Research Article

Nonnegative Infinite Matrices that Preserve (p, q) -Convexity of Sequences

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This paper deals with matrix transformations that preserve the (p, q) -convexity of sequences. The main result gives the necessary and sufficient conditions for a nonnegative infinite matrix A to preserve the (p, q) -convexity of sequences. Further, we give examples of such matrices for different values of p and q .

1. Introduction

If $p > 0, q > 0$, then the sequence $\{x_n\}$ of real numbers is said to be (p, q) -convex if

$$\Delta_{p,q}(x_n) = x_n - (p+q)x_{n-1} + pqx_{n-2} \geq 0 \quad (1)$$

for $n \geq 2$. The operator $\Delta_{p,q}$ generates the second-order difference Δ^2 when $p = q = 1$. Several authors [1–3] have proved various results on the convex sequences defined by $\Delta^2 x_n \geq 0$. Other authors [4, 5] have studied the classes of sequences satisfying $\Delta_{1,q}(x_n) \geq 0$. Also, the necessary and sufficient conditions for a sequence to be a (p, q) -convex sequence can be found in [6]. Moreover, some inequalities on (p, q) -convex sequences are given in [7, 8].

In [9–11], the authors discuss the matrix transformations that preserve (p, q) -convexity of sequences in the case of a lower triangular matrix with a particular type of matrix transformation. But the question of a general infinite matrix preserving (p, q) -convexity has not been considered anywhere in the literature. This paper deals with the necessary and sufficient conditions for a nonnegative infinite matrix to preserve (p, q) -convexity in both settings when $p \neq q$ and $p = q$.

2. Preliminaries

For any given sequence $\{x_n\}$, we can find a corresponding sequence $\{c_k\}$ such that

$$\begin{aligned} c_0 &= x_0, \\ c_1 &= x_1 - (p+q)c_0 \end{aligned} \quad (2)$$

and, for $k \geq 2$,

$$c_k = x_k - \sum_{i=0}^{k-1} (p^{k-i} + p^{k-i-1}q + \cdots + pq^{k-i-1} + q^{k-i})c_i, \quad (3)$$

which implies that $\{x_n\}$ can be represented by

$$\begin{aligned} x_0 &= c_0, \\ x_1 &= c_1 + (p+q)c_0, \end{aligned} \quad (4)$$

and, for $n \geq 2$,

$$\begin{aligned} x_n &= c_n + (p+q)c_{n-1} + (p^2 + pq + q^2)c_{n-2} + \cdots \\ &\quad + (p^n + p^{n-1}q + \cdots + pq^{n-1} + q^n)c_0 \\ &= c_n + \sum_{i=1}^n (p^i + p^{i-1}q + \cdots + pq^{i-1} + q^i)c_{n-i}. \end{aligned} \quad (5)$$

As a consequence, we get the following lemma. A variation of this lemma can be found in [6].

Lemma 1. If the sequence $\{x_n\}$ is given by representation (5), then $\Delta_{p,q}(x_n) = c_n$. Thus, the sequence $\{x_n\}$ is (p, q) -convex if and only if $c_n \geq 0$ for $n \geq 2$.

Proof. It suffices to show that $\Delta_{p,q}(x_n) = x_n - (p+q)x_{n-1} + pqx_{n-2} = c_n$ for $n \geq 2$. Using (5),

$$\begin{aligned} \Delta_{p,q}(x_n) &= \left(c_n + (p+q)c_{n-1} + (p^2 + pq + q^2)c_{n-2} \right. \\ &\quad + \cdots + \left(\sum_0^n p^{n-k} q^k \right) c_0 \Big) - (p+q) \left(c_{n-1} \right. \\ &\quad + (p+q)c_{n-2} + (p^2 + pq + q^2)c_{n-3} + \cdots \\ &\quad + \left(\sum_0^{n-1} p^{n-k-1} q^k \right) c_0 \Big) + pq \left(c_{n-2} + (p+q)c_{n-3} \right. \\ &\quad + (p^2 + pq + q^2)c_{n-4} + \cdots + \left. \left(\sum_0^{n-2} p^{n-k-2} q^k \right) c_0 \right). \end{aligned} \quad (6)$$

On the right side, we see that the coefficient of $c_n = 1$, and the coefficient of $c_{n-r} = 0$ for $r = 1, 2, \dots, n$. Thus,

$$\Delta_{p,q}(x_n) = c_n \quad \text{for } n \geq 2. \quad (7)$$

Hence, we have the previous lemma.

