

Research Article

On the Convex and Convex-Concave Solutions of Opposing Mixed Convection Boundary Layer Flow in a Porous Medium

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In this paper, we are concerned with the solution of the third-order nonlinear differential equation $f''' + ff'' + \beta f'(f' - 1) = 0$, satisfying the boundary conditions $f(0) = a \in \mathbb{R}$, $f'(0) = b < 0$, and $f'(t) \rightarrow \lambda$, as $t \rightarrow +\infty$, where $\lambda \in \{0, 1\}$ and $0 < \beta < 1$. The problem arises in the study of the opposing mixed convection approximation in a porous medium. We prove the existence, nonexistence, and the sign of convex and convex-concave solutions of the problem above according to the mixed convection parameter $b < 0$ and the temperature parameter $0 < \beta < 1$.

1. Introduction

Owing to their numerous applications in industrial manufacturing processes, the convection phenomena about heated or cooled surfaces embedded in fluid-saturated porous media have attracted considerable attention during the last few decades. In this paper, our interest focuses on the analysis of the boundary value problems $\mathcal{P}_{\lambda(a,b)}$

$$\begin{aligned} f''' + ff'' + \beta f'(f' - 1) &= 0 \\ f(0) &= a, \quad a \in \mathbb{R} \\ f'(0) &= b < 0 \\ f'(t) &\rightarrow \lambda \quad \text{as } t \rightarrow +\infty \end{aligned} \quad \mathcal{P}_{\lambda(a,b)}$$

where $\lambda \in \{0, 1\}$. This problem derives from the study of mixed convection boundary layer near a semi-infinite vertical plate embedded in a saturated porous medium, with a prescribed power law of the distance from the leading edge for the temperature. The parameter β is a temperature power-law profile and b is the mixed convection parameter, namely, $b = R_a/Pe - 1$, with R_a the Rayleigh number and Pe the Péclet

number. The interested reader can consult references [1, 2] for more details on the physical derivation and the numerical treatments.

Mathematical results about the problem $\mathcal{P}_{\lambda(a,b)}$ with $\lambda = 1$ can be found in [3–7]. The case where $a \geq 0$, $b \geq 0$, $\beta > 0$ and $\lambda \in \{0, 1\}$ was treated by Aïboudi and al. in [3], and the results obtained generalize the ones of [6]. In [4], Brighi and Hoernel established some results about the existence and uniqueness of convex and concave solution of $\mathcal{P}_{1(a,b)}$ where $-2 < \beta < 0$ and $b > 0$. These results can be recovered from [8], where the general equation $f''' + ff'' + \mathbf{g}(f') = 0$ is studied.

In [5], some theoretical results can be found about the problem $\mathcal{P}_{1(0,b)}$ with $-2 < \beta < 0$, $b = 1 + \varepsilon$, and $\varepsilon < -1$. In particular, the authors prove that there exist $\varepsilon_* \in (-1.807, -1.806)$ and $\varepsilon^* \in (-1.193, -1.192)$, such that

- (i) $\mathcal{P}_{1(0,b)}$ has no convex solution for any $\beta < 0$ and each $\varepsilon \in \varepsilon_*$.
- (ii) $\mathcal{P}_{1(0,b)}$ has a convex solution for each $\beta < 0$ and each $\varepsilon \in [\varepsilon^*, -1)$.

In [7] one can find an interesting new result about the existence of convex solutions of $\mathcal{P}_{1(0,b)}$ where $0 < \beta <$

1 under some conditions. In [5, 7], the method used by the authors allows them to prove the existence of a convex solution for the case $a = 0$ and seems difficult to generalize for $a \neq 0$.

The problem $\mathcal{P}_{\lambda(a,b)}$ with $\beta = 0$ is the well known Blasius problem. For a broad view, see [9]. See also [10].

Great interest is given to analytical studies of similarity solutions because of their applications in different fields, for example, in magnetohydrodynamic (see [11–13]) or in boundary layer flows (see [8, 14]).

