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Research Article

On the Relationship between the Inhomogeneous Wave and Helmholtz Equations in a Fractional Setting

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We study convergence of solutions of a space and time inhomogeneous fractional wave equation on the quarter-plane to the stationary regime described by solutions of the Helmholtz equation.

1. Introduction and Main Result

Consider an inhomogeneous wave equation on the half-line with a time-periodic forcing term

$$u_{tt}(x,t) - u_{xx}(x,t) = e^{i\lambda t}\psi(x), \quad t \geq 0, \ x \geq 0, \tag{1}$$

where $\psi(x)$ is, for simplicity, a C^∞ function with compact support. Closely related to it is the one-dimensional Helmholtz equation

$$v_{xx}(x) + \lambda^2 v(x) = -\psi(x). \tag{2}$$

On the one hand, for any solution $v(x)$ of the Helmholtz equation (2) the function $v(x)e^{i\lambda t}$ solves the wave equation (1); on the other, there are pairs of solutions $u(x,t)$ and $v(x)$ of (1) and (2), respectively, such that $u(x,t) - v(x)e^{i\lambda t} \longrightarrow 0$ as $t \longrightarrow \infty$.

Results of this type go back at least to [1] (for the wave equation in $\mathbb{R}^3$) and are known under the name of radiation principles. For an extensive treatment see the classical work [2]. The case of (1) will be recalled in Section 3.

In this paper we study a very simple question that clarifies, to us, the role of the fractional analogue, (5) below, of the Helmholtz equation (2): to what extent does a similar result hold in fractional settings? For concreteness, we chose an inhomogeneous fractional wave equation on a half-line

$$D_x^\alpha u(x,t) - {}^C D_t^\beta u(x,t) = \varepsilon(t) f(x), \quad x,t \geq 0. \tag{3}$$

Here D_x^α denotes the Riemann-Liouville fractional derivative, ${}^C D_t^\beta$ denotes the Caputo fractional derivative defined in Section 2, and $\varepsilon(t)$ is a fractional-harmonic function (see below), and $1 < \alpha, \beta < 2$. In various settings, (3) is a subject of active research; see [3] and related literature. The term *fractional diffusion-wave equation* is also used in the literature.

By a *fractional-harmonic* function $\varepsilon(t)$ we mean in this context a solution of the fractional differential equation (FDE)

$${}^C D_t^\beta \varepsilon(t) = \Omega \varepsilon(t), \tag{4}$$

where $\Omega \in \mathbb{C} \setminus 0$ is a fixed constant. Taking a periodic $\varepsilon(t)$, as in [4], does not seem to lead to a nice result.

By [5, (3.1.34)], as $\varepsilon(t)$ we can take any linear combination of Mittag-Leffler functions $E_\beta(\Omega t^\beta)$ and $tE_{\beta,2}(\Omega t^\beta)$ defined in (14) below. This is quite natural since in the limiting case $\beta = 2$, $E_2(-t^2) = \cos t$, and $tE_{\beta,2}(-t^2) = \sin t$. We will work out only the case $\varepsilon(t) = E_\beta(\Omega t^\beta)$; an arbitrary linear combination leads to similar results.

Let now $v(x)$ be some function satisfying the fractional Helmholtz equation

$$D_x^\alpha v(x) - \Omega v(x) = f(x), \quad x \geq 0. \tag{5}$$

The classically known solutions of (5) are given in Theorem 4 below. Note that more recently related multidimensional equations have been solved, e.g., in [6] by the spectral method and in [7] in the form of an integral representation.

Then

$$D_x^\alpha (v(x)\varepsilon(t)) - {}^C D_t^\beta (v(x)\varepsilon(t)) = \varepsilon(t)\left(D_x^\alpha v(x) - \Omega v(x)\right) = \varepsilon(t) f(x), \tag{6}$$

and so the function

$$u(x,t) = v(x) E_\beta\left(\Omega t^\beta\right) \tag{7}$$

satisfies the fractional inhomogeneous wave equation (3).