Also, in (5), the representation of x_n in terms of c_n can be written as follows:

$$\begin{aligned} x_n &= c_n + \sum_{i=0}^{n-1} (p^{n-i} + p^{n-i-1}q + \cdots + q^{n-i}) c_i \\ &= \begin{cases} c_n + \sum_{i=0}^{n-1} \left(\frac{p^{n-i+1} - q^{n-i+1}}{p-q} \right) c_i, & \text{if } p \neq q \\ c_n + \sum_{i=0}^{n-1} (n-i+1) p^{n-i} c_i, & \text{if } p = q \end{cases} \quad (8) \\ &= \begin{cases} \sum_{i=0}^n \left(\frac{p^{n-i+1} - q^{n-i+1}}{p-q} \right) c_i, & \text{if } p \neq q \\ \sum_{i=0}^n (n-i+1) p^{n-i} c_i & \text{if } p = q. \end{cases} \end{aligned}$$

Now, we give below some definitions. Let $A = [a_{n,k}]$ be a nonnegative infinite matrix defining a sequence to sequence transformation by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k. \quad (9)$$

Then, we define the matrices $[\alpha_{n,k}]$ and $[\beta_{n,k}]$ as

$$\begin{aligned} \alpha_{n,k} &= \sum_{j=k}^{\infty} p^{j-k} a_{n,j} = a_{n,k} + p a_{n,k+1} + p^2 a_{n,k+2} + \cdots, \\ \beta_{n,i} &= \sum_{k=i}^{\infty} q^{k-i} \alpha_{n,k} = \alpha_{n,i} + q \alpha_{n,i+1} + q^2 \alpha_{n,i+2} + \cdots \quad (10) \\ &= \sum_{k=i}^{\infty} q^{k-i} \left(\sum_{j=k}^{\infty} p^{j-k} a_{n,j} \right). \end{aligned}$$

Interchanging the order of summation, we get, for each $n = 0, 1, 2, \dots$, and $i = 0, 1, 2, \dots$,

$$\begin{aligned} \beta_{n,i} &= \sum_{j=i}^{\infty} \left(\sum_{k=i}^j q^{k-i} p^{j-k} \right) a_{n,j} \\ &= \sum_{j=i}^{\infty} (p^{j-i} + q p^{j-i-1} + q^2 p^{j-i-2} + \cdots + q^{j-i}) a_{n,j} \quad (11) \\ &= \begin{cases} \sum_{j=i}^{\infty} \left(\frac{p^{j-i+1} - q^{j-i+1}}{p-q} \right) a_{n,j}, & \text{if } p \neq q \\ \sum_{j=i}^{\infty} (j-i+1) p^{j-i} a_{n,j}, & \text{if } p = q. \end{cases} \end{aligned}$$

Furthermore, for $n \geq 2$,

$$\begin{aligned} \Delta_{p,q}(\beta_{n,i}) &= \beta_{n,i} - (p+q)\beta_{n-1,i} + pq\beta_{n-2,i} \\ &= \begin{cases} \sum_{j=i}^{\infty} \left(\frac{p^{j-i+1} - q^{j-i+1}}{p-q} \right) \Delta_{p,q}(a_{n,j}), & \text{if } p \neq q \\ \sum_{j=i}^{\infty} (j-i+1) p^{j-i} \Delta_{p,q}(a_{n,j}), & \text{if } p = q. \end{cases} \quad (12) \end{aligned}$$

In order for the matrix $[\beta_{n,i}]$ to be well-defined, we need the matrix $[a_{n,k}]$ to satisfy certain conditions which will depend on the values of p and q .

(I) When $p \neq q$, due to symmetry of p and q in the definition of $\beta_{n,i}$, it is sufficient to consider the following cases:

- (a) $0 < p, q < 1$
- (b) $0 < p < 1, q = 1$
- (c) $p > 1, q = 1$
- (d) $p > 1, 0 < q < 1$
- (e) $p, q > 1$

Case (a). For $0 < p, q < 1$, we require the matrix A to satisfy that, for each n ,

$$\sum_{k=1}^{\infty} k a_{n,k} < \infty. \quad (14)$$

Thus, using (11) and $p, q < 1$, we have

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty} (p^{j-i} + qp^{j-i-1} + \cdots + q^{j-i}) a_{n,j} \\ &< \sum_{j=i}^{\infty} (j-i+1) a_{n,j} = \sum_{j=i}^{\infty} (j-i) a_{n,j} + \sum_{j=i}^{\infty} a_{n,j} < \infty \quad (15) \\ &\quad \text{by (14).}\end{aligned}$$

Thus, $\beta_{n,i}$ is well-defined.

Case (b). For $0 < p < 1$, $q = 1$, we require the matrix A to satisfy that, for each n ,

$$\sum_{k=0}^{\infty} a_{n,k} < \infty. \quad (16)$$

Then using (11), we have

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty} \left(\frac{1-p^{j-i+1}}{1-p} \right) a_{n,j} \\ &= \frac{1}{1-p} \left((1-p) a_{n,i} + (1-p^2) a_{n,i+1} + \cdots \right) \\ &< \frac{1}{1-p} (a_{n,i} + a_{n,i+1} + \cdots), \text{ since } 0 < p < 1 \\ &< \infty \text{ by (16).}\end{aligned} \quad (17)$$

Thus, $\beta_{n,i}$ is well-defined.

