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Research Article

Best Proximity Point Theorems for Cyclic Relatively ρ-Nonexpansive Mappings in Modular Spaces

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In this paper we introduce the notion of proximal ρ-normal structure of pair of ρ-admissible sets in modular spaces. We prove some results of best proximity points in this setting without recourse to Zorn's lemma. We provide some examples to support our conclusions.

1. Introduction

Fixed point theory is powerful tools in different fields such as differential equations, dynamical systems, optimal control, and many other scientific branches; it treats equations of type $Tx = x$ where $T : X \longrightarrow X$ is a map of a nonempty set to itself.

Let $A, B \subset X$ and T a cyclic mapping on $A \cup B$; that is, $T : A \cup B \longrightarrow A \cup B$ and $T(A) \subseteq B$, $T(B) \subseteq A$; in this case, T does not necessarily possess a fixed point if, for instance, $A \cap B = \emptyset$. One often attempts to find a point x which is closest to Tx in the sense that the "distance" between x and Tx is equal to the distance between A and B; such a point x is said to be a best proximity point.

The first result of this kind is due to Fan [1] which is stated in locally convex Hausdorff topological vector space. Afterward, many extensions and generalizations were given; see, for instance, [2–6].

On the other hand, Eldred et al. in [7], after generalizing the geometric concept of normal structure for a pair of subsets (A, B) in Banach space introduced earlier by Brodski and Milman (see [8]), proved the existence of best proximity points for relatively nonexpansive mappings in Banach space. Recall that a map $T : A \cup B \longrightarrow A \cup B$ is called relatively nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x \in A$ and $y \in B$. This class of mapping is much larger than nonexpansive, because, it does not guarantee the continuity of T.

After that, Sankar and Veeramani in [9] without using Zorn's lemma, proved the existence of a best proximity point by using convergence theorem. Also, Espinola in [10] showed that under a suitable condition on the pair (A, B) the relatively nonexpansive assumption can be seen as nonexpansive one, which in fact guarantees the continuity of the map.

Recently, the best proximity points results was investigated by many authors and found extension and generalization for different class of mappings and spaces; for a recent account of the theory we refer the reader to [11–18]. In this paper, we extend the notion of proximal ρ-normal structure for a pair of ρ-admissible subsets (A, B) which is a generalization of Khamsi and Kozlowski definition. Also, we give existence results of a best proximity point in the setting of proximal ρ-admissible subsets in modular space. Our proofs do not invoke Zorn's lemma.

2. Preliminaries

We start by recalling some basic facts of modular space. For more details the reader can consult [19].

Let X be an arbitrary vector space.

Definition 1. A function $\rho : X \longrightarrow [0,\infty)$ is called a modular on X if for arbitrary $x, y \in X$,

(1) $\rho(x) = 0$ if and only if $x = 0$
(2) $\rho(\alpha x) = \rho(x)$ for every scalar α with $|\alpha| = 1$
(3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$
If the following property is satisfied,
(4) $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1$ and $\alpha, \beta \ge 0$
we say that ρ is a convex modular. A modular ρ defines a corresponding modular space, i.e., the vector space X_ρ given by

$$X_\rho = \{x \in X : \rho(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\}. \quad (1)$$

In general the modular ρ is not subadditive and therefore does not behave as a norm or a distance.

Definition 2. Let X_ρ be a modular space.

(1) We say that (x_n) is ρ-convergent to x and write $x_n \longrightarrow x(\rho)$ if and only if $\rho(x_n - x) \longrightarrow 0$.
(2) A sequence (x_n), where $x_n \in X_\rho$, is called ρ-Cauchy if $\rho(x_n - x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$.
(3) We say that X_ρ is ρ-complete if and only if any ρ-Cauchy sequence in X_ρ is ρ-convergent.
(4) A set $C \subset X_\rho$ is called ρ-closed if for any sequence (x_n) of C; the convergence $x_n \longrightarrow x(\rho)$ implies that x belongs to C.
(5) A set $C \subset X_\rho$ is called ρ-bounded if $\sup\{\rho(x - y) : x, y \in C\} < \infty$.
(6) A set $C \subset X_\rho$ is ρ-sequentially compact, if for any sequence (x_n) of C, there exists a convergent subsequence $(x_{n_k})_k$ of (x_n) such that $x_{n_k} \longrightarrow x(\rho)$ in C.
(7) We will say ρ satisfies Fatou property if

$$\rho(x) \le \liminf_{n\longrightarrow\infty} \rho(x_n) \text{ whenever } x_n \longrightarrow x(\rho). \quad (2)$$

We shall say that a pair (A, B) of sets in a modular space satisfies a property if each of the sets A and B has that property. Thus (A, B) is said to be ρ-closed if both A and B are ρ-closed, $(H, K) \subseteq (A, B) \Longleftrightarrow H \subseteq A$ and $K \subseteq B$, (A, B) is not reduced to one point which means that A and B are not singletons, etc. We shall also introduce the following notation:

$$\begin{aligned} \delta_\rho(A, B) &= \sup\{\rho(x - y) : x \in A, y \in B\}; \\ \delta_\rho(x, A) &= \sup\{\rho(x - y) : y \in A\}; \\ dist_\rho(A, B) &= \inf\{\rho(x - y) : x \in A, y \in B\}; \\ \gamma_\rho(A, B) &= \max\left\{\inf\left\{\delta_\rho(x, B) : x \in A\right\},\right. \\ &\quad \left.\inf\left\{\delta_\rho(y, A) : y \in B\right\}\right\}. \end{aligned} \quad (3)$$

The following definitions are extensions of Definition 5.7 in [19] and are more adapted for a pair of subsets (A, B).

Definition 3. Let (A, B) be a ρ-bounded pair.

