For each \( t \in [0, 1] \), it is easy to see that \( N_1(t, \cdot) :
$X \rightarrow 2^X$ is bounded and of type $(S,\iota)$. Let $\{x_n\} \subset X$ be such that $x_n \rightarrow x_0$, $t_n \rightarrow t_0$ as $n \rightarrow \infty$, $u_n' \in S_1x_n$, and $w_n' \in S_2x_n$ so that $\limsup_{n \rightarrow \infty} (T_{q(t_n)}^{0} x_n + g_n, x_n - x_0) \leq 0$, where $g_n = t_n w_n^* + (1-t_n)u_n^* + t_n C f_n^* x_n$. Let

$$z_n^* = T_{q(t_n)}^{0} x_n, \quad z_n = t_n w_n^* x_n \quad (51)$$

\forall n.

We show that $\{z_n^*\}$ and $\{z_n\}$ are bounded. Since $0 < \varepsilon_1 \leq q(t_n) \leq \varepsilon_2$ and $0 < t_1 \leq r_1 \leq t_2$ for all $n$, we conclude from Lemma 4 that $\{z_n^*\}$ is bounded. Since $z_n = x_n - q(t_n) f^{-1}(z_n^*)$ for all $n$ and $\{z_n^*\}$, $\{q(t_n)\}$ and $\{x_n\}$ are bounded, we get the boundedness of $\{z_n\}$. Since $C$ is compact, we assume without loss of generality that $Cz_n \rightarrow h_0^*$ as $n \rightarrow \infty$. By the boundedness of $\{x_n\}$, $S_1$ and $S_2$, we assume, by passing into subsequences if necessary, that $x_n \rightarrow x_0$, $w_n' \rightarrow w_0'$, and $u_n \rightarrow u_0'$ as $n \rightarrow \infty$. On the other hand, the pseudo-monotonicity of $S_1$ and $S_2$ gives

$$\liminf_{n \rightarrow \infty} \langle g_n^*, x_n - x_0 \rangle \geq 0. \quad (52)$$

Consequently, we get

$$\limsup_{n \rightarrow \infty} \langle z_n^*, x_n - x_0 \rangle \leq \limsup_{n \rightarrow \infty} \langle z_n^* + g_n^*, x_n - x_0 \rangle$$

$$- \liminf_{n \rightarrow \infty} \langle g_n^*, x_n - x_0 \rangle \leq \liminf_{n \rightarrow \infty} \langle g_n^*, x_n - x_0 \rangle \leq 0. \quad (53)$$

Since $T$ is positively homogeneous of order $a > 0$, it is not difficult to see that $T^{-1}: R(T) \rightarrow D(T)$ is maximal monotone and positively homogeneous of order $\alpha > 0$. It also holds that $(\Lambda T)^{-1}(x) = T^{-1}((1/\lambda)(x)) = \lambda^{-1/a} T^{-1}(x)$ for all $x \in R(T)$. In addition, we see that

$$z_n^* = T_{q(t_n)}^{0} x_n = (y_{t_n} T)_{q(t_n)} x_n$$

$$= \left( y_{t_n} T \right)^{-1} + q(t_n) f^{-1} \left( y_{t_n} T \right)^{-1} x_n$$

$$= \left( \frac{1}{y_{t_n}} T + q(t_n) f^{-1} \right)^{-1} x_n$$

$$= \left( \frac{1}{y_{t_n}^{1/a}} T^{-1} + q(t_n) y_{t_n}^{1/a} f^{-1} \right)^{-1} x_n$$

$$= y_{t_n}^{1/a} \left( T^{-1} + q(t_n) y_{t_n}^{1/a} f^{-1} \right)^{-1} x_n$$

$$= y_{t_n}^{1/a} T_{q(t_n) y_{t_n}^{1/a}} x_n \quad \forall n. \quad (54)$$

In fact, it is true that, $T_n^\alpha(x) = a^{\varepsilon \alpha} T_{e_n^\varepsilon}(x)$ for all $x \in X, \varepsilon > 0$ and $\alpha > 0$. For each $n$, letting $\lambda_n = y_{t_n}^{1/a}$, we get