For the cases (c), (d), and (e), we require the matrix A to satisfy that, for each n ,

$$\sum_{k=0}^{\infty} p^k a_{n,k} < \infty. \quad (18)$$

Case (c). When $p > 1$, $q = 1$, we have, as in the case (b),

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty} \left(\frac{p^{j-i+1} - 1}{p - 1} \right) a_{n,j} \\ &= \frac{p}{p-1} \sum_{j=i}^{\infty} \left(p^{j-i} - \frac{1}{p} \right) a_{n,j} < \frac{p}{p-1} \sum_{j=i}^{\infty} p^{j-i} a_{n,j} \quad (19) \\ &\leq \frac{p}{p-1} \sum_{j=i}^{\infty} p^j a_{n,j} < \infty \text{ by (18).}\end{aligned}$$

Thus, $\beta_{n,i}$ is well-defined.

Case (d). When $p > 1$, $0 < q < 1$, from (11),

$$\beta_{n,i} = \frac{1}{p-q} \sum_{j=i}^{\infty} (p^{j-i+1} - q^{j-i+1}) a_{n,j}. \quad (20)$$

Since $q < p$, using (18), we have $\sum_{j=i}^{\infty} q^{j-i} a_{n,j} < \sum_{j=i}^{\infty} p^{j-i} a_{n,j} < \infty$. Therefore,

$$\beta_{n,i} = \frac{p}{p-q} \sum_{j=i}^{\infty} p^{j-i} a_{n,j} - \frac{q}{p-q} \sum_{j=i}^{\infty} q^{j-i} a_{n,j} < \infty. \quad (21)$$

Thus $\beta_{n,i}$ is well-defined.

Case (e). When $p, q > 1$, we can assume without loss of generality that $p > q$.

Proceeding as in case (d), we see that $\beta_{n,i}$ is well-defined in this case also.

(II) When $p = q$, we consider the following cases:

- (f) $0 < p < 1$
- (g) $p = 1$ (22)
- (h) $p > 1$.

Case (f). For $0 < p < 1$, we require the matrix A to satisfy that, for each n ,

$$\sum_{k=1}^{\infty} k a_{n,k} < \infty. \quad (23)$$

Then, using (11), we have

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty} (j-i+1) p^{j-i} a_{n,j} < \sum_{j=i}^{\infty} (j-i+1) a_{n,j} \\ &= \sum_{j=i}^{\infty} (j-i) a_{n,j} + \sum_{j=i}^{\infty} a_{n,j} < \infty \text{ by (23).}\end{aligned} \quad (24)$$

Thus, $\beta_{n,i}$ is well-defined.

Case (g). When $p = 1$, $\Delta_{p,q}$ -convexity reduces to the well-known second-order convexity Δ^2 , which has been investigated in detail in [3].

Case (h). For $p > 1$, we require the matrix A to satisfy that, for each n ,

$$\sum_{k=1}^{\infty} k p^k a_{n,k} < \infty. \quad (25)$$

Then, using (11), we have

$$\begin{aligned}\beta_{n,i} &= \sum_{j=i}^{\infty} (j-i+1) p^{j-i} a_{n,j} \\ &\leq \sum_{j=i}^{\infty} (j-i) p^{j-i} a_{n,j} + \sum_{j=i}^{\infty} p^j a_{n,j} < \infty \text{ by (25).}\end{aligned} \quad (26)$$

Thus, $\beta_{n,i}$ is well-defined. □

3. Main Results

In this section, we prove the necessary and sufficient conditions for a nonnegative infinite matrix A to transform a (p, q) -convex sequence into a (p, q) -convex sequence showing that each column of the corresponding matrix $[\beta_{n,k}]$ is a (p, q) -convex sequence.

First, we consider the values of p and q , where $p \neq q$ results in the cases listed in (13).