We will say that (H, K) is proximal ρ-admissible pair of (A, B) if

$$\begin{aligned} H &= \bigcap_{i\in I} B_\rho(y_i, r_i) \cap A \text{ and} \\ K &= \bigcap_{i\in I} B_\rho(x_i, r_i') \cap B \end{aligned} \quad (4)$$

where $(x_i, y_i) \in A \times B$, $r_i, r_i' \ge d_\rho(A, B)$, I is an arbitrary index set, and $B_\rho(x, r) = \{y \in X_\rho : \rho(x - y) \le r\}$ the standard ρ-closed ball of X_ρ. The family of all proximal ρ-admissible pair of (A, B) will be denoted by $\mathcal{Q}(A, B)$.

If $(D_1, D_2) \subseteq (A, B)$ we write

$$\begin{aligned} co_A^{D_2}(D_1) &= \bigcap_{y\in D_2} B_\rho(y, \delta_\rho(y, D_1)) \cap A \\ co_B^{D_1}(D_2) &= \bigcap_{x\in D_1} B_\rho(x, \delta_\rho(x, D_2)) \cap B. \end{aligned} \quad (5)$$

Remark 4. Note that $(co_A^{D_2}(D_1), co_B^{D_1}(D_2)) \in \mathcal{Q}(A, B)$ and is the smallest ρ-admissible pair of (A, B) which contains (D_1, D_2). Indeed, let $(H, K) \in \mathcal{Q}(A, B)$ such that $(D_1, D_2) \subseteq (H, K)$; then $H = \bigcap_{y\in D_2} B_\rho(y, r_y) \cap A$ and for each $x \in D_1$ and $y \in D_2$ we have $\rho(x - y) \le r_y$. Hence, $\delta_\rho(y, D_1) \le r_y$ since $D_1 \subseteq H$, which implies that

$$\begin{aligned} co_A^{D_2}(D_1) &= \bigcap_{y\in D_2} B_\rho(y, \delta_\rho(y, D_1)) \cap A \\ &\subseteq \bigcap_{y\in D_2} B_\rho(y, r_y) \cap A. \end{aligned} \quad (6)$$

In the same manner, we obtain $co_B^{D_1}(D_2) \subseteq K$.

Definition 5. Let (A, B) be a ρ-bounded pair.

(1) $\mathcal{Q}(A, B)$ is said to satisfy the property $(\mathcal{R})$-proximal, if for any sequence

$$\left(\{A_n\}_{n\ge1}, \{B_m\}_{m\ge1}\right) \subseteq \mathcal{Q}(A, B), \quad (7)$$

which are nonempty and decreasing, has a nonempty intersection.
(2) $\mathcal{Q}(A, B)$ is said to be proximal ρ-normal, if for each proximal ρ-admissible pair (H, K) not reduced to one point of (A, B) for which $dist_\rho(H, K) = dist_\rho(A, B)$ and $\delta_\rho(H, K) > dist_\rho(H, K)$ there exists $(x, y) \in H \times K$ such that

$$\begin{aligned} &\delta_\rho(x, K) < \delta_\rho(H, K) \text{ and} \\ &\delta_\rho(y, H) < \delta_\rho(H, K). \end{aligned} \quad (8)$$

(3) We say that the pair (A, B) is proximal ρ-sequentially compactness provided that every sequence $(\{x_n\}_n, \{y_n\}_n)$ of (A, B) satisfying the condition that $\rho(x_n - y_n) \longrightarrow dist_\rho(A, B)$ has a convergent subsequence in (A, B).

Remark 6. Notice that the $\mathcal{Q}(A,A)$ is proximal ρ-normal (resp., has the $(\mathcal{R})$-proximal property) if and only if $\mathcal{Q}(A)$ is ρ-normal (resp., has the $(\mathcal{R})$-property) in the sense of Khamsi-Kozlowski (see [19, Definition 5.7]).

Definition 7. A map $T : A \cup B \longrightarrow A \cup B$ will be said cyclic relatively ρ-nonexpansive on $A \cup B$ if

(1) $T(A) \subseteq B$ and $T(B) \subseteq A$

(2) $\rho(Tx - Ty) \leq \rho(x - y)$ for $x \in A$, $y \in B$

We conclude this section by a modular version of Kirk's fixed point theorem [20] which follows as a corollary from our former result Theorem 10 (see Corollary 11 below).

Theorem 8 (see [19, Theorem 5.9]). *Let A be a ρ-bounded and ρ-closed nonempty subset of X_ρ which satisfies $(\mathcal{R})$-property. Assume that $\mathcal{Q}(A)$ is ρ-normal. If $T : A \longrightarrow A$ is ρ-nonexpansive, then T has a fixed point.*

3. Best Proximity Results with ρ-Normal Structure

In what follows, we investigate the validity of technical lemma due to Gillespie and Williams [21] for a pair of ρ-admissible subsets in modular space. This result can be considered as the main ingredient and will play an important role in this article.

Lemma 9. *Let (A,B) be a ρ-bounded pair of X_ρ. Let $T : A \cup B \longrightarrow A \cup B$ be a cyclic relatively ρ-nonexpansive mapping. Assume that $\mathcal{Q}(A,B)$ is proximal ρ-normal. Let $(H,K) \in \mathcal{Q}(A,B)$ be a nonempty and T-cyclic pair; i.e., $T(H) \subseteq K$ and $T(K) \subseteq H$ with $dist_\rho(H,K) = dist_\rho(A,B)$ not reduced to one point. Then, there exists a nonempty T-cyclic pair $(H_0,K_0) \in \mathcal{Q}(A,B)$ such that $(H_0,K_0) \subseteq (H,K)$ and*

$$\delta_\rho(H_0,K_0) \leq \frac{\delta_\rho(H,K) + \gamma_\rho(H,K)}{2}. \tag{9}$$