$$\langle z_n^*, x_n - x_0 \rangle = \langle \lambda_n T_{q(t_n)}^{0} x_n, x_n - x_0 \rangle$$

$$= \lambda_n \langle T_{q(t_n)}^{0} x_n, x_n - x_0 \rangle$$

$$= \lambda_n \langle T_{q(t_n)}^{0} x_n, x_n - x_0 \rangle$$

$$+ \lambda_n \langle T_{q(t_n)}^{0} x_n, x_n - x_0 \rangle \quad (55)$$

Since $(0, \infty) \times (t, x) \rightarrow T, x$ is continuous, it follows that $T_{q(t_n)}^{0} x_n \rightarrow T_{q(t_n)}^{0} x_0$ as $n \rightarrow \infty$, where $\lambda_n \rightarrow \lambda > 0$ as $n \rightarrow \infty$. By the monotonicity of $T_{q(t_n)}^{0} x_n$ for all $n$, we have

$$\liminf_{n \rightarrow \infty} \langle z_n^*, x_n - x_0 \rangle \geq 0; \quad (56)$$

that is,

$$0 \leq \liminf_{n \rightarrow \infty} \langle z_n^*, x_n - x_0 \rangle \leq \limsup_{n \rightarrow \infty} \langle z_n^*, x_n - x_0 \rangle \leq 0, \quad (57)$$

which implies $\langle z_n^*, x_n \rangle \rightarrow \langle z_0^*, x_0 \rangle$ as $n \rightarrow \infty$. Consequently, we arrive at

$$\limsup_{n \rightarrow \infty} \langle w_n^*, x_n - x_0 \rangle \leq 0.$$

Since both $S_1$ and $S_2$ are bounded and of type $(S,\iota)$, it follows that $x_n \rightarrow x_0$ as $n \rightarrow \infty$. As a result of this, we get

$$z_n^* = \lambda_n T_{q(t_n)}^{0} x_n \rightarrow \lambda_0 T_{q(t_n)}^{0} x_0 = z_0^*.$$

$$z_n = x_n - q(t_n) f^{-1}(z_n^*) \rightarrow x_0 - q(t_n) f^{-1}(T_{q(t_n)}^{0} x_0) = x_0 - q(t_n) f^{-1}(z_0^*)$$

as $n \rightarrow \infty$; that is, we have $\limsup_{n \rightarrow \infty} \langle z_n^*, z_n \rangle \leq \langle z_0^*, z_0 \rangle$. Since $\{T_n\}_{t \in \{0,1\}}$ is a pseudomonotone homotopy of maximal monotone operators, it follows that $z_0 \in D(t_0 T)$ and $z_0^* \in t_0 T(z_0)$. In conclusion, we obtain that $x_0 \in \partial G, w_0' \in S_1x_0, u_0' \in S_2x_0, z_0 \in D(T)$, and

$$z_0^* + t_0 w_0^* + (1-t_0) u_0^* + t_0 Cz_0$$

$$\in T_{q(t_0)}^{0} x_0 + t_0 S_1 x_0 + (1-t_0) S_2 x_0 + Cz_0. \quad (60)$$

Therefore, for any $\varepsilon \in (0,1]$, the family $\{N_1(t, \cdot)\}_{t \in \{0,1\}}$ is a homotopy of class $(S,\iota)$. Thus, $d(N_1(t, \cdot), G, 0)$ is independent of $t \in (0,1]$ and $\varepsilon \in (0, \varepsilon_0]$; that is,