Conversely, do solutions $u(x,t)$ of (3) behave, for $t \longrightarrow \infty$, as $v(x)E_\beta(\Omega t^\beta)$? Our main result shows that the answer is "not quite."

Before formulating the results, notice that, depending on the phase of the complex number Ω, the function $E_\beta(\Omega t^\beta)$ either is exponentially growing, or is bounded by a constant, or behaves as $O(t^{-\beta})$ as $t \longrightarrow \infty$; see [8, Theorems 1.3, 1.4]. Therefore any interesting result on the behavior of $u(x,t) - v(x)E_\beta(\Omega t^\beta)$ for $t \longrightarrow \infty$ ought to include information on asymptotic terms of order $O(t^{-\beta})$ and smaller.

The main results of this paper are as follows.

Theorem 1. *Let $1 < \beta \le \alpha < 2$, $\Omega \in \mathbb{C}\setminus 0$, and $\varepsilon(t) = E_\beta(\Omega t^\beta)$, and let $f(x)$ be a C^∞ function with compact support on $[0,\infty)$. Then there exist the following:*

(a) A solution $u(x,t)$ of the fractional wave equation (3) satisfying $u(x,0) = u_t(x,0) = 0$, $D_x^{\alpha-2}u(0+,t) = 0$

(b) A solution $v(x)$ of the fractional Helmholtz equation (5) satisfying $D_x^{\alpha-2}v(0+) = 0$,

such that, as $t \longrightarrow +\infty$,

$$\begin{aligned} u(x,t) - E_\beta\left(\Omega t^\beta\right) v(x) &= \frac{\sin\beta\pi}{\pi}\Gamma(\beta) C_{\beta-1}(x)\frac{1}{t^\beta} \\ &+ \frac{\sin(\beta+\beta/\alpha)\pi}{\pi}\Gamma\left(\beta+\frac{\beta}{\alpha}\right) C_{\beta-1+\beta/\alpha}(x)\frac{1}{t^{\beta+\beta/\alpha}} \\ &+ \frac{\sin 2\beta\pi}{\pi}\Gamma(2\beta) C_{2\beta-1}(x)\frac{1}{t^{2\beta}} + O\left(\frac{1}{t^{\beta+2\beta/\alpha}}\right), \end{aligned} \tag{8}$$

where the O symbol should be understood pointwise with respect to x and the functions $C_{\beta-1}$, $C_{\beta-1+\beta/\alpha}$, and $C_{2\beta-1}$ are given in (66), (67), and (68).

Thus, if Ω is such that $E_\beta(\Omega t^\beta) = O(t^{-\beta})$, the right-hand side in (8) is of the same order as $u(x,t)$.

Theorem 1 is proven in Section 4. The proof follows, with complications specific to fractional calculus, the outline of the case of the wave equation, Section 3, and of the paper [1].

The case $\beta > \alpha$ is much less natural. Indeed, a prototype for this case would be a heat equation with a *space*-harmonic source term and the limit as the space variable tends to infinity. We are not aware of any good result for such an equation which we would want to generalize to the fractional case. Nevertheless, if we impose a condition $(3/2)\alpha > \beta > \alpha$, we are able to prove the following statement; see Section 5.

Theorem 2. *Let $1 < \alpha < \beta < \min\{3\alpha/2, 2\}$, $\delta = \alpha(1-\beta^{-1})$, $\Omega \in \mathbb{C}\setminus 0$, and $\varepsilon(t) = E_\beta(\Omega t^\beta)$, and let $f(x)$ be a C^∞ function with compact support on $[0,\infty)$. Then there exist the following:*

