Research Article

On Solvability Theorems of Second-Order Ordinary Differential Equations with Delay

Nai-Sher Yeh

Department of Mathematics, Fu Jen Catholic University, Xinzhuang District, New Taipei City 24205, Taiwan

Correspondence should be addressed to Nai-Sher Yeh; 038300@mail.fju.edu.tw

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For each \( x_0 \in [0,2\pi) \) and \( k \in \mathbb{N} \), we obtain some existence theorems of periodic solutions to the two-point boundary value problem

\[
\begin{align*}
\ddot{u}(x) + k^2 u(x-x_0) + g(x, u(x-x_0)) &= h(x) \quad \text{in} \quad (0,2\pi), \\
u(0) - u(2\pi) &= u'(0) - u'(2\pi) = 0,
\end{align*}
\]

(1)

where \( h \in L^1(0,2\pi) \) is given and \( g : (0,2\pi) \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function which grows linearly in \( u \) as \( |u| \to \infty \), and \( h \in L^1(0,2\pi) \) may satisfy a generalized Landesman-Lazer condition

\[
(1+\text{sign}(\beta)) \int_0^{2\pi} h(x) v(x) dx < \int_{(x_0)\in[0,2\pi]} g^+(x)|v(x)|^{1-\beta} dx + \int_{(x_0)\in[0,2\pi]} g^-(x)|v(x)|^{1-\beta} dx \quad \forall v \in N(L) \setminus \{0\}.
\]

Here \( N(L) \) denotes the subspace of \( L^1(0,2\pi) \) spanned by \( \sin kx \) and \( \cos kx \), \( -1 < \beta \leq 0 \), \( g^+(x) = \liminf_{u\to\infty} (g(x,u)u/|u|^{1-\beta}) \), and \( g^-(x) = \liminf_{u\to-\infty} (g(x,u)u/|u|^{1-\beta}) \).

1. Introduction

Let \( x_0 \in [0,2\pi) \) and \( k \in \mathbb{N} \) be fixed. We consider the following two-point boundary value problems:

\[
\begin{align*}
\ddot{u}(x) + k^2 u(x-x_0) + g(x, u(x-x_0)) &= h(x) \quad \text{in} \quad (0,2\pi), \\
u(0) - u(2\pi) &= u'(0) - u'(2\pi) = 0,
\end{align*}
\]

(1)

where \( h \in L^1(0,2\pi) \) is given and \( g : (0,2\pi) \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function; that is, \( g(x,u) \) is continuous in \( u \in \mathbb{R} \), for a.e. \( x \in (0,2\pi) \), and satisfies, for each \( r > 0 \), the fact that there exists an \( a_r \in L^1(0,2\pi) \) such that

\[
|g(x,u)| \leq a_r(x)
\]

(3)

for a.e. \( x \in (0,2\pi) \) and all \( |u| \leq r \). Concerning the growth condition of the nonlinear term \( g \) to (1) and (2), we assume that

(H) there exist constants \(-1 < \beta \leq 0, r_0 > 0, \) and \( a,b,c,d \in L^1(0,2\pi), \) \( a,b \geq 0 \) and \( a(x) \leq 2k + 1 \) for a.e. \( x \in (0,2\pi) \) with strict inequality on a positive measurable subset of \( (0,2\pi) \), such that for a.e. \( x \in (0,2\pi) \) and all \( u \geq r_0 \)

\[
c(x)|u|^{-\beta} \leq g(x,u) \leq a(x)|u| + b(x);
\]

(4)

and for a.e. \( x \in (0,2\pi) \) and all \( u \leq -r_0 \)

\[
-a(x)|u| - b(x) \leq g(x,u) \leq d(x)|u|^{-\beta};
\]

(5)

(G) there exist constants \(-1 < \beta \leq 0, r_0 > 0, \) and \( a,b,c,d \in L^1(0,2\pi), \) \( a,b \geq 0 \) and \( a(x) \leq 2k - 1 \) for a.e. \( x \in (0,2\pi) \) with strict inequality on a positive measurable subset of \( (0,2\pi) \), such that for a.e. \( x \in (0,2\pi) \) and all \( u \geq r_0 \)

\[
c(x)|u|^{-\beta} \leq g(x,u) \leq a(x)|u| + b(x);
\]