The main goal of this paper is to study the question of existence and nonexistence of the solutions of $\mathcal{P}_{\lambda(a,b)}$ with $0 < \beta < 1$ and $\lambda \in \{0, 1\}$. We will focus our attention on convex and convex-concave solutions of the equation

$$f''' + ff'' + \beta f'(f' - 1) = 0. \tag{1}$$

As usual, to get a convex or convex-concave solution of $\mathcal{P}_{\lambda(a,b)}$, we use the shooting technique which consists of finding the values of a parameter $c \geq 0$ for which the solution of (1) satisfying the initial conditions $f(0) = a$, $f'(0) = b$, and $f''(0) = c$ exists on $[0, +\infty)$ and is such that $f'(t) \rightarrow \lambda$ as $t \rightarrow +\infty$. We denote by f_c the solution of the following initial value problem and by $[0, T_c)$ the right maximal interval of existence:

$$\begin{aligned} f''' + ff'' + \beta f'(f' - 1) &= 0 \\ f(0) &= a \\ f'(0) &= b < 0 \\ f''(0) &= c \geq 0 \end{aligned} \tag{2}$$

2. On Blasius Equation

In this section, we recall some basic properties of the subsolutions and ε -subsolutions of the Blasius equation. Let $I \subset \mathbb{R}$ be an interval and $f : I \rightarrow \mathbb{R}$ be a function.

Definition 1. We say that f is a subsolution of the Blasius equation $f''' + ff'' = 0$ if f is of class C^3 and if $f''' + ff'' \leq 0$ on I .

Definition 2 (let $\varepsilon > 0$). We say that f is an ε -subsolution of the Blasius equation $f''' + ff'' = 0$ if f is of class C^3 and if $f''' + ff'' \leq -\varepsilon$ on I .

Proposition 3 (let $t_0 \in \mathbb{R}$). *There does not exist nonpositive concave subsolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. See [8], Proposition 2.11. □

Proposition 4 (let $\varepsilon > 0$ and $t_0 \in \mathbb{R}$). *There does not exist any ε -subsolution of the Blasius equation on the interval $[t_0, +\infty)$.*

Proof. See [8], Proposition 2.18. □

3. Preliminary Results

Proposition 5. *Let f be a solution of (1) on some maximal interval $I = (T_-, T_+)$.*

- (1) *If F is any antiderivative of f on I , then $(f''e^F)' = -\beta f'(f' - 1)e^F$.*
- (2) *Assume that $T_+ = +\infty$ and that $f'(t) \rightarrow \lambda \in \mathbb{R}$ as $t \rightarrow +\infty$. If moreover f is of constant sign at infinity, then $f''(t) \rightarrow 0$ as $t \rightarrow +\infty$.*
- (3) *If $T_+ = +\infty$ and if $f'(t) \rightarrow \lambda \in \mathbb{R}$ as $t \rightarrow +\infty$, then $\lambda = 0$ or $\lambda = 1$.*
- (4) *If $T_+ < +\infty$, then f'' and f' are unbounded near T_+ .*
- (5) *If there exists a point $t_0 \in I$ satisfying $f''(t_0) = 0$ and $f'(t_0) = \mu$, where $\mu = 0$ or 1 , then, for all $t \in I$, we have $f(t) = \mu(t - t_0) + f(t_0)$.*

Proof. The first item follows immediately from (1). For the proof of items (2)-(5), see [8], Proposition 3.1 with $g(x) = \beta x(x - 1)$. □

Lemma 6. *Let $\beta \in (0, 1]$ and f be a solution of (1) on some maximal interval $I = (T_-, T_+)$. If there exists $t_0 \in I$ such that*

$$f'(t_0) > 1 \text{ and} \tag{2}$$

$$f(t_0)(1 - f'(t_0)) \leq f''(t_0) \leq 0,$$

then $T_+ = +\infty$ and $f'(t) \rightarrow 1$ as $t \rightarrow +\infty$. Moreover, $f'' < 0$ on $[t_0, +\infty)$.

Proof. See [3], Lemma 9. □

4. The Boundary Value Problem in the Convex and Convex-Concave Case with $0 < \beta < 1$

In the following, we take $a, b \in \mathbb{R}$ and $\lambda \in \{0, 1\}$ with $b < 0$ and $0 < \beta < 1$. We are interested here in convex and convex-concave solutions of the boundary value problem $\mathcal{P}_{\lambda(a,b)}$. As mentioned in the introduction, we will use the shooting method to find these solutions. Define the following sets:

$$\begin{aligned} C_1 &= \{c \geq 0 : f'_c \leq 0 \text{ and } f''_c \geq 0 \text{ on } [0, T_c)\}, \\ C_2 &= \{c \geq 0 : \exists t_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t. } f'_c < 0 \text{ on } (0, t_c), f'_c > 0 \text{ on } (t_c, t_c + \varepsilon_c) \text{ and } f''_c > 0 \text{ on } (0, t_c + \varepsilon_c)\}, \\ C_3 &= \{c \geq 0 : \exists s_c \in [0, T_c), \exists \varepsilon_c > 0 \text{ s.t. } f''_c > 0 \text{ on } (0, s_c), f''_c < 0 \text{ on } (s_c, s_c + \varepsilon_c) \text{ and } f'_c < 0 \text{ on } (0, s_c + \varepsilon_c)\}. \end{aligned} \tag{3}$$