(a) A solution $u(x,t)$ of the fractional wave equation (3) satisfying $u(x,0) = u_t(x,0) = 0$,

$$\begin{aligned} u(x,0) &= \int_0^x (x-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(\Omega(x-\xi)^\alpha\right) f(\xi)\,d\xi, \\ u_t(x,0) &= \int_0^x (x-\xi)^{\delta-1} E_{\alpha,\delta}\left(\Omega(x-\xi)^\alpha\right) f(\xi)\,d\xi, \\ D_x^{\alpha-2}u(0+,t) &= D_x^{\alpha-1}u(0+,t) = 0; \end{aligned} \tag{9}$$

(b) a solution $v(x)$ of the fractional Helmholtz equation (5) satisfying $D_x^{\alpha-2}v(0+) = D_x^{\alpha-1}v(0+) = 0$,

such that, as $t \longrightarrow +\infty$,

$$\begin{aligned} u(x,t) - E_\beta\left(\Omega t^\beta\right) v(x) &= \frac{\sin\beta\pi}{\pi}\Gamma(\beta) B_{\beta-1}(x)\frac{1}{t^\beta} \\ &+ \frac{\sin 2\beta\pi}{\pi}\Gamma(2\beta) B_{2\beta-1}(x)\frac{1}{t^{2\beta}} + O\left(\frac{1}{t^{3\beta}}\right), \end{aligned} \tag{10}$$

where the O symbol should be understood pointwise with respect to x and the functions $B_{\beta-1}$, $B_{2\beta-1}$ are given in (89) and (90).

The functions u and v constructed in the proofs of the above theorems are such that the corresponding fractional derivatives exist in the sense of definitions (11) and (13) below, in which the integral is the Lebesgue integral and the derivatives are taken in the sense of elementary calculus. In this paper we do not address the questions of uniqueness of solutions and appropriate functional spaces where such uniqueness would hold; therefore we phrase our results in the language "there exists a solution..." rather than "the solution...."

It would be curious to have an interpretation of the functions $C_{\beta-1}(x)$ and $B_{\beta-1}(x)$.

2. Preliminaries

In this section we collect some definitions and results from fractional calculus in order to make the paper self-contained.

The *Riemann-Liouville fractional derivative* of order $\alpha > -1$, $\alpha \notin \mathbb{Z}$, of a function $f(x)$ is defined ([5, §2.1] for $\alpha > 0$ and, in different notation, [9] for $-1 < \alpha < 0$) as

$$\left(D^\alpha f\right)(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_0^x \frac{f(\xi)}{(x-\xi)^{\alpha-n+1}}d\xi, \quad n = [\alpha]+1,\ x > 0. \tag{11}$$

An additional subscript x in D_x^α will emphasize that the differentiation is with respect to x.

Remark 3. Note that the usual formula for differentiation under the integral sign

$$\frac{d}{dx}\left(\int_0^x \varphi(x,\xi)\,d\xi\right) = \varphi(x,x) + \int_0^x \frac{\partial}{\partial x}\varphi(x,\xi)\,d\xi \quad (12)$$

is not applicable in (11) because the derivative of the integrand is not integrable at zero.

The *Caputo fractional derivative* of order $\beta > 0$, $\beta \notin \mathbb{Z}$, of a function $f(t)$ can be defined [5, §2.4] by

$$\left({}^{C}D^{\beta} f\right)(t) = \frac{1}{\Gamma(m-\beta)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\beta-m+1}}d\tau, \qquad m = [\beta]+1,\ t>0. \quad (13)$$

An additional subscript t in ${}^{C}D_t^{\beta}$ will emphasize that the differentiation is with respect to t.

Some analytical conditions need to be imposed on the function f to guarantee the existence of its fractional derivatives; see [5].

The following *Mittag-Leffler functions* treated in detail in [5, §1.8] and [8, §1.2] play the same role for FDEs as does the function e^x in the theory of integer-order differential equations:

$$E_{\alpha}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)}, \qquad E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)}. \quad (14)$$

There is the following result on FDEs with the Riemann-Liouville derivative.