(6)

and for a.e. \( x \in (0,2\pi) \) and all \( u \leq -r_0 \)

\[
-a(x)|u| - b(x) \leq g(x,u) \leq d(x)|u|^{-\beta};
\]

(7)
respectively, and a generalized Landesman–Lazer condition
\begin{equation}
0 < \int_{x(\gamma)>0} g_\beta^+(x) |v(x)|^{1-\beta} \, dx + \int_{x(\gamma)<0} g_\beta^-(x) |v(x)|^{1-\beta} \, dx,
\end{equation}
for all \( v \in N(L) \setminus \{0\} \), may be satisfied. Here \( N(L) \) denotes the subspace of \( L^1(0, 2\pi) \) spanned by \( \sin kx \) and \( \cos kx \), \( \beta \in \mathbb{R} \), \( g^\beta(x) = \lim_{\gamma \to -\infty} g(x, u)/|u|^{1-\beta} \), and \( g_\beta(x) = \lim_{\gamma \to -\infty} \mathcal{L}(g(x, u) u/|u|^{1-\beta}) \). Under assumptions and either with or without the Landesman–Lazer condition
\begin{equation}
\int_0^{2\pi} h(x) \, v(x) \, dx < \int_{x(\gamma)>0} g_\beta^+(x) |v(x)|^{1-\beta} \, dx + \int_{x(\gamma)<0} g_\beta^-(x) |v(x)|^{1-\beta} \, dx,
\end{equation}
for all \( v \in N(L) \setminus \{0\} \), the solvability of the problem (1) has been extensively studied if the nonlinearity \( g(x, u) \) has at most linear growth in \( u \) as \( |u| \to \infty \) (see [1–13] for the case \( x_0 = 0 \) and [14–16] for the general case) or grows superlinearly in \( u \) in one of directions \( u \to \infty \) and \( u \to -\infty \) and may be bounded in the other (see [8, 17] for the case \( x_0 = 0 \) and [14] for the general case when \( k = 0 \)). Based on the well-known Leray-Schauder continuation method (see [18, 19]), we obtain solvability theorems to (1) (resp., (2)) when \( g(x, u) \) satisfies \((H)\) (resp., \((G)\)) and either (8) with \(-1 < \beta < 0\) or (9) with \( \beta = 0 \) is satisfied, which extends the results of [15] for the nonresonance case, and has been established in [9] for the case \( x_0 = 0 \) and \( g(x, u) \) grows sublinearly in \( u \) as \( |u| \to \infty \) with \(-1 < \beta \leq 1 \). Unfortunately, it is still unknown when \( k \in \mathbb{N} \), \( g(x, u) \) grows linearly in \( u \) as \( |u| \to \infty \) and the assumption of (8) is replaced by
\begin{equation}
\int_0^{2\pi} h(x) \, v(x) \, dx = 0
\end{equation}
for all \( v \in N(L) \setminus \{0\} \) with \( \beta > 0 \). In the following we will make use of real Banach spaces \( L^p(0, 2\pi) \), \( C[0, 2\pi] \) and Sobolev spaces \( W^{k, p}(0, 2\pi) \) and \( H^1(0, 2\pi) \). The norms of \( L^p(0, 2\pi) \), \( C[0, 2\pi] \) and \( H^1(0, 2\pi) \) are denoted by \( \|u\|_{L^p}, \|u\|_{C[0, 2\pi]} \) and \( \|u\|_{H^1} \), respectively. By a solution of (1), we mean a periodic function \( u : \mathbb{R} \to \mathbb{R} \) of period \( 2\pi \) which belongs to \( W^{k, p}(0, 2\pi) \) and satisfies the differential equation in (1) a.e. \( x \in (0, 2\pi) \).