Remark 7. It is easy to prove that C_2 and C_3 are disjoint nonempty open subsets of $[0, +\infty)$ and that there exist $c_0 >$

$c_* > 0$ such that $C_2 = (c_0, +\infty)$, $C_3 = [0, c_*)$, and $C_1 \cup C_2 \cup C_3 = [0, +\infty)$ (see Appendix A of [8] with $g(x) = \beta x(x - 1)$ and $\beta > 0$).

Lemma 8 (let $\beta > 0$). *Then, f_c is a convex solution of the boundary value problem $\mathcal{P}_{0(a,b)}$ if and only if $c \in C_1$.*

Proof. See Appendix A of [8] with $g(x) = \beta x(x - 1)$ and $\beta > 0$. □

Lemma 9 (let $\beta > 0$). *If $c \in C_3$, then $T_c < +\infty$. Moreover, f_c is convex-concave, decreasing and $f'_c(t) \rightarrow -\infty$ as $t \rightarrow T_c$.*

Proof. If $c \in C_3$ then there exists $s_c \in [0, T_c)$ such that $f'_c(s_c) < 0$ and $f''_c(s_c) = 0$. From Proposition 5, items (1) and (3), we have $f''_c(t) < 0$ and $f'_c(t) < 0$ for all $t \in (s_c, T_c)$, and $f'_c(T) \rightarrow -\infty$ as $t \rightarrow T_c$. Thus, f_c is convex-concave solution on $[0, T_c)$ and $f'_c(t) \rightarrow -\infty$ as $t \rightarrow T_c$.

Let us assume that $T_c = +\infty$; then there exists $t_0 \in (s_c, +\infty)$ such that f'_c and f_c are negative on $t \in (t_0, +\infty)$ and we obtain $f'''_c + f_c f''_c = -\beta f'_c(f'_c - 1) < 0$ on $(t_0, +\infty)$. Hence, f_c is a nonpositive concave subsolution of the Blasius equation on $(t_0, +\infty)$. This contradicts the Proposition 3 and thus $T_c < +\infty$. □

Remark 10. From Proposition 5, items (1), (3), and (5), if $c \in C_2$, then there are only three possibilities for the solution of the initial value problem $\mathcal{P}_{(a,b,c)}$:

- (1) f_c is convex and $f'_c(t) \rightarrow +\infty$ as $t \rightarrow T_c$ (with $T_c \leq +\infty$).
- (2) There exists a point $t_0 \in [0, T_c)$ such that $f''_c(t_0) = 0$ and $f'_c(t_0) > 1$.
- (3) f_c is a convex solution of $\mathcal{P}_{1(a,b)}$.

The next proposition shows that case (1) cannot hold.

Proposition 11 (let $\beta > 0$). *There does not exist $c \geq 0$, such that f_c is convex on its right maximal interval of existence $[0, T_c)$ and $f'_c(t) \rightarrow +\infty$ as $t \rightarrow T_c$.*

Proof. Assume that f_c is convex on its right maximal interval of existence $[0, T_c)$ and $f'_c(t) \rightarrow +\infty$ as $t \rightarrow T_c$. Then there exists $t_0 \in [0, T_c)$ such that, for all $t \in [t_0, T_c)$, $f'_c(t) > 1$ and

$$\begin{aligned} f'''_c(t) + f_c(t) f''_c(t) &= -\beta f'_c(t) (f'_c(t) - 1) \\ &< -\beta f'_c(t_0) (f'_c(t_0) - 1) = -\varepsilon. \end{aligned} \tag{4}$$

Consequently, f_c is a ε -subsolution of the Blasius equation on $[t_0, T_c)$. Therefore from Proposition 4 we have $T_c < +\infty$.