Proof. Set $r = (1/2)(\delta_\rho(H,K) + \gamma_\rho(H,K))$. If $\delta_\rho(H,K) = dist_\rho(H,K)$ one can choose $(H_0,K_0) = (H,K)$. We assume that $\delta_\rho(H,K) > dist_\rho(H,K)$. Since $\mathcal{Q}(A,B)$ is proximal ρ-normal, we have

$$\gamma_\rho(H,K) < \delta_\rho(H,K) \tag{10}$$

and hence $\gamma_\rho(H,K) < r$. Thus, there exists $(x_1,y_1) \in H \times K$ such that

$$\begin{aligned} &\delta(x_1,K) < r \text{ and} \\ &\delta(y_1,H) < r. \end{aligned} \tag{11}$$

Let

$$\begin{aligned} D^H &= \bigcap_{y \in K} B_\rho(y,r) \cap H \text{ and} \\ D^K &= \bigcap_{x \in H} B_\rho(x,r) \cap K; \end{aligned} \tag{12}$$

then $(D^H,D^K) \neq \emptyset$ since $(x_1,y_1) \in D^H \times D^K$.

Let $\mathcal{F}$ denote the set of all nonempty pairs $\{(E_\alpha,F_\alpha)\}_{\alpha\in\Lambda}$ of $\mathcal{Q}(A,B)$ which are subsets of (A,B) such that T is cyclic on $E_\alpha \cup F_\alpha$ and $(D^H,D^K) \subseteq (E_\alpha,F_\alpha)$ with $dist_\rho(E_\alpha,F_\alpha) = dist_\rho(A,B)$ for all $\alpha \in \Lambda$. Obviously, $\mathcal{F}$ is nonempty since $(A,B) \in \mathcal{F}$. Defining (L_1,L_2) by

$$\begin{aligned} L_1 &= \bigcap_\alpha E_\alpha \text{ and} \\ L_2 &= \bigcap_\alpha F_\alpha \end{aligned} \tag{13}$$

it is clear that $(L_1,L_2) \neq \emptyset$ since $(D^H,D^K) \subset (L_1,L_2)$ and T is cyclic on $L_1 \cup L_2$, (E_α,F_α) is proximal ρ-admissible for each α so it is (L_1,L_2), and it is easy to check that $dist_\rho(L_1,L_2) = dist_\rho(A,B)$; thus $(L_1,L_2) \in \mathcal{F}$.

Let $M_1 = D^H \cup T(L_2)$ and $M_2 = D^K \cup T(L_1)$; it is claimed that

$$\begin{aligned} co_A^{M_2}(M_1) &= L_1 \text{ and} \\ co_B^{M_1}(M_2) &= L_2. \end{aligned} \tag{14}$$

Indeed, $M_1 \subset L_1$, $M_2 \subset L_2$ and the pair (L_1,L_2) is proximal ρ-admissible; then

$$\left(co_A^{M_2}(M_1), co_B^{M_1}(M_2)\right) \subseteq (L_1,L_2) \tag{15}$$

since $(co_A^{M_2}(M_1), co_B^{M_1}(M_2))$ is the smallest ρ-admissible pair which contains (M_1,M_2). Also,

$$\begin{aligned} T\left(co_A^{M_2}(M_1)\right) &\subseteq T(L_1) \text{ and} \\ T\left(co_B^{M_1}(M_2)\right) &\subseteq T(L_2) \end{aligned} \tag{16}$$

which implies

$$\begin{aligned} T\left(co_A^{M_2}(M_1)\right) &\subseteq M_2 \text{ and} \\ T\left(co_B^{M_1}(M_2)\right) &\subseteq M_1. \end{aligned} \tag{17}$$

Note that $dist_\rho(T(L_1),T(L_2)) = dist_\rho(L_1,L_2)$ since T is relatively ρ-nonexpansive mapping. And, since $(M_1,M_2) \subseteq (co_A^{M_2}(M_1), co_B^{M_1}(M_2))$ we get

$$dist_\rho\left(co_A^{M_2}(M_1), co_B^{M_1}(M_2)\right) = dist_\rho(A,B), \tag{18}$$

and hence $(co_A^{M_2}(M_1), co_B^{M_1}(M_2)) \in \mathcal{F}$; that is,

$$\left(co_A^{M_2}(M_1), co_B^{M_1}(M_2)\right) = (L_1,L_2). \tag{19}$$

Define

$$\begin{aligned} H_0 &= \bigcap_{y \in L_2} B_\rho(y,r) \cap L_1 \text{ and} \\ K_0 &= \bigcap_{x \in L_1} B_\rho(x,r) \cap L_2. \end{aligned} \tag{20}$$

We claim that (H_0,K_0) is the desired pair. Since $(D^H,D^K) \subseteq (H_0,K_0)$ the pair (H_0,K_0) is nonempty; also $(H_0,K_0) \in \mathcal{Q}(A,B)$.

Note that for each $x \in H_0$ and $y \in K_0$ we have

$$\rho(x-y) \le r \Longrightarrow \delta_\rho(H_0, K_0) \le r. \tag{21}$$

Next, we show that T is cyclic on $H_0 \cup K_0$ to complete the proof. Let $x \in H_0$; then,

$$\rho(Tx - Ty) \le \rho(x-y) \le r \quad (\forall y \in L_2) \tag{22}$$

since T is relatively ρ-nonexpansive. Thus, $T(L_2) \subset B_\rho(Tx, r)$.