### 2. Existence Theorems

For each \( v \in W^{2, 1}(0, 2\pi) \) with \( v(0) - v(2\pi) = v'(0) - v'(2\pi) = 0 \) and \( k \in \mathbb{N} \), we write \( \overline{v} = \sum_{0 \leq j \neq k} P_j v, \overline{v} = \sum_{j \neq k} P_j v, \) and \( \bar{v} = \sum_{0 \leq j \neq k} P_j v \). Here \( P_j v \) denotes the projection of \( v \) on the eigenspace of \( d^2/dx^2 \) spanned by \( \sin jx \) and \( \cos jx \) for \( j \in \mathbb{N} \cup \{0\} \). Just as an application of [11, Lemma 2] or [1, Lemma 2.2], we can modify slightly the proof of [15, Lemma 1] to obtain the next lemma.

**Lemma 1.** Let \( k \in \mathbb{N} \cup \{0\} \) and \( \Gamma \) be a nonnegative \( L^1(0, 2\pi) \)-function such that for a.e. \( x \in (0, 2\pi) \), \( \Gamma(x) \leq 2k+1 \) with strict inequality on a positive measurable subset of \((0, 2\pi)\). Then there exists a constant \( K_1 > 0 \) such that
\begin{equation}
\int_0^{2\pi} (\bar{u}(x) - u(x)) \cdot \left( u''(x) + k^2 u(x) - \Gamma(x) \right) \, dx
\end{equation}
whenever \( p \in L^1(0, 2\pi) \) with \( 0 \leq p(x) \leq \Gamma(x) \) for a.e. \( x \in (0, 2\pi) \) and \( u \in W^{2, 1}(0, 2\pi) \) is a periodic function of period \( 2\pi \) with \( u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0 \).

**Proof.** Just as in [20, Lemma 1], we can modify slightly the proof of [11, Lemma 2] or [1, Lemma 2.2] to obtain the fact that there exists a constant \( K_1 > 0 \) such that
\begin{equation}
\int_0^{2\pi} (\bar{u}'(x))^2 - \left( k^2 + p(x) \right) (\bar{u}(x))^2 \, dx
\end{equation}
\begin{equation}
\int_0^{2\pi} (\bar{u}'(x))^2 - \left( k^2 + p(x) \right) (\bar{u}(x))^2 \, dx
\end{equation}
whenever \( p \in L^1(0, 2\pi) \) with \( 0 \leq p(x) \leq \Gamma(x) \) for a.e. \( x \in (0, 2\pi) \) and \( u \in W^{2, 1}(0, 2\pi) \) with \( u(0) - u(2\pi) = u'(0) - u'(2\pi) = 0 \). Let us extend \( u(x) \) and \( p(x) \) \( 2\pi \) periodically in \( x \) to all of \( \mathbb{R} \) and then use the same notations for the periodic extensions as for the original functions. In this case, we have
\begin{equation}
\int_0^{2\pi} (\bar{u}'(x))^2 \, dx = \int_0^{2\pi} (\bar{u}'(x))^2 \, dx
\end{equation}
and
\begin{equation}
\int_0^{2\pi} \left[ u''(x) + \left( k^2 + p(x) \right) u(x) \right] \left( \bar{u}(x) - u(x) \right) \, dx
\end{equation}
\begin{equation}
\int_0^{2\pi} \left[ u''(x) + \left( k^2 + p(x) \right) u(x) \right] \left( \bar{u}(x) - u(x) \right) \, dx
\end{equation}

\[
\begin{align*}
&\geq \int_{0}^{2\pi} (\bar{u}'(x))^2 \, dx - \frac{1}{2} \int_{0}^{2\pi} (\bar{w}'(x))^2 \, dx \\
&+ (\bar{u}'(x-x_0))^2 \, dx + \frac{1}{2} \int_{0}^{2\pi} (k^2 + p(x)) \\
&\cdot [\bar{u}(x-x_0)^2 - (\bar{u}(x))^2 - (\bar{u}(x-x_0))^2] \, dx \\
&+ \frac{1}{2} \int_{0}^{2\pi} (k^2 + p(x)) \\
&\cdot [\bar{u}(x-x_0) + \bar{u}(x-x_0) - \bar{u}(x)]^2 \, dx \quad \geq \frac{1}{2} \\
&\left( k^2 + p(x) \right) (\bar{u}(x)) \, dx + \frac{1}{2} \\
&\left( k^2 + p(x) \right) (\bar{u}(x-x_0)) \, dx - \frac{1}{2} \\
&\left( k^2 + p(x) \right) (\bar{u}(x-x_0)) \, dx + \frac{1}{2} \\
&\left[ \bar{u}(x-x_0) + \bar{u}(x-x_0) - \bar{u}(x) \right]^2 \, dx.
\end{align*}
\]