Furthermore, there exists $t_1 \in [t_0, T_c)$ such that $f_c(t_1) = \alpha > 0$ and $f'_c(t_1) > 1$, and then $f'''_c(t) + f_c(t) f''_c(t) < 0$ and $f_c(t) > f_c(t_1) = \alpha$ for all $t \in [t_1, T_c)$. Thus,

$$f'''_c(t) < -\alpha f''_c(t) \tag{5}$$

for all $t \in [t_1, T_c)$. Next, integrating (5) on $[t_1, t]$ for $t_1 < t < T_c$, we obtain $f''_c(t) - f''_c(t_1) < -\alpha(f'_c(t) - f'_c(t_1))$ and using Proposition 5, item (4), yields a contradiction as $t \rightarrow T_c$. □

5. The $a \leq 0$ Case

Lemma 12 (let $0 < \beta < 1$ and $a \leq 0$). *If $c \geq 0$ and if there exists $t_0 \in [0, T_c)$ such that $f''_c(t_0) = 0$ and $f'_c(t_0) > 1$, then $f_c(t_0) > 0$.*

Proof. Let $c \geq 0$ and assume that there exists $t_0 \in [0, T_c)$ such that $f''_c(t_0) = 0$ and $f'_c(t_0) > 1$.

Let us consider the function $H_c = f''_c + f_c(f'_c - \beta)$. Since $H'_c = (1 - \beta)f'_c{}^2 \geq 0$ on $[0, T_c)$, then H_c is nondecreasing on $[0, T_c)$ and hence

$$\begin{aligned} 0 \leq H_c(0) &= c + a(b - \beta) < H_c(t_0) \\ &= f_c(t_0) (f'_c(t_0) - \beta). \end{aligned} \tag{6}$$

Thus, $f_c(t_0) > 0$. □

For the rest of this section we will set $a^* = -\sqrt{(1 - b^2)/(\beta - 2b)}$.

Proposition 13 (let $0 < \beta < 1$). *If either $b \leq -1$ or $b \in (-1, 0]$ and $a \leq a^*$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has no convex solution.*

Proof. Suppose that $b \leq -1$ and that f_c is a convex solution of the boundary value problem $\mathcal{P}_{1(a,b)}$. Then, there exists $t_* > 0$ such that $f_c(t_*) = 0$.

Let $K_c = 2f_c f''_c - f_c{}^2 + f_c^2(2f'_c - \beta)$. From (1), we obtain $K'_c = 2(2 - \beta)f_c f_c{}^{\prime 2} < 0$ on $(0, t_*)$. Therefore, K_c is decreasing on $(0, t_*)$ and hence $K_c(0) > K_c(t_*)$. It follows that

$$f_c{}^{\prime 2}(t_*) > -2ac + b^2 + a^2(\beta - 2b) \geq b^2, \tag{7}$$

which implies that $f'_c(t_*) > 1$. This is a contradiction. The same contradiction is obtained where $b \in (-1, 0]$ and $a \leq a^*$. □

Theorem 14. *Let $0 < \beta < 1$ and $a, b \in \mathbb{R}$ with $b < 0$ and $a \leq 0$ and $0 < \beta < 1$.*

- (1) *The boundary value problem $\mathcal{P}_{0(a,b)}$ has at least one convex solution.*
- (2) *If either $b \leq -1$ or $b \in (-1, 0]$ and $a \leq a^*$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has no convex solution and has infinitely many convex-concave solutions.*

Proof. The first result follows from Remark 7 and Lemma 8. The second result follows from Remark 7, Remark 10, Proposition 11, Proposition 13, and Lemma 6. □

6. The $a > 0$ Case

Let $a, b \in \mathbb{R}$ with $b < 0$ and $a > 0$. We assume $0 < \beta < 1$ and consider the solution f_c of the initial value problem $P_{(a,b,c)}$ on the right maximal interval of existence $[0, T_c)$.

Let us set $b^* = \max\{-(1/2)a^2, -\beta/(1 - \beta)\}$.

Lemma 15 (let $0 < \beta < 1$. let $c \geq 0$). *If $b \in (b^*, 0)$ and if there exists $t_* \in (0, T_c)$ such that t_* is the first point where $f_c(t_*) = 0$, then $f'_c(t_*) < 0$ and $f''_c(t_*) < 0$.*

Proof. Let $t_* \in (0, T_c)$ be such that $f_c > 0$ on $[0, t_*)$ and $f_c(t_*) = 0$. Suppose that $f_c'' > 0$ on $[0, t_*)$. Then, necessarily, we have $f_c' < 0$ on $[0, t_*)$. Moreover, since f_c' is increasing and $b > b^*$, we also have $f_c' > -\beta/(1 - \beta)$ on $[0, t_*)$.