Theorem 2. For $p \neq q$, a nonnegative infinite matrix A satisfying (14), (16), or (18), corresponding to the cases listed in (13), preserves (p, q) -convexity of sequences if and only if, for $n = 2, 3, 4, \dots$,

- (i) $\Delta_{p,q}(\beta_{n,0}) = 0$
- (ii) $\Delta_{p,q}(\beta_{n,1}) = 0$
- (iii) $\Delta_{p,q}(\beta_{n,i}) \geq 0$ for $i \geq 2$

where the matrix $[\beta_{n,i}]$ is defined by

$$\beta_{n,i} = \sum_{j=i}^{\infty} \left(\frac{p^{j-i+1} - q^{j-i+1}}{p - q} \right) a_{n,j}. \quad (27)$$

Proof. First, we prove a result on the transformed sequence of any (p, q) -convex sequence $\{x_n\}$. Now, we have, from (8),

$$x_n = \sum_{i=0}^n \left(\frac{p^{n-i+1} - q^{n-i+1}}{p - q} \right) c_i, \quad (28)$$

where $c_i \geq 0$ for $i \geq 2$ by Lemma 1. Then, the n th term of the transformed sequence is

$$(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k = \sum_{k=0}^{\infty} a_{n,k} \sum_{i=0}^k \left(\frac{p^{k-i+1} - q^{k-i+1}}{p - q} \right) c_i. \quad (29)$$

Interchanging the order of summation,

$$\begin{aligned} (Ax)_n &= \sum_{i=0}^{\infty} c_i \sum_{k=i}^{\infty} \frac{p^{k-i+1} - q^{k-i+1}}{p - q} a_{n,k} \\ &= c_0 \sum_{k=0}^{\infty} \frac{p^{k+1} - q^{k+1}}{p - q} a_{n,k} + c_1 \sum_{k=1}^{\infty} \frac{p^k - q^k}{p - q} a_{n,k} \\ &\quad + \sum_{i=2}^{\infty} c_i \sum_{k=i}^{\infty} \frac{p^{k-i+1} - q^{k-i+1}}{p - q} a_{n,k}. \end{aligned} \quad (30)$$

From (11), we have

$$(Ax)_n = c_0 \beta_{n,0} + c_1 \beta_{n,1} + \sum_{i=2}^{\infty} c_i \beta_{n,i}. \quad (31)$$

Then, for $n \geq 2$,

$$\begin{aligned} \Delta_{p,q}(Ax)_n &= (Ax)_n - (p + q)(Ax)_{n-1} + pq(Ax)_{n-2} \\ &= \left(c_0 \beta_{n,0} + c_1 \beta_{n,1} + \sum_{i=2}^{\infty} c_i \beta_{n,i} \right) \\ &\quad - (p + q) \left(c_0 \beta_{n-1,0} + c_1 \beta_{n-1,1} + \sum_{i=2}^{\infty} c_i \beta_{n-1,i} \right) \\ &\quad + pq \left(c_0 \beta_{n-2,0} + c_1 \beta_{n-2,1} + \sum_{i=2}^{\infty} c_i \beta_{n-2,i} \right) \\ &= c_0 [\beta_{n,0} - (p + q) \beta_{n-1,0} + pq \beta_{n-2,0}] \\ &\quad + c_1 [\beta_{n,1} - (p + q) \beta_{n-1,1} + pq \beta_{n-2,1}] \\ &\quad + \sum_{i=2}^{\infty} c_i [\beta_{n,i} - (p + q) \beta_{n-1,i} + pq \beta_{n-2,i}]. \end{aligned} \quad (32)$$

Thus, for any (p, q) -convex sequence $\{x_n\}$,

$$\begin{aligned} \Delta_{p,q}(Ax)_n &= c_0 \Delta_{p,q}(\beta_{n,0}) + c_1 \Delta_{p,q}(\beta_{n,1}) \\ &\quad + \sum_{i=2}^{\infty} c_i \Delta_{p,q}(\beta_{n,i}). \end{aligned} \quad (33)$$

Now, to prove the sufficiency of the conditions given in the theorem, assume that (i), (ii), and (iii) are true. Then, by (33),

$$\Delta_{p,q}(Ax)_n \geq 0. \quad (34)$$

Thus, the sequence $(Ax)_n$ is also (p, q) -convex.

Conversely, assume that the matrix A preserves (p, q) -convexity of the sequences. Suppose that the condition (i) fails to hold. Then there exists an integer $N \geq 2$ such that

$$\Delta_{p,q}(\beta_{N,0}) = L \neq 0. \quad (35)$$

Consider the following sequence:

$$u = \left\{ -L, \frac{-(p^2 - q^2)}{p - q} L, \frac{-(p^3 - q^3)}{p - q} L, \dots \right\}. \quad (36)$$