Combining (12) with (13), we have
\[
\begin{align*}
\int_{0}^{2\pi} (\bar{u}(x-x_0) - \bar{u}(x)) \\
\cdot (u''(x) + k^2 u(x-x_0) + p(x) u(x-x_0)) \, dx \\
\geq \frac{1}{2} \int_{0}^{2\pi} (\bar{u}'(x-x_0))^2 - (k^2 + p(x)) \\
\cdot (\bar{u}(x-x_0))^2 + \frac{1}{2} \int_{0}^{2\pi} (k^2 + p(x)) \\
\cdot (\bar{u}(x-x_0))^2 - (\bar{u}'(x-x_0))^2 \, dx \\
\geq K_1 \| u \|_{H^1(\theta)}^2 + K_1 \| u \|_{H^1}^2.
\end{align*}
\]

\textbf{Lemma 2.} Let \( k \in \mathbb{N} \) and \( \Gamma \) be a nonnegative \( L^1(0, 2\pi) \)-function such that for a.e. \( x \in (0, 2\pi) \), \( \Gamma(x) \leq 2k - 1 \) with strict inequality on a positive measurable subset of \( (0, 2\pi) \). Then there exists a constant \( K_2 > 0 \) such that
\[
\begin{align*}
\int_{0}^{2\pi} (\bar{u}(x-x_0) - \bar{u}(x)) \\
\cdot (u''(x) + k^2 u(x-x_0) + p(x) u(x-x_0)) \, dx \\
\geq K_2 \| u \|_{H^1}^2.
\end{align*}
\]

Whenever \( p \in L^1(0, 2\pi) \) with \( 0 \leq p(x) \leq \Gamma(x) \) for a.e. \( x \in (0, 2\pi) \) and \( u \in W^{2,1}(0, 2\pi) \) with \( \nu(0) - \nu(2\pi) = 0 \), we have \( \int_{0}^{2\pi} (\bar{u}'(x))^2 - (k^2 + p(x)) (\bar{u}(x))^2 \, dx \geq 0 \) and
\[
\begin{align*}
\left( k^2 + p(x) \right) (\bar{u}(x)) \, dx + \frac{1}{2} \\
\left( k^2 + p(x) \right) (\bar{u}(x-x_0)) \, dx - \frac{1}{2} \\
\left( k^2 + p(x) \right) (\bar{u}(x-x_0)) \, dx + \frac{1}{2} \\
\left[ \bar{u}(x-x_0) + \bar{u}(x-x_0) - \bar{u}(x) \right]^2 \, dx.
\end{align*}
\]

\textbf{Theorem 3.} Let \( k \in \mathbb{N} \cup \{0\} \) and \( g : (0, 2\pi) \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function satisfying (H). Then for each \( h \in L^1(0, 2\pi) \) problem (1) has a solution \( u \), provided that either (8) with \( -1 < \beta < 0 \) or (9) with \( \beta = 0 \) holds.