$$d(N_1(t, \cdot), G, 0) = d(N_1(1, \cdot), G, 0)$$

$$= d(T_{t_1} + S_1 + C t_1, G, 0) \quad (61)$$

\forall t \in (0,1].
On the other hand, by the definition of $d$, we have that
\[
d(N(t,\cdot),G,0) = d(T^\epsilon_t + tS_1 + \epsilon \{ T + S \} C_t, G,0) \quad (62)
\]
is independent of $\epsilon \in (0,1]$ and $t \in (0,\epsilon_0]$. In particular, for $t = 1$, we have $d(N(t,\cdot),G,0) = d(T + S + C, G,0)$ for all $t \in (0,1]$. But, for $t = 0$, we see that $N(0,x) = S_1 x$ for all $x \in X$. To complete the proof, it is sufficient to show that
\[
d(T + S + C, G,0) = d(S_2, G,0). \quad (63)
\]
For each $\epsilon > 0$, we consider the homotopy
\[
T^\epsilon_t x = T^\epsilon x + t (S_1 x + C_\epsilon x) + (1-t) S_2 x, \quad (t, x) \in [0,1] \times G.
\]
Suppose that there exist $\epsilon_n \downarrow 0^+$, $t_n \in [0,1]$, $x_n \in \partial G$, $u_n^* \in S_1 x_n$ and $u_n^* \in S_2 x_n$ such that
\[
T^\epsilon_{t_n} x_n + t_n (u_n^* + C_{\epsilon_n} x_n) + (1-t_n) u_n^* = 0 \quad \forall n. \quad (65)
\]
We assume without loss of generality that $t_n \to t_0 \in [0,1]$, $x_n \to x_0$, $u_n^* \to u_0^*$ and $u_n^* \to u_0^*$ as $n \to \infty$. By the $\Gamma^\epsilon_t$ condition on $C$, we get
\[
\left\| T^\epsilon_{t_n} x_n \right\| \leq \tau \left\| x_n - \epsilon_n \Gamma^{-1}_{t_n} (T^\epsilon_{t_n} x_n) \right\| + \kappa_2 \quad (66)
\]
where $\kappa_2$ is an upper bound for the sequence $\{ \| u_n^* \| + \| x_n^* \| \}$. This shows the boundedness of $\{ T^\epsilon_{t_n} x_n \}$ and $\{ T^\epsilon_{t_n} x_n \}$. By the maximality of $T^\epsilon_t$ along with Lemma 5, the compactness of $C$ and the $(S_1)$ condition on $S_1$ and $S_2$ and analogous arguments to those in the proof of Theorem 9, we conclude that $x_0 \in D(T) \cap \partial G$, $v_0^* \in T x_0$, $w_0^* \in S_1 x_0$, and $u_0^* \in S_2 x_0$ so that $v_0^* = 0$ and $u_0^* = 0$, that is, $0 \in (T + t_0 S_1 + C_0) x_0$ if $x \in \partial G$. However, this is impossible. In addition, the boundary condition on $M$ in (v) implies that $\{ N(t,\cdot) \}_{t \in [0,1]}$ is an admissible homotopy; that is, $d(T_{t} + S_1 + C_{\epsilon} G,0) = d(T_{t} + S_2, G,0)$ for all $\epsilon \in (0,\epsilon_0]$. Since $0 \in G$ and $S_2$ satisfies the condition $(u^*_t, x) \geq \| x \|^2$ for all $x \in X$ and $u^* \in S_2 x$ and $0 \in T(0)$, it follows that $0 \neq t(T_{t} + S_2 x + (1-t) S_1 x)$ for all $x \in \partial G$ and $t \in [0,1]$. Finally, we conclude that $d(N(t,\cdot),G,0)$ is independent of all $t \in [0,1]$. This completes the proof.

2.3. Degree Theory For $T + C + S$ With $S$ Pseudomonotone. In this section we present a generalization of the theory developed in the previous section for operators of type $T + C + S$, where $S : X \to 2^{X^*}$ is bounded pseudomonotone and $T, C$ satisfy the conditions of Section 2.1. For each $\epsilon > 0$, it is well-known that $S + \epsilon I$ is bounded and of type $(S_1)$. As a result of this, we may apply the arguments used in the proof of Lemma 7 to show that $d(T + S + C + \epsilon I, G, f^*)$ is well-defined and constant for all sufficiently small $\epsilon > 0$ provided that $f^* \notin (T + S + C) (D(T) \cap \partial G)$, where $d$ is given in Definition 6. We thus give the following definition.