Then $\{u_n\}$ is a (p, q) -convex sequence because, using (2) and Lemma 1,

$$\begin{aligned} c_0 &= u_0 = -L, \\ c_1 &= u_1 - (p + q)c_0 = 0 \end{aligned} \quad (37)$$

and, for $i \geq 2$,

$$\begin{aligned} c_i &= \Delta_{p,q}(u_i) = u_i - (p + q)u_{i-1} + pq u_{i-2} \\ &= \frac{-(p^{i+1} - q^{i+1})}{p - q} L + (p + q) \frac{(p^i - q^i)}{p - q} L \\ &\quad - pq \frac{(p^{i-1} - q^{i-1})}{p - q} L = 0. \end{aligned} \quad (38)$$

$$\begin{aligned} \Delta_{p,q}(Au)_N &= c_0 \Delta_{p,q}(\beta_{N,0}) + c_1 \Delta_{p,q}(\beta_{N,1}) \\ &\quad + \sum_{i=2}^{\infty} c_i \Delta_{p,q}(\beta_{N,i}) = -L^2 < 0, \end{aligned} \quad (39)$$
$$v = \left\{ 0, -L, \frac{-(p^2 - q^2)}{p - q}L, \frac{-(p^3 - q^3)}{p - q}L, \dots \right\}, \quad (40)$$
$$\begin{aligned} c_0 &= 0, \\ c_1 &= -L, \\ c_i &= 0 \quad \text{for } i \geq 2. \end{aligned} \tag{41}$$
$$\Delta_{p,q}(\beta_{N,i}) = L < 0. \quad (42)$$
$$x = \left\{ 0, \dots, 0, 1, \frac{p^2 - q^2}{p - q}, \frac{p^3 - q^3}{p - q}, \dots \right\}. \quad (43)$$
$$\begin{aligned} c_i &= 0 \quad \text{for } 0 \leq i \leq j-1; \\ c_j &= 1; \end{aligned} \tag{44}$$

$$c_i = \Delta_{p,q}(x_i) = 0 \quad \text{as in (38)}. \quad (45)$$

$$\begin{aligned} \Delta_{p,q}(Ax)_N &= c_0 \Delta_{p,q}(\beta_{N,0}) + c_1 \Delta_{p,q}(\beta_{N,1}) \\ &+ \sum_{i=2}^{\infty} c_i \Delta_{p,q}(\beta_{N,i}) = c_j \Delta_{p,q}(\beta_{N,j}) = L \quad (46) \\ &< 0, \end{aligned}$$
$$\beta_{n,i} = \sum_{j=i}^{\infty} (j-i+1) p^{j-i} a_{n,j}. \quad (47)$$
$$x_n = \sum_{i=0}^n (n-i+1) p^{n-i} c_i, \quad (48)$$
$$(Ax)_n = \sum_{k=0}^{\infty} a_{n,k} x_k = \sum_{k=0}^{\infty} a_{n,k} \left(\sum_{i=0}^k (k-i+1) p^{k-i} c_i \right). \quad (49)$$
$$\begin{aligned} (Ax)_n &= \sum_{i=0}^{\infty} c_i \sum_{k=i}^{\infty} (k-i+1) p^{k-i} a_{n,k} \\ &= c_0 \sum_{k=0}^{\infty} (k+1) p^k a_{n,k} + c_1 \sum_{k=1}^{\infty} k p^{k-1} a_{n,k} \quad (50) \\ &\quad + \sum_{i=2}^{\infty} c_i \sum_{k=i}^{\infty} (k-i+1) p^{k-i} a_{n,k}. \end{aligned}$$
$$(Ax)_n = c_0\beta_{n,0} + c_1\beta_{n,1} + \sum_{i=2}^{\infty} c_i\beta_{n,i}. \quad (51)$$
$$\begin{aligned} \Delta_{p,p}(Ax)_n &= (Ax)_n - 2p(Ax)_{n-1} + p^2(Ax)_{n-2} \\ &= \left(c_0\beta_{n,0} + c_1\beta_{n,1} + \sum_{i=2}^{\infty} c_i\beta_{n,i} \right) \\ &\quad - 2p \left(c_0\beta_{n-1,0} + c_1\beta_{n-1,1} + \sum_{i=2}^{\infty} c_i\beta_{n-1,i} \right) \\ &\quad + p^2 \left(c_0\beta_{n-2,0} + c_1\beta_{n-2,1} + \sum_{i=2}^{\infty} c_i\beta_{n-2,i} \right). \end{aligned} \quad (52)$$

Thus, for any (p, p) -convex sequence $\{x_n\}$,

$$\begin{aligned} \Delta_{p,p}(Ax)_n &= c_0 \Delta_{p,p}(\beta_{n,0}) + c_1 \Delta_{p,p}(\beta_{n,1}) \\ &\quad + \sum_{i=2}^{\infty} c_i \Delta_{p,p}(\beta_{n,i}). \end{aligned} \quad (53)$$

Now, to prove the sufficiency of the conditions given in the theorem, assume that (i), (ii), and (iii) are true. Then by (53),

$$\Delta_{p,p}(Ax)_n \geq 0. \quad (54)$$

Thus, the sequence $(Ax)_n$ is also (p, p) -convex.

Conversely, assume that the matrix A preserves (p, p) -convexity of sequences.