\textbf{Proof.} Let \( \alpha \in \mathbb{R} \) be fixed and \( 0 < \alpha < 2k + 1 \). We consider the boundary value problems
\[
u''(x) + k^2 \nu(x-x_0) + (1-t) \alpha \nu(x-x_0)
+ t g(x, \nu(x-x_0)) = th(x) \quad \text{in} \ (0, 2\pi),
\]
\[
u(0) - \nu(2\pi) = u'(0) - u'(2\pi) = 0
\]
for \( 0 \leq t \leq 1 \), which becomes the original problem when \( t = 1 \). Since \( 0 < \alpha < 2k + 1 \), we observe from Lemma 1 that (17) has only a trivial solution when \( t = 0 \). To apply the Leray-Schauder continuation method, it suffices to show that solutions to (17) for \( 0 < t < 1 \) have an a priori bound in \( H^1(0, 2\pi) \). To this end, let \( \theta : \mathbb{R} \to \mathbb{R} \) be a continuous function.
such that \( 0 \leq \theta \leq 1, \) \( \theta(u) = 0 \) for \( |u| \leq r_0, \) and \( \theta(u) = 1 \) for \( |u| \geq 2r_0. \) We define \( c(x) = \max[a_0(x), b(x), c(x)], |d(x)|, \)

\[ g_1(x, u) \]

\[ = \begin{cases} 
\min \{ g(x, u) + e(x) |u|^{-\beta}, \ a(x) u \theta(u) \} & \text{if } u \geq 0 \\
\max \{ g(x, u) - e(x) |u|^{-\beta}, \ a(x) u \theta(u) \} & \text{if } u \leq 0,
\end{cases} \tag{18} \]

and \( g_2(x, u) = g(x, u) - g_1(x, u). \) Then \( g_1, g_2 : (0, 2\pi) \times \mathbb{R} \rightarrow \mathbb{R} \) are Carathéodory functions, such that for a.e. \( x \in (0, 2\pi) \) and \( u \in \mathbb{R}, \ u \neq 0 \)

\[ 0 \leq \frac{g_1(x, u)}{u} \leq a(x), \tag{19} \]

\[ |g_2(x, u)| \leq e(x) |u|^{-\beta} + e(x). \tag{20} \]

If \( u \) is a possible solution to (17) for some \( 0 < t < 1, \) then using (19), (20), and Lemma 1, we have

\[
0 = \int_0^{2\pi} \left( \bar{u}(x) - \bar{u}(x - x_0) \right) \left[ u''(x) + k^2 u(x - x_0) \right.
\]

\[ + (1-t) a u(x - x_0) \]

\[ + t g_1(x, u(x - x_0) - t h(x) \right) dx \]

\[
\geq K_1 \left[ \| u \|^2_{L^1} - \left( \| e \|^2_{L^2} + \| e \|^2_{C^0} + \| h \|^2_{L^1} \right) (\| u \|^2_{L^1} + \| u \|^2_{C^0} + \| u \|^2_{C^0}) \right.
\]

\[ + \| u \|^2_{C^0} \]

\[ \geq K_1 \left[ \| u \|^2_{L^1} - C_1 (\| u \|^2_{C^0} + 1) (\| u \|^2_{L^1} + \| u \|^2_{C^0}) \right.
\]

\[ + \| u \|^2_{C^0} \]

\[ = \left. \frac{C_1}{K_1} \left( \| u \|^2_{C^0} + 1 \right) \left( \| u \|^2_{L^1} + \| u \|^2_{C^0} \right) \right. \tag{21} \]

which implies that

\[
\left\| u \right\|^2_{L^1} \leq \frac{C_1}{K_1} \left( \| u \|^2_{C^0} + 1 \right) \left( \| u \|^2_{L^1} + \| u \|^2_{C^0} \right)
\]

\[
\leq C_2 \left( \| u \|^2_{L^1} + \| u \|^2_{C^0} \right) \tag{22} \]