Recall that $D^H = \bigcap_{y \in K} B_\rho(y, r) \cap H$; then if $z \in D^H$ we have for all $w \in K$

$$\rho(z-w) \le r \tag{23}$$

and since $(H,K) \in \mathscr{F}$, we get $L_2 \subset K$; then $L_2 \subset B(z,r)$. It is clear that $Tx \in L_2$; that is,

$$Tx \in B_\rho(z,r) \Longrightarrow z \in B_\rho(Tx, r) \tag{24}$$

and hence $D^H \subset B_\rho(Tx, r)$, which implies

$$L_1 = co_A^{M_2}\left(D^H \cup T(L_2)\right) \subseteq B_\rho(Tx, r) \cap A; \tag{25}$$

this deduces that $Tx \in K_0$; that is, $T(H_0) \subseteq K_0$. Similarly, we can see that $T(K_0) \subseteq H_0$. Since $(L_1, L_2) \subseteq (H, K)$ we get $(H_0, K_0) \subseteq (H, K)$. □

Theorem 10. *Let (A,B) be a nonempty ρ-bounded and ρ-closed pair in a modular space X_ρ. Moreover, assume that $\mathcal{Q}(A,B)$ has the proximal ρ-normal structure and the property $(\mathscr{R})$-proximal.*

If T is cyclic relatively ρ-nonexpansive on $A \cup B$, then there exists $(x,y) \in A \times B$ such that

$$\rho(x - Tx) = \rho(y - Ty) = dist_\rho(A,B). \tag{26}$$

Proof. Let $\mathscr{F}$ denote the set of all nonempty pairs (E,F) of $\mathcal{Q}(A,B)$ which are subsets of (A,B) such that T is cyclic on $E \cup F$ and $dist_\rho(E,F) = d_\rho$, where $d_\rho = dist_\rho(A,B)$.

$\mathscr{F}$ is nonempty since $(A,B) \in \mathscr{F}$. Define $\widetilde{\delta}_\rho : \mathscr{F} \longrightarrow [0,\infty)$ by

$$\widetilde{\delta}_\rho\left(D^A, D^B\right) = \inf\left\{\delta_\rho(E,F) : (E,F) \in \mathscr{F} \text{ and } (E,F) \subseteq \left(D^A, D^B\right)\right\}. \tag{27}$$

Set $(D_1^A, D_1^B) = (A,B)$; by definition of $\widetilde{\delta}_\rho$, there exists $(D_2^A, D_2^B) \in \mathscr{F}$ such that $(D_2^A, D_2^B) \subseteq (D_1^A, D_1^B)$, $dist_\rho(D_2^A, D_2^B) = dist_\rho(D_1^A, D_1^B) = d_\rho$, and

$$\delta_\rho\left(D_2^A, D_2^B\right) < \widetilde{\delta}_\rho\left(D_1^A, D_1^B\right) + 1 \tag{28}$$

and suppose that $(D_k^A, D_k^B)_{k=1,2,\ldots,n}$ are constructed for $n \ge 1$. Again, by definition of $\widetilde{\delta}_\rho$, there exists $(D_{n+1}^A, D_{n+1}^B) \subseteq (D_n^A, D_n^B)$ such that

$$\delta_\rho\left(D_{n+1}^A, D_{n+1}^B\right) < \widetilde{\delta}_\rho\left(D_n^A, D_n^B\right) + \frac{1}{n} \tag{29}$$

and $dist_\rho(D_{n+1}^A, D_{n+1}^B) = d_\rho$. Since $\mathcal{Q}(A,B)$ has the property $(\mathscr{R})$-proximal, $(D_\infty^A, D_\infty^B) \neq \emptyset$ where

$$D_\infty^A = \bigcap_{n\ge1} D_n^A \text{ and } D_\infty^B = \bigcap_{n\ge1} D_n^B, \tag{30}$$

note that $T(D_\infty^A) = T(\bigcap_n D_n^A) \subseteq \bigcap_n T(D_n^A) \subseteq \bigcap_n D_n^B = D_\infty^B$; in the same manner $T(D_\infty^B) \subseteq D_\infty^A$. Also, $(D_\infty^A, D_\infty^B) \in \mathcal{Q}(A,B)$ since $(D_n^A, D_n^B) \in \mathcal{Q}(A,B)$ for all $n \ge 1$. Also, we have

$$\begin{aligned} d_\rho(A,B) &\le d_\rho\left(D_\infty^A, D_\infty^B\right) \\ &= \inf\left\{\rho(x-y) : (x,y) \in D_\infty^A \times D_\infty^B\right\} \\ &= \inf\left\{\rho(x-y) : (x,y) \in \bigcap_{n\ge1} D_n^A \times \bigcap_{n\ge1} D_n^B\right\} \\ &= \inf\left\{\rho(x-y) : (x,y) \in D_n^A \times D_m^B \ (\forall n,m \in \mathbb{N}^*)\right\} \\ &= \inf_{n,m\in\mathbb{N}^*} d_\rho\left(D_n^A, D_m^B\right) \\ &\le d_\rho\left(D_n^A, D_n^B\right) = d_\rho(A,B) \end{aligned} \tag{31}$$

and then $(D_\infty^A, D_\infty^B) \in \mathscr{F}$.

If $\delta_\rho(D_\infty^A, D_\infty^B) = dist_\rho(D_\infty^A, D_\infty^B)$, then for each $(x,y) \in D_\infty^A \times D_\infty^B$ we get

$$\rho(x - Tx) = \rho(y - Ty) = dist_\rho\left(D_\infty^A, D_\infty^B\right). \tag{32}$$

Assume that $\delta_\rho(D_\infty^A, D_\infty^B) > dist_\rho(D_\infty^A, D_\infty^B)$.

First Case. If one of the pair (D_∞^A, D_∞^B) is reduced to one point, say, for example, $D_\infty^B = \{y\}$, since T is cyclic relatively ρ-nonexpansive on $D_\infty^A \cup D_\infty^B$ we get for all $x \in D_\infty^A$

$$\rho(Tx - Ty) = \rho(y - Ty) \le \rho(x-y) \tag{33}$$

which implies that

$$\rho(y - Ty) = \rho(x - Tx) = dist_\rho\left(D_\infty^A, D_\infty^B\right) \tag{34}$$

for $x = Ty$; note that $Tx = y$ for each $x \in D_\infty^A$.