Suppose that the condition (i) fails to hold. Then there exists an integer $N \geq 2$ such that

$$\Delta_{p,p}(\beta_{N,0}) = L \neq 0. \quad (55)$$

Consider the following sequence:

$$u = \{-L, -2pL, -3p^2L, \dots\}. \quad (56)$$

It is easy to see, using (2) and Lemma 1, that u is a (p, p) -convex sequence with

$$\begin{aligned} c_0 &= u_0 = -L, \\ c_i &= 0 \quad \text{for } i \geq 1. \end{aligned} \quad (57)$$

Thus, from (53), for the transformed sequence $\{(Au)_n\}$,

$$\begin{aligned} \Delta_{p,p}(Au)_N &= c_0 \Delta_{p,p}(\beta_{N,0}) + c_1 \Delta_{p,p}(\beta_{N,1}) \\ &\quad + \sum_{i=2}^{\infty} c_i \Delta_{p,p}(\beta_{N,i}) = -L^2 < 0, \end{aligned} \quad (58)$$

which contradicts that $\{(Au)_n\}$ must be (p, p) -convex.

Next, suppose that the condition (ii) is not true. This case can be settled by a similar argument by considering the following sequence:

$$v = \{0, -L, -2pL, -3p^2L, \dots\}, \quad (59)$$

which implies that

$$\begin{aligned} c_0 &= 0, \\ c_1 &= -L, \\ c_i &= 0 \quad \text{for } i \geq 2. \end{aligned} \quad (60)$$

Now, suppose that the condition (iii) is not true. Then there exists an integer $j \geq 2$ such that the j th column-sequence of the matrix $[\beta_{n,k}]$ is not (p, p) -convex. That is, for some $N \geq 2$,

$$\Delta_{p,p}(\beta_{N,j}) = L < 0. \quad (61)$$

Consider the (p, p) -convex sequence:

$$\begin{aligned} x &= \{0, \dots, 0, 1, 2p, 3p^2, \dots\}. \\ &\quad \downarrow \quad \downarrow \downarrow \downarrow \\ &\quad x_0 \quad x_{j-1} x_j x_{j+1} \end{aligned} \quad (62)$$

We see that, as in the proof of Theorem 2,

$$\Delta_{p,p}(Ax)_N = L < 0, \quad (63)$$

which contradicts that $\{Ax\}$ is a (p, p) -convex sequence.

We see that the result on the convexity of sequences given in [3, p. 331] is a particular case of Theorem 3 when $p = q = 1$. Also, this theorem generalizes the necessary and sufficient conditions for a triangular matrix given in [9, p. 4]. $\square$

4. Examples

We give below examples of (p, q) -convexity preserving matrices for each of the cases (a) through (h) given in (13) and (22).

Example for Case (a). Considering $0 < p, q < 1$, and $p \neq q$, we can assume, without loss of generality, that $p < q$. Let the matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = \begin{cases} p^n, & \text{if } k = 0, \\ \frac{p^n q^k}{k}, & \text{if } k \geq 1. \end{cases} \quad (64)$$

Then, for each n ,

$$\sum_{k=1}^{\infty} k a_{n,k} = \sum_{k=1}^{\infty} p^n q^k = p^n \left(\frac{q}{1-q} \right) < \infty. \quad (65)$$

Thus, by (14), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots$. The matrix A satisfies the three conditions of Theorem 2 because, for $n \geq 2$, using (12),

$$\Delta_{p,q}(\beta_{n,i}) = \sum_{j=i}^{\infty} \left(\frac{p^{j-i+1} - q^{j-i+1}}{p - q} \right) \Delta_{p,q}(a_{n,j}), \quad (66)$$

in which

$$\begin{aligned} \Delta_{p,q}(a_{n,j}) &= a_{n,j} - (p+q)a_{n-1,j} + pqa_{n-2,j} \\ &= \begin{cases} p^n - (p+q)p^{n-1} + pqp^{n-2}, & \text{if } j = 0, \\ \frac{q^j}{j} (p^n - (p+q)p^{n-1} + pqp^{n-2}), & \text{if } j \geq 1 \end{cases} \\ &= 0. \end{aligned} \quad (67)$$

Therefore, the matrix A preserves (p, q) -convexity of sequences.

Example for Case (b). Considering $0 < p < 1$, $q = 1$, let the matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = p^k. \quad (68)$$

Then, for each n ,

$$\sum_{k=0}^{\infty} a_{n,k} = \sum_{k=0}^{\infty} p^k = \frac{1}{1-p} < \infty. \quad (69)$$

Thus, by (16), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots$. The matrix A satisfies the three conditions of Theorem 2 because, for $n \geq 2$, using (12),

$$\Delta_{p,1}(\beta_{n,i}) = \sum_{j=i}^{\infty} \left(\frac{p^{j-i+1} - 1}{p-1} \right) \Delta_{p,1}(a_{n,j}), \quad (70)$$

in which

$$\begin{aligned} \Delta_{p,1}(a_{n,j}) &= a_{n,j} - (p+1)a_{n-1,j} + pa_{n-2,j} \\ &= p^j - (p+1)p^j + p^{j+1} = 0. \end{aligned} \quad (71)$$

Therefore, the matrix A preserves $(p, 1)$ -convexity of sequences.