Definition 10. Let $G$ be a nonempty, bounded, and open subset of $X$. $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone, $S : X \to 2^{X^*}$ be bounded pseudomonotone, and $C : D(C) \to X^*$ be compact with $D(T) \subseteq D(C)$ and belonging to the class $\Gamma^\epsilon_t$. Assume, further, that $f^* \notin (T + S + C) (D(T) \cap \partial G)$. Then the degree mapping $d$ for $T + S + C$ at $f^*$ with respect to $G$ is defined by
\[
d(T + S + C, G, f^*) = \lim_{\epsilon \to 0} d(T + S + C + \epsilon I, G, f^*), \quad (67)
\]
where $d(T + S + C + \epsilon I, G, f^*)$ denotes the degree mapping constructed in Section 2.1.

The following theorem gives some basic properties and homotopy invariance results analogous to those of Theorem 9.

Theorem 11. Let $G$ be a nonempty, bounded, and open subset of $X$. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone, $S : X \to 2^{X^*}$ be bounded pseudomonotone, and $C : D(C) \to X^*$ be compact with $D(T) \subseteq D(C)$ and belonging to the class $\Gamma^\epsilon_t$. Then the following properties hold:

(i) (Normalization) $d(f,G,0) = 1$ if $0 \in G$ and $d(f,G,0) = 0$ if $0 \notin \partial G$.

(ii) (Existence) If $f^* \notin (T + S + C) (D(T) \cap \partial G)$ and $d(T + S + C, G, f^*) \neq 0$, then $f^* \in (T + S + C) (D(T) \cap G)$. If $T + S$ is of type (S), then $f^* \in (T + C + S) (D(T) \cap G)$.

(iii) (Decomposition) Let $G_1$ and $G_2$ be nonempty and disjoint open subsets of $G$ such that $f^* \notin (T + S + C) (D(T) \cap (G_1 \cup G_2))$. Then
\[
d(T + S + C, G, f^*) = d(T + S + C + G_1, f^*) + d(T + S + C + G_2, f^*). \quad (68)
\]

(iv) (Translation invariance) Let $f^* \notin (T + S + C) (D(T) \cap \partial G)$. Then we have
\[
d(T + S + C - f^*, G,0) = d(T + S + C, G, f^*). \quad (69)
\]

(v) Let $M(t,x) = T x + tS_1 x + C x + (1-t) S_2 x$, $(t,x) \in [0,1] \times (D(T) \cap \partial G)$, and $S_1 : X \to 2^{X^*}$ be bounded pseudomonotone such that $0 \notin \overline{M(t,D(T) \cap \partial G)}$ for all $t \in [0,1]$. Then $d(M(t,\cdot),G,0)$ is independent of $t \in [0,1]$.

(vi) Let $N(t,x) = T x + S_2 x + C x + (1-t) S_2 x$, $(t,x) \in [0,1] \times (D(T) \cap \partial G)$, and $T : X \supset D(T) \to X^*$ be densely defined maximal monotone and positively homogeneous of order $\alpha > 0$, $S_1 : X \to 2^{X^*}$ be
bounded pseudomonotone, $S_2 : X \to 2^{X'}$ be bounded and of type $(S_+)$, and $0 \notin \overline{N}(t, D(T) \cap \partial G)$ for all $t \in [0,1]$. Assume, further, that $0 \notin M(t, D(T) \cap \partial G)$ for all $t \in [0,1]$. Then $d(N(t, \cdot), G, 0)$ is independent of $t \in [0,1]$.