Second Case. The pair (D_∞^A, D_∞^B) is not reduced to one point; by Lemma 9, there exists $(D_A^*, D_B^*) \subseteq (D_\infty^A, D_\infty^B)$

$$\delta_\rho\left(D_A^*, D_B^*\right) \le \frac{\delta_\rho\left(D_\infty^A, D_\infty^B\right) + \gamma_\rho\left(D_\infty^A, D_\infty^B\right)}{2} \tag{35}$$

which implies

$$\begin{aligned} \delta_\rho\left(D_A^*, D_B^*\right) &\le \delta_\rho\left(D_\infty^A, D_\infty^B\right) \\ &\le \delta_\rho\left(D_n^A, D_n^B\right) \\ &\le \widetilde{\delta}_\rho\left(D_n^A, D_n^B\right) + \frac{1}{n} \\ &\le \delta_\rho\left(D_A^*, D_B^*\right) + \frac{1}{n} \end{aligned} \tag{36}$$

since $(D_A^*, D_B^*) \subseteq (D_n^A, D_n^B)$ for any $n \geq 1$. If we let $n \longrightarrow \infty$, we get $\delta_\rho(D_A^*, D_B^*) = \delta_\rho(D_\infty^A, D_\infty^B)$. By (35) we get

$$\delta_\rho\left(D_\infty^A, D_\infty^B\right) \leq \gamma_\rho\left(D_\infty^A, D_\infty^B\right) \tag{37}$$

and this is in contradiction with the assumption that $\mathcal{Q}(A, B)$ is proximal ρ-normal. This completes the proof. □

If we set $A = B$, we get Theorem 8.

Corollary 11. *Let A be a ρ-bounded and ρ-closed nonempty subset of X_ρ. Assume that $\mathcal{Q}(A, A)$ is ρ-normal and satisfies the property $(\mathcal{R})$-proximal. If $T : A \longrightarrow A$ is ρ-nonexpansive, then T has a fixed point.*

We conclude by the following example.

Example 12. Let the real space $X = \{x = (x_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}^*} : \sum_{n\geq 1} |x_n|^{1/2} < \infty\}$, and define the modular functional $\rho : X \longrightarrow [0, \infty]$ by

$$\rho(x) = \sum_{n=1}^{\infty} |x_n|^{1/2}, \quad \text{for all } x = (x_n)_{n\geq 1} \in X \tag{38}$$

Suppose that $\{e_n\}$ is the canonical basis of X and let

$$\begin{aligned} A &= \left\{\frac{1}{2}e_1\right\} \cup \{e_3 + e_n : n \in \mathbb{N} \setminus \{0, 1, 3\}\} \text{ and} \\ B &= \left\{\frac{1}{2}e_3\right\} \cup \{e_1 + e_n : n \in \mathbb{N} \setminus \{0, 1\}\}. \end{aligned} \tag{39}$$

Then, (A, B) is ρ-bounded, ρ-closed in X_ρ, and not convex. Note that A (resp., B) is not ρ-sequentially compact, because the sequence $\{e_3 + e_n\}_{n\neq\{0,1,3\}}$ (resp., $\{e_1 + e_n\}_{n\neq\{0,1\}}$) does not have any ρ-convergent subsequence in A (resp., in B).

We have $\rho((1/2)e_1 - (1/2)e_3) = \sqrt{2}$; also, for all $x \in A$ and $n \in \mathbb{N}^*$, $\rho(x - e_1 - e_n) \geq \sqrt{2}$, which implies that $dist_\rho(A, B) = \sqrt{2}$.

$\mathcal{Q}(A, B)$ satisfies the property $(\mathcal{R})$-proximal; indeed, let $(\{H_n\}_{n\geq 1}, \{K_m\}_{m\geq 1})$ be a sequence of $\mathcal{Q}(A, B)$ which are nonempty and decreasing.

(1) If, for each $n \in \mathbb{N}^*$, $K_n = \bigcap_{j\in J_n} B_\rho(e_3 + e_{j,n}, r_{j,n}) \cap B$, so, for all $j \in J_n$, $(1/2)e_3 \in B_\rho(e_3 + e_{j,n}, 1 + \sqrt{1/2}) \cap B \subset \bigcap_{n\geq 1} K_n$, because $\rho(e_3 + e_{j,n} - (1/2)e_3) = 1 + \sqrt{1/2}$, and since $K_n \neq \emptyset$ for each $n \in \mathbb{N}^*$, we have $r_{j,n} \geq 1 + \sqrt{1/2}$. Hence, $\bigcap_{n\geq 1} K_n \neq \emptyset$.

(2) If, for each $n \in \mathbb{N}^*$, $K_n = \bigcap_{k\in J_n'} B_\rho((1/2)e_1, r_{k,n}) \cap B$, so $(1/2)e_3 \in B_\rho((1/2)e_1, \sqrt{2}) \cap B \subset \bigcap_{n\geq 1} K_n$, since $K_n \neq \emptyset$ for all $n \in \mathbb{N}^*$. Hence $\bigcap_{n\geq 1} K_n \neq \emptyset$.

(3) If there exists $n \in \mathbb{N}^*$ such that

$$\begin{aligned} K_n = &\left(\bigcap_{j\in J_n} B_\rho\left(e_3 + e_{j,n}, r_{j,n}\right)\right) \\ &\cap \left(\bigcap_{k\in J_n'} B_\rho\left(\frac{1}{2}e_1, r_{k,n}\right)\right) \cap B, \end{aligned} \tag{40}$$

we have $(1/2)e_3 \in B_\rho(e_3 + e_{j,n}, 1 + \sqrt{1/2}) \cap B_\rho((1/2)e_1, \sqrt{2}) \cap B \subset \bigcap_{n\geq 1} K_n$, and hence $\bigcap_{n\geq 1} K_n \neq \emptyset$.