Let $E_c = f_c'' + f_c f_c'$. From (1), we have $E_c' = (1 - \beta)f_c''^2 + \beta f_c'$. Consequently, $E_c' < 0$ on $[0, t_*)$ and since $E_c(t_*) = f_c''(t_*) \geq 0$, it follows that $E_c > 0$ on $[0, t_*)$. Integrating from 0 to t_* gives

$$0 < \int_0^{t_*} E_c(t) dt = f_c'(t_*) - b - \frac{1}{2}a^2. \tag{8}$$

Thus $f_c'(t_*) > b + (1/2)a^2 \geq 0$ which is a contradiction.

Therefore, there exists $t_0 \in [0, t_*)$ such that $f_c'' > 0$ on $(0, t_0)$ and $f_c''(t_0) = 0$. From Proposition 5, items (1) and (5), we have either $f_c'(t_0) < 0$ or $f_c'(t_0) > 1$. The second case cannot happen. Assume, for the sake of contradiction, that $f_c'(t_0) > 1$. Then $f_c'' \geq 0$ and $f_c > 0$ on $[0, t_0]$, so that we have $f_c(t_0)(1 - f_c'(t_0)) \leq f_c''(t_0) \leq 0$. From Lemma 6, we obtain that $T_c = +\infty$, $f_c'(t) \rightarrow 1$ as $t \rightarrow +\infty$, and $f_c'' < 0$ on $[t_0, +\infty)$. It follows that f_c is a positive convex-concave solution of the boundary value problem $\mathcal{P}_{1(a,b)}$ on $[0, +\infty)$, which contradicts the existence of t_* . Consequently, we have $f_c'(t_0) < 0$. This implies that $f_c' < 0$ on $[0, t_0]$ and that $c \in C_3$. By virtue of Lemma 9, we see that f_c'' remains negative after t_0 . The proof is complete. \square

Lemma 16 (let $0 < \beta < 1$). *If there exists $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$ and $f_c'(t_0) > 1$, then $f_c(t_0) > 0$.*

Proof. Assume that there exists $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$, $f_c'(t_0) > 1$, and $f_c(t_0) < 0$. Then, there would exist $t_1 < t_0$ such that $f_c(t_1) = 0$.

Let $H_c = f_c'' + f_c(f_c' - \beta)$. From (1), we have $H_c' = (1 - \beta)f_c''^2 \geq 0$ on $[0, T_c)$. Therefore, H_c is nondecreasing on $[0, T_c)$. Since $H_c(t_0) = f_c(t_0)(f_c'(t_0) - \beta) < 0$, we get $f_c''(t_1) = H_c(t_1) < 0$. But, this and Proposition 5, item (1), imply that f_c'' remains negative on (t_1, T_c) , a contradiction. Hence $f_c(t_0) > 0$. \square

Lemma 17. *If $0 < \beta < 1$ and $b \in (b^*, 0)$, then there exists $c_0 \in C_2$ such that if $c \geq c_0$ then f_c is a convex-concave solution of $\mathcal{P}_{1(a,b)}$.*

Proof [let $c \in C_2$]. From Remark 10 and Proposition 11, we see that either f_c is a convex solution of $\mathcal{P}_{1(a,b)}$ or there exists $t_0 \in [0, T_c)$ such that $f_c''(t_0) = 0$ and $f_c'(t_0) > 1$. Now, as we have seen in the proof of Lemma 15, in the second case, f_c is a convex-concave solution of $\mathcal{P}_{1(a,b)}$.