for some constants \( C_1, C_2 > 0 \) independent of \( u. \) It remains to show that solutions to (17) for \( 0 < t < 1 \) have an a priori bound in \( H^1(0, 2\pi). \) We argue by contradiction and suppose that there exists a sequence \( \{ u_n \} \) of periodic functions with period \( 2\pi \) and a corresponding sequence \( \{ t_n \} \) in \( (0, 1) \) such that \( u_n \) is a solution to (17) with \( t = t_n \) and \( \| u_n \|_{H^1} \geq n \) for all \( n. \) Let \( v_n = u_n/\| u_n \|_{H^1} ; \) then \( \| v_n \|_{H^1} = 1 \) for all \( n \in \mathbb{N}, \) and by (22) we have \( \| v_n \|_{H^1} \rightarrow 0 \) as \( n \rightarrow \infty. \) Since \( \| v_n \|_{H^1} = 1 \) and \( \| P_k v_n \|_{H^1} \leq \| v_n \|_{H^1} + \| v_n \|_{C^0} \| P_k \|_{C^0} \| v_n \|_{C^0} \) for all \( n \in \mathbb{N}, \) we have a bounded sequence \( \{ P_k v_n \} \) in \( H^1(0, 2\pi). \) For simplicity, we may assume that \( v_n \) converges to \( v \in H^1(0, 2\pi) \) for some \( v \in N(L) \) with \( \| v \|_{H^1} = 1. \) In particular, \( v_n \rightarrow v \) in \( C[0, 2\pi]. \) Clearly, \( v(x - x_0) \in N(L) \) and \( \| v(x - x_0) \|_{L^1} = \| v \|_{L^1}. \) It follows that \( u_n(x) \rightarrow \infty \) for each \( x \in \mathbb{R} \) with \( v(x) > 0, \) and \( u_n(x) \rightarrow -\infty \) for each \( x \in \mathbb{R} \) with \( v(x) < 0. \) Since \( \int_0^{2\pi} u_n''(x) P_k u_n(x - x_0)dx + \int_0^{2\pi} k^2 u_n(x) P_k u_n(x - x_0)dx = 0 \) and \( \| P_k u_n \|_{L^2}^2 = \| P_k u_n(x - x_0) \|_{L^2}^2, \) we have

\[
\int_0^{2\pi} u_n''(x) P_k u_n(x - x_0)dx + \int_0^{2\pi} k^2 u_n(x) P_k u_n(x - x_0)dx
\]

\[
\cdot P_k u_n(x - x_0)dx = \int_0^{2\pi} u_n''(x) dx
\]

\[
\cdot \cdot \cdot P_k u_n(x - x_0)dx + \int_0^{2\pi} k^2 u_n(x) P_k u_n(x - x_0)dx
\]

\[
+ \int_0^{2\pi} k^2 \left[ u_n(x - x_0) - u_n(x) \right] dx
\]

\[
\cdot P_k u_n(x - x_0)dx
\]

\[
= \int_0^{2\pi} k^2 \left[ u_n(x - x_0) - u_n(x) \right] dx
\]

\[
= \int_0^{2\pi} k^2 \left[ P_k u_n(x - x_0) - P_k u_n(x) \right] dx
\]

\[
\cdot P_k u_n(x - x_0)dx \geq 0.
\]

Multiplying each side of (17) by \( P_k v_n(x - x_0), \) and then integrating them over \([0, 2\pi]\) when \( u = u_n \) and \( t = t_n, \) we get

\[
t_n \int_0^{2\pi} g(x, u_n(x - x_0)) P_k v_n(x - x_0)dx
\]

\[
\leq (1 - t_n) \alpha \int_0^{2\pi} u_n(x - x_0) P_k v_n(x - x_0)dx
\]

\[
+ t_n \int_0^{2\pi} g(x, u_n(x - x_0)) P_k v_n(x - x_0)dx
\]

\[
\leq t_n \int_0^{2\pi} h(x) P_k v_n(x - x_0)dx.
\]

By (19) and the assumption of \(-1 < \beta \leq 0, \) we have

\[
g_1(x, u_n(x - x_0)) P_k v_n(x - x_0) \| u_n \|^2_{H^1}
\]

\[
= \frac{g_1(x, u_n(x - x_0))}{u_n(x - x_0)} u_n(x - x_0) P_k v_n(x - x_0)
\]

\[
\cdot \| u_n \|^2_{H^1} \geq \frac{g_1(x, u_n(x - x_0))}{u_n(x - x_0)} u_n(x - x_0)
\]

\[
\cdot \| u_n \|^2_{H^1} \geq \frac{1}{2} \left[ u_n(x - x_0) - P_k u_n(x - x_0) \right]^2 \| u_n \|^2_{H^1}
\]

\[
\cdot \cdot \cdot a \left( u_n^2(x - x_0) \right)^2 \| u_n \|^2_{H^1}
\]