Analogously, we obtain that $\bigcap_{n\geq 1} H_n \neq \emptyset$ replacing $e_3 + e_{j,n}$ and $(1/2)e_1$ by $e_1 + e_{j,n}$ and $(1/2)e_3$, respectively.

$\mathcal{Q}(A, B)$ has the proximal ρ-normal structure. Indeed, let (H, K) be a proximal ρ-admissible pair of (A, B) not reduced to one point for which $dist_\rho(H, K) = dist_\rho(A, B) = \sqrt{2}$ and $\delta_\rho(H, K) > dist_\rho(H, K)$; then $(1/2)e_1 \in H$ and $(1/2)e_3 \in K$. So, there exist $p, q \in \mathbb{N} \setminus \{0, 1, 3\} \times \mathbb{N} \setminus \{0, 1\}$ such that

$$\begin{aligned} &e_3 + e_p \in H \text{ and} \\ &e_1 + e_q \in K. \end{aligned} \tag{41}$$

Therefore, $\delta_\rho((1/2)e_1, K) = 1 + \sqrt{1/2}$ and $\delta_\rho((1/2)e_3, H) = 1 + \sqrt{1/2}$; then we get

$$\begin{aligned} \delta_\rho(H, K) &\geq \rho\left(e_3 + e_p - e_1 - e_q\right) \\ &> \max\left\{\delta_\rho\left(\frac{1}{2}e_1, K\right), \delta_\rho\left(\frac{1}{2}e_3, H\right)\right\}. \end{aligned} \tag{42}$$

Let $T : A \cup B \longrightarrow A \cup B$ be a mapping defined by

$$\begin{aligned} Ty &= \frac{1}{2}e_1 \quad \text{if } y \in B \text{ and} \\ Tx &= \begin{cases} \frac{1}{2}e_3 & \text{if } x = \frac{1}{2}e_1 \\ e_1 + e_2 & \text{if } x \in A \setminus \left\{\frac{1}{2}e_1\right\}. \end{cases} \end{aligned} \tag{43}$$

T is cyclic and for each $y \in B$ and $x = (1/2)e_1$

$$\rho(Tx - Ty) = \rho\left(\frac{1}{2}e_1 - \frac{1}{2}e_3\right) = \sqrt{2} \leq \rho(x - y), \tag{44}$$

and for each $x \in A \setminus \{(1/2)e_1\}$, $y \in B$

$$\begin{aligned} \rho(Tx - Ty) &= \rho\left(\frac{1}{2}e_1 - e_1 - e_2\right) = 1 + \sqrt{\frac{1}{2}} \\ &\leq \rho(x - y). \end{aligned} \tag{45}$$

Then, T is cyclic relatively ρ-nonexpansive on $A \cup B$. Therefore, all assumptions of Theorem 10 are satisfied, so T has a best proximity point; in particular

$$\begin{aligned} \rho\left(\frac{1}{2}e_3 - T\left(\frac{1}{2}e_3\right)\right) &= \rho\left(\frac{1}{2}e_1 - T\left(\frac{1}{2}e_1\right)\right) \\ &= dist_\rho(A, B). \end{aligned} \tag{46}$$

4. Best Proximity for Proximal ρ-Sequentially Compact Pair

In this section, we use (A_0, B_0) to denote the proximal pair obtained from (A, B) upon setting

$$A_0 = \left\{x \in A : \rho\left(x - y'\right) = dist_\rho(A, B) \text{ for some } y' \in B\right\}$$
$$B_0 = \left\{y \in B : \rho\left(x' - y\right) = dist_\rho(A, B) \text{ for some } x' \in A\right\}. \tag{47}$$

Lemma 13. *Let (A, B) be a nonempty ρ-bounded and proximal ρ-sequentially compactness pair in a modular space X_ρ for which ρ satisfies Fatou property. Then (A_0, B_0) is a nonempty ρ-sequentially compact pair of (A, B) such that $dist_\rho(A_0, B_0) = dist_\rho(A, B)$.*

Proof. It is clear that $dist_\rho(A_0, B_0) = dist_\rho(A, B)$. Let (x_n) and (y_n) be two sequences in A and B, respectively, such that

$$\rho(x_n - y_n) \longrightarrow dist_\rho(A, B). \tag{48}$$

Since (A, B) is a proximal ρ-compactness pair, there exist subsequences (x_{n_k}) and (y_{n_k}) of (x_n) and (y_n), respectively, such that $x_{n_k} \longrightarrow x \in A$ and $y_{n_k} \longrightarrow y \in B$ as $k \longrightarrow \infty$. Since ρ has Fatou property

$$\rho(x - y) \le \liminf_{k \longrightarrow \infty} \rho\left(x_{n_k} - y_{n_k}\right) = dist_\rho(A, B). \tag{49}$$

This implies that A_0 is nonempty, since $x \in A_0$. Similarly, we can see that B_0 is nonempty. The ρ-sequentially compact of A_0 is vacuous since for each sequence (x_n) of A_0 has a convergent subsequence for which this limit is in A_0 because A_0 is ρ-closed in A. Indeed, let $(x_n) \subset A_0$ such that $x_n \longrightarrow a$; then there exists a sequence (y_n) in B_0 such that

$$\rho(x_n - y_n) \longrightarrow dist_\rho(A, B) \tag{50}$$

and the proximal ρ-compactness of (A, B) implies the existence of subsequences (x_{n_k}) and (y_{n_k}) of (x_n) and (y_n), respectively, such that $x_{n_k} \longrightarrow x \in A$ and $y_{n_k} \longrightarrow y \in B$. Since ρ has Fatou property,

$$\rho(x - y) \le \liminf_{k \longrightarrow \infty} \rho\left(x_{n_k} - y_{n_k}\right) = dist_\rho(A, B). \tag{51}$$