Let $c \in C_2$ be such that f_c is a convex solution of $\mathcal{P}_{1(a,b)}$. Therefore, we have $b < f_c' < 1$ on $[0, +\infty)$ and, from Lemma 15, we have $f_c > 0$. It follows that

$$\begin{aligned} (f_c'' + f_c(f_c' - 1))' &= (1 - \beta)f_c'(f_c' - 1) \\ &\geq -\frac{1}{4}(1 - \beta) \end{aligned} \tag{9}$$

on $[0, +\infty)$. Integrating between 0 and $t \geq 0$, and using the fact that $f_c > 0$, we obtain

$$\begin{aligned} f_c''(t) &\geq -\frac{1}{4}(1 - \beta)t + a(b - 1) + c \\ &\quad - f_c(t)(f_c'(t) - 1) \\ &\geq -\frac{1}{4}(1 - \beta)t + a(b - 1) + c. \end{aligned} \tag{10}$$

Integrating once again we get

$$\forall t \geq 0, \tag{11}$$

$$1 > f_c'(t) \geq -\frac{1}{8}(1 - \beta)t^2 + (a(b - 1) + c)t + b. \tag{12}$$

Let us set $P_c(t) = -(1/8)(1 - \beta)t^2 + (a(b - 1) + c)t + b - 1$. We have $P_c(t) < 0$ for all $t \geq 0$. It means that P_c has no positive roots. Thus c cannot be too large, because, on the contrary, its discriminant $\Delta = (a(b - 1) + c)^2 + (1/2)(1 - \beta)(b - 1)$ and $a(b - 1) + c$ would be positive, and hence the polynomial P_c would have two positive roots, a contradiction.

Therefore, there exists $c_0 > 0$ such that f_c is convex-concave solution of the problem $\mathcal{P}_{1(a,b)}$ for $c \geq c_0$. This completes the proof. \square

Theorem 18. *Let $a, b \in \mathbb{R}$, with $b < 0$, $a > 0$, and $0 < \beta < 1$.*

- (1) *The boundary value problem $\mathcal{P}_{0(a,b)}$ has at least one convex solution. If in addition $b \in (b^*, 0)$, then any convex solution of $\mathcal{P}_{0(a,b)}$ is positive.*
- (2) *If $b \in (b^*, 0)$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has infinitely many positive convex-concave solutions.*

Proof. The first part of (1) follows from Remark 7 and Lemma 8. The second part follows from Lemma 15, because if there was a point t_* such that $f_c > 0$ on $[0, t_*)$ and $f_c(t_*) = 0$ then $f_c''(t_*) < 0$, a contradiction. The second result follows from Remark 7, Remark 10, Proposition 11, Lemma 16, and Lemma 6. \square

7. Conclusion

In this work, in particular in Theorems 14 and 18, we have presented some new and important results about the boundary value problems $\mathcal{P}_{0(a,b)}$ and $\mathcal{P}_{1(a,b)}$, which we summarize below. The parameters β and b satisfy $0 < \beta < 1$ and $b < 0$. The constants a_* and b_* are defined in Sections 5 and 6.

- (1) For $a \leq 0$:
 - (a) The boundary value problem $\mathcal{P}_{0(a,b)}$ has at least one convex solution.
 - (b) If either $b \leq -1$ or $b \in (-1, 0]$ and $a \leq a^*$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has no convex solution and has infinitely many convex-concave solutions.
- (2) For $a > 0$:

- (a) If $b \in (b^*, 0)$, then the boundary value problem $\mathcal{P}_{0(a,b)}$ has at least one positive convex solution.
- (b) If $b \in (b^*, 0)$, then the boundary value problem $\mathcal{P}_{1(a,b)}$ has infinitely many positive convex-concave solutions.

Numerical simulations prompt us to formulate the following conjecture.

Conjecture 19. Let $a, b \in \mathbb{R}$, with $b \leq -1$, $a > 0$, and $0 < \beta < 1$. The boundary value problem $\mathcal{P}_{1(a,b)}$ has no convex solution.

To finish, we give the following proposition concerning the case $a = 0$.

Proposition 20 (let $\beta < 2$). If $b \leq -1$, then the boundary value problem $\mathcal{P}_{1(0,b)}$ has no convex solution.

Proof. Assume that f_c is a convex solution of the boundary value problem $\mathcal{P}_{1(0,b)}$. Then, there exists $t_* \geq 0$, such that $f_c < 0$ on $(0, t_*)$, $f_c(t_*) = 0$, and $f_c'(t_*) > 0$. Consider again the function

$$K_c = 2f_c f_c'' - f_c'^2 + f_c^2 (2f_c' - \beta). \quad (13)$$

We have $K_c' = 2(2 - \beta)f_c f_c'^2 < 0$ on $(0, t_*)$. Thus, K_c is a decreasing function and hence $K_c(0) > K_c(t_*)$. It follows that $f_c'^2(t_*) > b^2$ which implies that $f_c'(t_*) > 1$, which is a contradiction. \square

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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