for a.e. \( x \in (0, 2\pi). \) Combining (22) with (25), we get that \( g_1(x, u_n(x - x_0)) P_k v_n(x - x_0) \| u_n \|^2_{H^1} \) is bounded from below
by an $L^1(0,2\pi)$-function independent of $n$. By (20) and the assumption of $-1 < \beta \leq 0$, we have

$$
\begin{align*}
|g_2(x, u_n(x - x_0)) P_k v_n(x - x_0) - (u_n(x - x_0))|^{1-\beta} & \leq |e(x)| |u_n(x - x_0)|^{1-\beta} + e(x) \left| P_k v_n(x - x_0) \right|
\cdot \|u_n\|^\beta_{L^\beta} \\
\cdot \|P_k v_n\|_{L^\beta} & \leq \left[ e(x) \right] |v_n(x - x_0)|^{1-\beta} + e(x)
\cdot \|P_k v_n(x - x_0)\|
\end{align*}
$$

(26)

for a.e. $x \in (0, 2\pi)$. In particular, $g_2(x, u_n(x - x_0)) P_k v_n(x - x_0)\|u_n\|^\beta_{L^\beta}$ is bounded from below by an $L^1(0,2\pi)$-function independent of $n$, which implies that $g(x, u_n(x - x_0)) P_k v_n(x - x_0\|u_n\|^\beta_{L^\beta}$ is also so,

$$
\int_{(x-x_0)} \liminf_{n \to \infty} g(x, u_n(x - x_0)) P_k v_n(x - x_0\|u_n\|^\beta_{L^\beta} dx = 0,
$$

and

$$
\begin{align*}
g(x, u_n(x - x_0)) P_k v_n(x - x_0) \|u_n\|^\beta_{L^\beta} & = 
\frac{g(x, u_n(x - x_0)) u_n(x - x_0) |v_n(x - x_0)|^{1-\beta}}{|u_n(x - x_0)|^{1-\beta}} \\
\cdot \liminf_{n \to \infty} g(x, u_n(x - x_0)) P_k v_n(x - x_0\|u_n\|^\beta_{L^\beta} dx
\end{align*}
$$

(27)

for all $n \in \mathbb{N}$ with $u_n(x - x_0) \neq 0$. Here $\text{sign}(\omega) = 1$ if $\omega > 0$, $\text{sign}(\omega) = 0$ if $\omega = 0$, and $\text{sign}(\omega) = -1$ if $\omega < 0$. Applying Fatou's lemma to the integral $\int_0^{2\pi} g(x, u_n(x - x_0)) P_k v_n(x - x_0\|u_n\|^\beta_{L^\beta} dx$, we have

$$
\begin{align*}
& \int_{(x-x_0)} \liminf_{n \to \infty} g(x, u_n(x - x_0)) u_n(x - x_0) |v_n(x - x_0)|^{1-\beta} dx \\
& + \int_{(x-x_0)} \liminf_{n \to \infty} g(x, u_n(x - x_0)) u_n(x - x_0) |v_n(x - x_0)|^{1-\beta} dx
\end{align*}
$$

which is a contradiction when either (8) with $-1 < \beta < 0$ or (9) with $\beta = 0$ is satisfied. Hence, the proof of this theorem is complete.

By slightly modifying the proof of Theorem 3, we can apply Lemma 2 to obtain an existence theorem to (2) when condition (H) is replaced by (G) and either (8) with $-1 < \beta < 0$ or (9) with $\beta = 0$ is satisfied, which has been established in [20] for the case $x_0 = 0$ when (9) with $\beta = 0$ is satisfied and in [9] for the case $x_0 = 0$ when (8) with $\beta = -1$ is satisfied.

**Theorem 4.** Let $k \in \mathbb{N}$ and $g : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function satisfying (G). Then for each $h \in L^1(0, 2\pi)$ problem (2) has a solution $u$, provided that either (8) with $-1 < \beta < 0$ or (9) with $\beta = 0$ holds.

**Conflicts of Interest**

There are no conflicts of interest involved.

**References**