Then $x \in A_0$; the uniqueness of the limit implies that $x = a$. Hence (A_0, B_0) is ρ-sequentially compact pair. □

Theorem 14. *Let (A, B) be a nonempty ρ-bounded and proximal ρ-sequentially compactness pair in a modular space X_ρ for which ρ has the Fatou property. Moreover, assume that $\mathcal{Q}(A, B)$ has the proximal ρ-normal structure.*

If T is cyclic relatively ρ-nonexpansive on $A \cup B$, then there exists $(x, y) \in A \times B$ such that

$$\rho(x - Tx) = \rho(y - Ty) = dist(A, B). \tag{52}$$

Proof. Let $x_0 \in A_0$; then there exists $y_0 \in B$ such that

$$\rho(Tx_0 - Ty_0) \le \rho(x_0 - y_0) = d_\rho \tag{53}$$

That is, $Tx_0 \in B_0$. Hence $T(A_0) \subseteq B_0$, similarly, $T(B_0) \subseteq A_0$ and T is cyclic relatively ρ-nonexpansive on $A_0 \cup B_0$.

Let $\mathcal{F}$ denote the set of all nonempty ρ-closed pairs (E, F) of $\mathcal{Q}(A, B)$ which are subsets of (A, B) such that T is cyclic on $E \cup F$ and $\rho(x - y) = d_\rho$ for some $(x, y) \in E \times F$, where $d_\rho = dist_\rho(A, B)$. Thus, $\mathcal{F}$ is nonempty since $(A, B) \in \mathcal{F}$.

Define $\widetilde{\delta}_\rho : \mathcal{F} \longrightarrow [0, \infty)$ by

$$\widetilde{\delta}_\rho\left(D^A, D^B\right) = \inf\left\{\delta_\rho(E, F) : (E, F) \in \mathcal{F} \text{ and } (E, F) \subseteq \left(D^A, D^B\right)\right\}. \tag{54}$$

Set $(D_1^A, D_1^B) = (A, B)$; by definition of $\widetilde{\delta}_\rho$, there exists $(D_2^A, D_2^B) \in \mathcal{F}$ such that $(D_2^A, D_2^B) \subseteq (D_1^A, D_1^B)$, $dist_\rho(D_2^A, D_2^B) = d_\rho$, and

$$\delta_\rho\left(D_2^A, D_2^B\right) < \widetilde{\delta}_\rho\left(D_1^A, D_1^B\right) + 1 \tag{55}$$

Suppose that $(D_k^A, D_k^B)_{k=1,2,\ldots,n}$ are constructed for $n \ge 1$. Again, by definition of $\widetilde{\delta}_\rho$, there exists $(D_{n+1}^A, D_{n+1}^B) \subseteq (D_n^A, D_n^B)$ such that

$$\delta_\rho\left(D_{n+1}^A, D_{n+1}^B\right) < \widetilde{\delta}_\rho\left(D_n^A, D_n^B\right) + \frac{1}{n} \tag{56}$$

and $dist_\rho(D_{n+1}^A, D_{n+1}^B) = d_\rho$. Since (A_0, B_0) is ρ-sequentially compact, $(D_\infty^A, D_\infty^B) \neq \emptyset$, where

$$D_\infty^A = \bigcap_{n\ge1} D_n^A \text{ and } D_\infty^B = \bigcap_{n\ge1} D_n^B. \tag{57}$$

Indeed, one can choose two sequences (x_n) and (y_n) such that $(x_n, y_n) \in (D_n^A \cap A_0) \times (D_n^B \cap B_0)$ for each $n \ge 1$ and

$$\rho(x_n - y_n) \longrightarrow d_\rho \tag{58}$$

Using the proximal ρ-compactness of (A_0, B_0), there exists (x_{n_k}) of (x_n) and (y_{n_k}) of (y_n) such that $x_{n_k} \longrightarrow x(\rho)$ and $y_{n_k} \longrightarrow y(\rho)$; let $p \ge 1$ and define two subsets of A_0 and B_0 as follows:

$$C_p^A = \left\{x_{n_k} : k \ge p\right\} \text{ and } C_p^B = \left\{y_{n_k} : k \ge p\right\} \tag{59}$$

Hence $x \in \bigcap_p C_p^A$ and $y \in \bigcap_p C_p^B$. Thus, $x \in \bigcap_{n\ge1} D_n^A = \bigcap_{k\ge1} D_{n_k}^A = D_\infty^A$ and $y \in \bigcap_{n\ge1} D_n^B = \bigcap_{k\ge1} D_{n_k}^B = D_\infty^B$. Also, ρ satisfies Fatou property and we get

$$\rho(x - y) = dist_\rho\left(D_\infty^A, D_\infty^B\right) = dist_\rho(A, B). \tag{60}$$

Note that

$$T\left(D_{\infty}^{A}\right)=T\left(\bigcap_{n}D_{n}^{A}\right)\subseteq\bigcap_{n}T\left(D_{n}^{A}\right)\subseteq\bigcap_{n}D_{n}^{B}=D_{\infty}^{B} \quad (61)$$

In the same manner $T(D_{\infty}^{B}) \subseteq D_{\infty}^{A}$. Hence, $(D_{\infty}^{A}, D_{\infty}^{B}) \in \mathscr{Q}(A,B)$ since $(D_{n}^{A}, D_{n}^{B}) \in \mathscr{Q}(A,B)$ for all $n \geq 1$, which implies $(D_{\infty}^{A}, D_{\infty}^{B}) \in \mathscr{F}$.