Example for Case (c). Considering $p > 1$, $q = 1$, let matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = \frac{1}{p^{2k}}. \quad (72)$$

Then, for each n ,

$$\sum_{k=0}^{\infty} p^k a_{n,k} = \sum_{k=0}^{\infty} \frac{1}{p^k} = \frac{1}{1-1/p} < \infty. \quad (73)$$

Thus, by (18), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots$. The matrix A satisfies the three conditions of Theorem 2 because, for $n \geq 2$, as in the previous example (b),

$$\begin{aligned} \Delta_{p,1}(a_{n,j}) &= a_{n,j} - (p+1)a_{n-1,j} + pa_{n-2,j} \\ &= \frac{1}{p^{2j}} - (p+1)\frac{1}{p^{2j}} + p\frac{1}{p^{2j}} = 0. \end{aligned} \quad (74)$$

Therefore, the matrix A preserves $(p, 1)$ -convexity of sequences.

Example for Case (d). Considering $p > 1$, $0 < q < 1$, let matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = p^{n-2k}. \quad (75)$$

Then, for each n ,

$$\sum_{k=0}^{\infty} p^k a_{n,k} = \sum_{k=0}^{\infty} p^{n-k} = \frac{p^{n+1}}{p-1} < \infty. \quad (76)$$

Thus, by (18), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots$. The matrix A satisfies the three conditions of Theorem 2 because, for $n \geq 2$, using (12),

$$\Delta_{p,q}(\beta_{n,i}) = \sum_{j=i}^{\infty} \left(\frac{p^{j-i+1} - q^{j-i+1}}{p-q} \right) \Delta_{p,q}(a_{n,j}), \quad (77)$$

in which

$$\begin{aligned} \Delta_{p,q}(a_{n,j}) &= a_{n,j} - (p+q)a_{n-1,j} + pqa_{n-2,j} \\ &= \frac{1}{p^{2j}} (p^n - (p+q)p^{n-1} + pqp^{n-2}) = 0. \end{aligned} \quad (78)$$

Therefore, the matrix A preserves (p, q) -convexity of sequences.

Example for Case (e). Considering $p, q > 1$ and $p \neq q$, we can assume, without loss of generality, that $p > q$. Let the matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = p^{n-2k} q^k. \quad (79)$$

Then, for each n ,

$$\sum_{k=0}^{\infty} p^k a_{n,k} = p^n \sum_{k=0}^{\infty} \left(\frac{q}{p} \right)^k = \frac{p^{n+1}}{p-q} < \infty. \quad (80)$$

Thus, by (18), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots$. The matrix A satisfies the three conditions of Theorem 2 because, for $n \geq 2$, as in the previous example (d),

$$\begin{aligned} \Delta_{p,q}(a_{n,j}) &= a_{n,j} - (p+q)a_{n-1,j} + pqa_{n-2,j} \\ &= \frac{q^j}{p^{2j}} (p^n - (p+q)p^{n-1} + pqp^{n-2}) = 0. \end{aligned} \quad (81)$$

Therefore, the matrix A preserves (p, q) -convexity of sequences.

Example for Case (f). Considering $0 < p = q < 1$, let the matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = \begin{cases} p^n, & \text{if } k = 0, \\ \frac{p^{n+k}}{k}, & \text{if } k \geq 1. \end{cases} \quad (82)$$

Then, for each n ,

$$\sum_{k=1}^{\infty} ka_{n,k} = \sum_{k=1}^{\infty} p^{n+k} = p^n \sum_{k=1}^{\infty} p^k = p^{n+1} \left(\frac{1}{1-p} \right) < \infty. \quad (83)$$

Thus, by (23), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots$. The matrix A satisfies the three conditions of Theorem 3 because, for $n \geq 2$, using (12),

$$\Delta_{p,p}(\beta_{n,i}) = \sum_{j=i}^{\infty} (j-i+1) p^{j-i} \Delta_{p,p}(a_{n,j}), \quad (84)$$

in which

$$\begin{aligned} \Delta_{p,p}(a_{n,j}) &= a_{n,j} - 2pa_{n-1,j} + p^2a_{n-2,j} \\ &= \begin{cases} p^n - 2pp^{n-1} + p^2p^{n-2}, & \text{if } j = 0, \\ \frac{p^j}{j} (p^n - 2pp^{n-1} + p^2p^{n-2}), & \text{if } j \geq 1 \end{cases} = 0. \end{aligned} \quad (85)$$

Therefore, the matrix A preserves (p, p) -convexity of sequences.