Proof. The proofs for (i) through (iv) follow as in the analogous items in the proof of Theorem 9. We shall give sketches of the proofs of (v) and (vi). To prove (v), for each $\epsilon > 0$, we consider the homotopy inclusion

$$M_\epsilon(t, x) = Tx + t(S_1 x + Cx + \epsilon Jx),$$

(70)

$$(t, x) \in [0,1] \times \left(D(T) \cap \overline{G}\right).$$

Following the arguments used in the proof of (v) of Theorem 9, it can be shown that there exists $\epsilon_0 > 0$ such that $0 \notin M_\epsilon(t, D(T) \cap \partial G)$ for all $t \in [0,1]$ and $\epsilon \in (0, \epsilon_0]$. Otherwise, we would get $0 \in M(t, D(T) \cap \partial G)$ for some $t_0 \in [0,1]$, which is impossible. On the other hand, for each $\epsilon \in (0, \epsilon_0]$, we see that

$$M_\epsilon(t, x) = Tx + t(S_1 x + Cx + \epsilon Jx) + (1 - t)(S_2 x + \epsilon Jx),$$

(71)

$$(t, x) \in [0,1] \times \left(D(T) \cap \overline{G}\right).$$

Since $S_1 + \epsilon J$ and $S_2 + \epsilon J$ are bounded operators of type $(S_+)$, the proof of (v) of Theorem 9 implies that $d(M_\epsilon(t, \cdot), G, 0)$ is independent of $t \in [0,1]$; that is,

$$d(M_\epsilon(t, \cdot), G, 0) = d(T + S_1 + C + \epsilon J, G, 0)$$

= $d(T + S_1 + C, G, 0)$

(72)

for all $t \in [0,1]$ and $\epsilon \in (0, \epsilon_0]$. As a result of this, we get

$$d(M(t, \cdot), G, 0) = \lim_{\epsilon \to 0^+} d(M_\epsilon(t, \cdot), G, 0)$$

= $d(T + S_1 + C, G, 0)$

(73)

for all $t \in [0,1]$. This proves that $d(M(t, \cdot), G, 0)$ is independent of $t \in [0,1]$ provided that $0 \notin M(t, D(T) \cap \partial G)$ for all $t \in [0,1]$. The proof of (vi) can be completed in analogous manner. The details are omitted here.

3. An Existence Theorem

As a consequence of the degree theory developed in Section 2, the following theorem gives a new existence result on the solvability of operator inclusions of the type $Tu + Su + Cu \ni f^*$ in $D(T)$ provided that $T + S$ is of type $(S_+)$ or $S$ is bounded of type $(S_+)$. Let $T : X \to D(T)$ be maximal monotone with $0 \in T(0)$, $S : X \to 2^{X'}$ be bounded pseudomonotone, and $C : X \to D(C) \to X^*$ be compact with $D(T) \subseteq D(C)$ and belonging to the class $\Gamma_\nu$. Let $f^* \in X^*$. Assume, further, that there exists $R > 0$ such that

$$\langle v^* + w^* + Cx - f^*, x \rangle > 0$$

(74)

for all $x \in D(T) \cap \partial B_R(0)$, $v^* \in Tx$, and $w^* \in Sx$. Then $f^* \in \overline{T + S + C}(D(T) \cap \partial B_R(0))$. Furthermore, $\overline{R(T + S + C)} = X^*$ provided that $T + S + C$ is coercive.

Proof. Let $\epsilon > 0$. We shall show that $0 \notin K_\epsilon(t, D(T) \cap \partial B_R(0))$ for all $t \in [0,1]$, where

$$K_\epsilon(t, x) = \{(Sx + \epsilon Jx - f^*) + (1 - t)(Sx + \epsilon Jx), x \in [0,1] \times (D(T) \cap \overline{B}_R(0)) \}.$$

Since $0 \in T(0)$, by using the boundary condition on $T + S + C$, we see that

$$\langle t(v^* + w^* + Cx - f^*), x \rangle + \langle (1 - t)v^* + \epsilon Jx, x \rangle \geq \langle (1 - t)v^* + \epsilon Jx, x \rangle$$

= $\langle 1 - t \rangle \langle v^*, x \rangle + \epsilon \langle Jx, x \rangle \geq \epsilon \|x\|^2 = \epsilon R^2 > 0$

for all $t \in [0,1], x \in D(T) \cap \partial B_R(0)$, $v^* \in Tx$, and $w^* \in Sx$; that is, for each $\epsilon > 0$, it follows that $0 \notin K_\epsilon(t, D(T) \cap \partial B_R(0))$ for all $t \in [0,1]$. Since $S \in \epsilon + S + \epsilon J$ are continuous, bounded, and of type $(S_+)$, (v) of Theorem 9 implies that $K_\epsilon(t, \cdot)$ is an admissible homotopy. Therefore, for each $\epsilon > 0$, we obtain

$$d(K_\epsilon(t, \cdot), B_R(0), 0) = d(\epsilon J, B_R(0), 0) = 1$$

(77)

for all $t \in [0,1]$.