In this step, we can use the same argument as Theorem 14 to prove that

$$\delta_{\rho}\left(D_{\infty}^{A}, D_{\infty}^{B}\right)=dist_{\rho}\left(D_{\infty}^{A}, D_{\infty}^{B}\right) \quad (62)$$

Hence we get for each $(x,y) \in D_{\infty}^{A} \times D_{\infty}^{B}$

$$\rho(x-Tx)=\rho(y-Ty)=dist_{\rho}\left(D_{\infty}^{A}, D_{\infty}^{B}\right) \quad (63)$$

which completes the proof. □

Corollary 15. *Let A be a ρ-bounded and ρ-sequentially compact nonempty subset of X_{ρ} which satisfies Fatou property. Assume that $\mathscr{Q}(A,A)$ is ρ-normal. If $T : A \longrightarrow A$ is ρ-nonexpansive, then T has a fixed point.*

We conclude by the following example.

Example 16. Let the real space $X = \{x=(x_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}^*} : \sum_{n\geq 1}|x_n|^{1/2} < \infty\}$ and define the modular functional $\rho : X \longrightarrow [0,\infty]$ by

$$\rho(x)=\sum_{n=1}^{\infty}\left|x_n\right|^{1/2}, \quad \text{for all } x=\left(x_n\right)_{n\geq 1}\in X \quad (64)$$

Suppose that $\{e_n\}$ is the canonical basis of X and let

$$A=\left\{\frac{1}{2}e_1\right\}\cup\left\{e_2+e_n : n\in\mathbb{N}\setminus\{0,1,2\}\right\} \text{ and}$$
$$B=\left\{e_1, e_2, \frac{1}{16}e_3\right\}. \quad (65)$$

Then, (A,B) is ρ-bounded and not convex. Let $u = (1/2)e_1$ in A; we have $\rho(u-e_1) = \sqrt{1/2}$, also for each $(x,y) \in A \times B$, $\rho(x-y) \geq \sqrt{1/2}$, which implies that $dist_{\rho}(A,B) = \sqrt{1/2}$.

Also, for all $(x_n) \subset A$ and $(y_n) \subset B$ such that $\lim_n \rho(x_n - y_n) = \sqrt{1/2}$ there exists $n_0 \in \mathbb{N}$ such that

$$x_n=\frac{1}{2}e_1 \text{ and}$$
$$y_n=e_1 \quad (66)$$

for each $n \geq n_0$, so (x_n) and (y_n) are ρ-convergent sequences and the pair (A,B) is proximal ρ-sequentially compactness. However, A is not ρ-sequentially compact since the sequence $\{e_2+e_n\}_{n\neq\{0,1,2\}}$ does not have any ρ-convergent subsequence in A.

$\mathscr{Q}(A,B)$ has the proximal ρ-normal structure. Indeed, let (H,K) be a proximal ρ-admissible pair of (A,B) not reduced to one point for which $dist_{\rho}(H,K) = dist_{\rho}(A,B) = \sqrt{1/2}$; then $(1/2)e_1 \in H$ and $e_1 \in K$. Also, $\delta_{\rho}(H,K) > dist_{\rho}(H,K)$, so there exits $m \notin \{0,1,2\}$ such that $e_2+e_m \in H$ and $e_2 \in K$ or $(1/16)e_3 \in K$.

If $K = \{e_1, (1/16)e_3\}$ we obtain $\delta_{\rho}((1/2)e_1, K) = 1/4 + \sqrt{1/2}$ and

$$\delta_{\rho}\left(\frac{1}{16}e_3, H\right)=\begin{cases}1+\dfrac{\sqrt{15}}{4} & \text{if } H=\left\{\dfrac{1}{2}e_1, e_2+e_3\right\}\\ 2+\dfrac{1}{4} & \text{otherwise,}\end{cases} \quad (67)$$

and hence $\delta_{\rho}(H,K) \geq \rho(e_2+e_m-e_1) > \max\{\delta_{\rho}((1/2)e_1, K), \delta_{\rho}((1/16)e_3, H)\}$.

If $e_2 \in K$, then, $\delta_{\rho}((1/2)e_1, K) = 1+\sqrt{1/2}$ and $\delta_{\rho}(e_2, H) = 1+\sqrt{1/2}$. Hence, we have

$$\delta_{\rho}(H,K)\geq\rho\left(e_2+e_m-e_1\right)$$
$$>\max\left\{\delta_{\rho}\left(\frac{1}{2}e_1, K\right), \delta_{\rho}\left(e_2, H\right)\right\}. \quad (68)$$

Let $T : A \cup B \longrightarrow A \cup B$ be a mapping defined by

$$Ty=\frac{1}{2}e_1 \quad \text{if } y\in B, \text{ and}$$
$$Tx=\begin{cases}e_1 & \text{if } x=\dfrac{1}{2}e_1\\ \dfrac{1}{16}e_3 & \text{if } x\in A\setminus\left\{\dfrac{1}{2}e_1\right\},\end{cases} \quad (69)$$

So, for each $y \in B$ and $x = (1/2)e_1$, we get $\rho(Tx-Ty) = \sqrt{1/2} \leq \rho(x-y)$, and, for each $y \in B$ and $x \in A \setminus \{(1/2)e_1\}$, we obtain, $\rho(Tx-Ty) = (1+2\sqrt{2})/4 \leq \rho(x-y)$.

Then, T is cyclic relatively ρ-nonexpansive on $A \cup B$. Therefore, all assumptions of Theorem 14 are satisfied, so T has a best proximity point; in particular

$$\rho\left(e_1-Te_1\right)=\rho\left(\frac{1}{2}e_1-T\left(\frac{1}{2}e_1\right)\right)=dist_{\rho}(A,B). \quad (70)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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