Examples for Case (g). They can be found in [3], since $\Delta_{1,1}$ is the same as the second-order convexity Δ^2 .

Example for Case (h). Considering $p = q > 1$, let the matrix $A = [a_{n,k}]$ be defined by

$$a_{n,k} = \begin{cases} p^n(n+2), & \text{if } k = 0, \\ \frac{p^{n-2k}(n+2)}{k}, & \text{if } k \geq 1. \end{cases} \quad (86)$$

Therefore, for each n ,

$$\begin{aligned} \sum_{k=1}^{\infty} k p^k a_{n,k} &= \sum_{k=1}^{\infty} p^{n-k} (n+2) = p^n (n+2) \sum_{k=1}^{\infty} \left(\frac{1}{p}\right)^k \\ &= (n+2) \frac{p^n}{p-1} < \infty. \end{aligned} \quad (87)$$

Thus, by (23), $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots$. The matrix A satisfies the three conditions of Theorem 3 because, for $n \geq 2$, using (12),

$$\Delta_{p,p}(\beta_{n,i}) = \sum_{j=i}^{\infty} (j-i+1) p^{j-i} \Delta_{p,p}(a_{n,j}), \quad (88)$$

in which

$$\begin{aligned} \Delta_{p,p}(a_{n,j}) &= a_{n,j} - 2pa_{n-1,j} + p^2 a_{n-2,j} \\ &= \begin{cases} p^n(n+2) - 2p^n(n+1) + p^n n & \text{if } j = 0, \\ \frac{(n+2)p^{n-2j}}{j} - 2\frac{(n+1)p^{n-2j}}{j} + \frac{np^{n-2j}}{j}, & \text{if } j \geq 1. \end{cases} \quad (89) \\ &= 0. \end{aligned}$$

Therefore, the matrix A preserves the convexity of sequences.

We conclude this paper by giving an example of an infinite matrix which does not preserve (p, q) -convexity of sequences.

It is interesting to notice that the Borel matrix preserves the $(1, 1)$ -convexity of sequences [3, p. 336], but it does not preserve (p, p) -convexity when $p \neq 1$.

The Borel matrix $B = [b_{n,k}]$ is defined by

$$b_{n,k} = \frac{n^k}{e^n k!}. \quad (90)$$

Then, for each n ,

$$\sum_{k=1}^{\infty} k b_{n,k} = \frac{n}{e^n} \sum_{k=1}^{\infty} \frac{n^{k-1}}{(k-1)!} = n < \infty, \quad (91)$$

$$\sum_{k=1}^{\infty} k p^k b_{n,k} = \frac{(np)}{e^n} \sum_{k=1}^{\infty} \frac{(np)^{k-1}}{(k-1)!} = \frac{(np)}{e^n} e^{np} < \infty. \quad (92)$$

Thus, for each of the cases, $0 < p < 1$ and $p > 1$, we see that (23) and (25) are satisfied and hence $\beta_{n,i}$ is well-defined for $n = 0, 1, 2, \dots$ and $i = 0, 1, 2, \dots$.

From (11),

$$\beta_{n,i} = \sum_{j=i}^{\infty} (j-i+1) p^{j-i} b_{n,j}. \quad (93)$$

Therefore,

$$\begin{aligned} \beta_{n,0} &= \sum_{j=0}^{\infty} (j+1) p^j \frac{n^j}{e^n j!} \\ &= \frac{1}{e^n} \left(\sum_{j=0}^{\infty} \frac{j(pn)^j}{j!} + \sum_{j=0}^{\infty} \frac{(pn)^j}{j!} \right) \\ &= \frac{1}{e^n} (pne^{pn} + e^{pn}) = e^{n(p-1)} (pn+1), \end{aligned} \quad (94)$$

which implies that

$$\begin{aligned} \Delta_{p,p}(\beta_{n,0}) &= \beta_{n,0} - 2p\beta_{n-1,0} + p^2\beta_{n-2,0} \\ &= e^{n(p-1)} (pn+1) - 2pe^{(n-1)(p-1)} (p(n-1)+1) \\ &\quad + p^2 e^{(n-2)(p-1)} (p(n-2)+1) \\ &= \frac{e^{n(p-1)}}{e^{2p}} \left((pn+1)(e^p - pe)^2 + 2p^2 e(e^p - pe) \right) \\ &> 0, \end{aligned} \quad (95)$$

since $e^p - pe > 0$ when $p \neq 1$. Thus, the condition (i) of Theorem 3 fails in the case of Borel matrix.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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