That is, $d(T + S + C + \epsilon J, B_R(0), f^*) = 1$. By (ii) of Theorem 9, we conclude that $f^* \in \overline{T + S + C + \epsilon J}(D(T) \cap \partial B_R(0))$; that is, for each $\epsilon_n \downarrow 0$, there exist $x_n \in D(T) \cap \partial B_R(0)$, $v_n^* \in Tx_n$, and $w_n^* \in S\epsilon_n$ such that

$$v_n^* + w_n^* + Cx_n + \epsilon_n Jx_n = f^* \quad \forall n.$$  

(78)

Since $\{x_n\}$ is bounded, we have $\epsilon_n Jx_n \to 0$ as $n \to \infty$, which implies that $f^* \in \overline{T + S + C + \epsilon J}(D(T) \cap \partial B_R(0))$. If $T + S + C$ is coercive, then for each $f^* \in X^*$ there exists $R = R(f^*) > 0$ such that the boundary condition holds. This implies that $f^* \in \overline{T + S + C + \epsilon J}(D(T) \cap \partial B_R(0))$. Since $f^* \in X^*$ is arbitrary, we conclude that $\overline{R(T + S + C)} = X^*$. The proof is complete.

The arguments used in the proof of Theorem 12 follow the existence result on the surjectivity of $T + S + C$ provided that either $S$ is bounded and of type $(S_+)$ or $T + S$ is of type $(S)$. Let $T : X \to D(T) \subseteq 2^{X'}$ be maximal monotone with $0 \in T(0)$, $S : X \to 2^{X'}$ be bounded pseudomonotone, and $C : X \to D(C) \to X^*$ be compact with $D(T) \subseteq D(C)$ and belonging to the class $\Gamma_\nu$. Let $f^* \in X^*$. Assume, further, that $T + S + C$ is coercive. Then $T + S + C$ is surjective provided that $S$ is bounded of type $(S_+)$ or $T + S$ is of type $(S)$.
\textbf{Proof.} Let \( f^* \in X^* \). Suppose \( T + S + C \) is coercive; that is, there exists \( \phi : [0, \infty) \to (-\infty, \infty) \) and \( \phi(t) \to \infty \) as \( t \to \infty \) such that
\[
\langle v^* + w^* + Cx, x \rangle \geq \phi(\|x\|) \|x\|
\]
\[\forall x \in D(T), \ v^* \in Tx, \ w^* \in Sx. \tag{79}\]
Then, there exists \( R = R(f^*; \|u\|) \) such that
\[
\langle v^* + w^* + Cx - f^*, x \rangle > 0 \tag{80}\]
for all \( x \in D(T) \cap \partial B_R(0) \), \( v^* \in Tx \), and \( w^* \in Sx \). Assume that \( T + S \) is of type \( (S) \). By Theorem 12, we conclude that \( f^* \in \frac{(T + S + C)(D(T) \cap B_R(0))}{(T + S + C)(D(T) \cap B_R(0))} \); that is, there exists \( x_n \in D(T) \cap B_R(0) \), \( v_n^* \in Tx_n \), and \( w_n^* \in Sx_n \) such that \( v_n^* + w_n^* + Cx_n \to f^* \) as \( n \to \infty \). Since \( C \) is compact, we assume without loss of generality that \( Cx_n \to g_n \) as \( n \to \infty \); that is, \( v_n^* + w_n^* \to f^* - g_n \). Since \( T + S \) is of type \( (S) \), it follows that \( x_n \to x_0 \) as \( n \to \infty \). By the maximality of \( T + S + C \) along with Lemma 5, the sublinearity condition, and arguments used in the proof of Lemma 7, we conclude that \( x_0 \in D(T) \cap B_R(0) \) and \( f^* \in Tx_0 + Sx_0 + Cx_0 \); that is, \( f^* \in (T + S + C)(D(T) \cap B_R(0)) \). Since \( f^* \in X^* \) is arbitrary, we conclude that \( T + S + C \) is surjective. The case when \( S \) is bounded and of type \( (S_2) \) is analogous. The details are omitted here. \qed

Theorem 12 is a new result and Corollary 13 gives a surjectivity result for operators of the type \( T + S + C \). For further existence results involving operators of the type \( T + S \), the reader is referred to Kenmochi [17], Le [18], and Asfaw [19]. For various examples on pseudomonotone and quasimonotone operators, we cite the paper due to Mustonen [20].

4. An Example

Let \( H = L^2(0, T; V) \) and \( V = W_0^{1, 2}(\Omega) \). It is well-known that \( H \) and \( V \) are real Hilbert spaces with duality pairing between \( u \in H \) and \( v \in H \) denoted by \( \langle u, v \rangle \) which is given by
\[
\langle u, v \rangle = \int_0^T \langle u(t), v(t) \rangle_V \, dt, \quad u \in H, \ v \in V, \tag{81}\]
where \( \langle u(t), v(t) \rangle_V \) denotes the duality pairing between \( u(t) \) and \( v(t) \) in \( V \). We shall apply the existence theorem(s) derived with the aid of the degree theory developed in this paper to establish existence of weak solution(s) in \( H \) for nonlinear problem given by
\[
\frac{\partial u}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, t, u, \nabla u) + g(x, t, u, \nabla u) = f(x, t) \tag{82}\]
where \( Q = \Omega \times (0, T) \) and \( f \in L^2(Q) \) and the functions \( a_i \) \((i = 1, 2, \ldots, N)\) and \( g \) satisfy the following measurability and sublinearity conditions:
\[
(C_1) \ a_i(x, t, \eta, \zeta) \ (i = 1, 2, \ldots, N) \text{ is Carathéodory function; that is, } (x, t) \to a_i(x, t, \eta, \zeta) \text{ is measurable for almost all } (\eta, \zeta) \in \mathbb{R}^N \text{ and } (\eta, \zeta) \to a_i(x, t, \eta, \zeta) \text{ is continuous for almost all } (x, t) \in \Omega \times [0, T]. \tag{83}\]
\[
(C_2) \text{ There exists } c_2 > 0 \text{ such that } \sum_{i=1}^N a_i(x, \eta, \zeta) \geq c_2 \text{ for all } (x, t) \in Q, \ (\eta, \zeta) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{84}\]
\[
(C_3) \ g \ : \ Q \times \mathbb{R} \to \mathbb{R} \text{ is Carathéodory function and there exist } c_3, k_2 \in L^2(Q) \text{ such that } g(x, t, \eta, \zeta) \eta \geq c_3 |\eta|^2 + k_2(x, t), \tag{85}\]
where \( u \) is understood in the sense of distributions; that is,
\[
\int_0^T u'(t) \psi(t) \, dt = -\int_0^T u(t) \psi'(t) \, dt \quad \forall \psi \in C_c^\infty(0, T). \tag{86}\]

Next we give the following theorem.

\textbf{Theorem 15.} Let \( f \in L^2(Q) \). Assume that conditions \((C_1)\) through \((C_3)\) are satisfied. Then \( (82) \) admits at least one weak solution.

\textbf{Proof.} Let \( S : H \to H \) be given by
\[
\langle Su, v \rangle = \sum_{i=1}^N \int_Q a_i(x, t, u, \nabla u) \frac{\partial v}{\partial x_i} \, dx \, dt, \tag{87}\]
where \( v \in H, \ u \in H \).
By using (C₁) and (C₂), it is well-known that S is bounded continuous of type (S₁). For the proof of these facts and other relevant properties of pseudomonotone and (S₁) type differential operators, the reader is referred to the papers by Browder [21], Berkovits and Mustonen [22], Hu and Papageorgiou [7], Landes and Mustonen [23], and the references therein. Let C : H → D(C) → H be defined by

$$\langle Cu, v \rangle = \int_Q g(x, t, u, V u)v(x, t) \, dx \, dt, \quad v \in H, \tag{88}$$

where u ∈ D(C) = \{ y ∈ H : y′ ∈ H \} and L : H → D(L) → H such that \( \langle Lu, v \rangle = \int_0^T (u'(t), v(t)) \, dt \) for all \( u \in D(L) = \{ y ∈ H : y(0) = y(T) \} \), that is, \( D(L) \subseteq D(C). \) It is well-known that L is a densely defined maximal monotone operator. The proof of this result is due to Brézis which can be found in the book by Zeidler [13, Theorem 32.1, pp. 897–899]. Since \( D(C) \) is compactly embedded in \( L^2(Q) \), it is known that \( \Gamma \) is a completely continuous operator; that is, \( \Gamma \) is a compact operator. Further reference on operators of the type \( \Gamma \) and existence results for parabolic problems, the reader is referred to the papers by Georgiou [7], Landes and Mustonen [23], and the references therein. Let \( \Gamma = \Gamma_0 + \Gamma_1 + \ldots + \Gamma_n \) and \( \Gamma_n \) be the boundary condition in \( \Omega \). Next we show the boundary condition in \( \Omega \). To this end, by using conditions (C₁) through (C₃) and monotonicity of \( L \), we have

$$\langle Cu, v \rangle \leq 2\varepsilon \|u\| \|v\| + \|k_2\|_{L^2(Ω)} \|v\| \tag{90}$$

for all \( u \in H \) and \( v \in H \). Consequently, taking supremum overall \( v \in H \) with \( \|v\| \leq 1 \), we conclude that \( \|Cu\| \leq \tau \|u\| + \sigma \) for all \( u \in H \), where \( \tau = 2\varepsilon \) and \( \sigma = \|k_2\|_{L^2(Ω)} \); that is, \( C \) belongs to \( \Gamma^\prime \). Next we show the boundary condition in \( \Omega \). To this end, by using conditions (C₁) through (C₃) and monotonicity of \( L \), we get

$$\langle Lu + Su + Cu, u \rangle$$

for all \( u \in H \). Since the right side of the above inequality approaches \( \gamma \) as \( \|u\| \rightarrow \infty \), for each \( f \in L^2(Ω) \) there exists \( R = R(f) > 0 \) such that

$$\langle Lu + Su + Cu - f, u \rangle > 0 \tag{92}$$

for all \( u \in H \). Since the right side of the above inequality approaches \( \gamma \) as \( \|u\| \rightarrow \infty \), for each \( f \in L^2(Ω) \) there exists \( R = R(f) > 0 \) such that

$$\langle Lu + Su + Cu - f, u \rangle > 0 \tag{92}$$

for all \( u \in H \) and \( \gamma \) is the right-hand side of the above inequality. Consequently, taking supremum overall \( v \in H \) with \( \|v\| \leq 1 \), we conclude that \( \|Cu\| \leq \tau \|u\| + \sigma \) for all \( u \in H \), where \( \tau = 2\varepsilon \) and \( \sigma = \|k_2\|_{L^2(Ω)} \); that is, \( C \) belongs to \( \Gamma^\prime \). Next we show the boundary condition in \( \Omega \). To this end, by using conditions (C₁) through (C₃) and monotonicity of \( L \), we get

$$\langle Lu + Su + Cu, u \rangle$$

for all \( u \in H \). Consequently, taking supremum overall \( v \in H \) with \( \|v\| \leq 1 \), we conclude that \( \|Cu\| \leq \tau \|u\| + \sigma \) for all \( u \in H \), where \( \tau = 2\varepsilon \) and \( \sigma = \|k_2\|_{L^2(Ω)} \); that is, \( C \) belongs to \( \Gamma^\prime \).

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of the paper.

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