Theorem 4 (see [5, §3.1], [9]). *If $\alpha > 0$, $\alpha \notin \mathbb{Z}$, $n = [\alpha]+1$, $f(x)$ is an integrable function on $[0,b)$ for some b, and $\lambda \in \mathbb{C}$ is a constant, then the initial-value problem*

$$\left(D^{\alpha}y\right)(x) - \lambda y(x) = f(x); \qquad \left(D^{\alpha-k}y\right)(0+) = b_k, \quad k = 1,\dots,n, \quad (15)$$

has a unique solution

$$y(x) = \sum_{k=1}^{n} b_k x^{\alpha-k}E_{\alpha,\alpha-k+1}(\lambda x^{\alpha}) + \int_0^x (x-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda(x-\xi)^{\alpha}\right)f(\xi)\,d\xi. \quad (16)$$

This theorem indicates the correct way to set up the initial conditions for equations with the Riemann-Liouville fractional derivative. By contrast, [5, §3.5], one sets up the initial value problem for an FDE with Caputo derivative, say, $({}^{C}D^{\beta}y)(t) = f(t, y(t))$ in the usual way: $y^{(k)}(0) = b_k$, $k = 0,\dots,[\beta]$.

3. Case of the Classical Inhomogeneous Wave Equation

In this section we recall the case of the usual wave equation in order to motivate the fractional version of the argument and especially the formulas (30)-(31).

It is well-known (see, e.g., [10, Lecture 13]) that the function

$$u(x,t) = \frac{1}{2}\int_0^t ds\int_{x-(t-s)}^{x+(t-s)} dy f(y,s) \quad (17)$$

is a solution of the initial value problem

$$u_{tt} - u_{xx} = f(x,t), \quad t \ge 0,\ -\infty < x < \infty; \qquad u(x,0) = 0, \qquad u_t(x,0) = 0. \quad (18)$$

Define the Laplace transforms with respect to t:

$$\tilde{u}(x,\gamma) = \int_0^{\infty} e^{-t\gamma}u(x,t)\,dt; \qquad \tilde{f}(x,\gamma) = \int_0^{\infty} e^{-t\gamma}f(x,t)\,dt. \quad (19)$$

Then (17), after simple manipulations, yields

$$\tilde{u}(x,\gamma) = \frac{1}{2\gamma}e^{\gamma x}\int_x^{\infty} d\xi e^{-\gamma\xi}\tilde{f}(\xi,\gamma) + \frac{1}{2\gamma}e^{-\gamma x}\int_{-\infty}^{x} d\eta e^{\gamma\eta}\tilde{f}(\eta,\gamma). \quad (20)$$

By the method of images we know that if $f(x,t)$ is originally defined for $x \ge 0$ and extended to the negative axis by $f(-x,t) = -f(x,t)$, then $u(x,t)$ from (17) solves the boundary-value problem on the half-line:

$$u_{tt} - u_{xx} = f(x,t), \quad t \ge 0,\ x \ge 0; \qquad u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad u(0,t) = 0. \quad (21)$$

In this case (20) becomes

$$\tilde{u}(x,\gamma) = \frac{e^{\gamma x}}{2\gamma}\int_x^{\infty} d\xi e^{-\gamma\xi}\tilde{f}(\xi,\gamma) + \frac{e^{-\gamma x}}{2\gamma}\int_0^{x} d\xi e^{\gamma\xi}\tilde{f}(\xi,\gamma) - \frac{e^{-\gamma x}}{2\gamma}\int_0^{\infty} d\xi e^{-\gamma\xi}\tilde{f}(\xi,\gamma). \quad (22)$$

If the forcing term $f(x,t)$ in (21) is periodic in time,

$$f(x,t) = \psi(x)e^{i\lambda t}, \qquad \tilde{f}(x,\gamma) = \frac{1}{\gamma - i\lambda}\psi(x), \quad (23)$$

we get

$$\tilde{u}(x,\gamma)=\frac{1}{2\gamma(\gamma-i\lambda)}\left[e^{\gamma x}\int_x^\infty d\xi e^{-\gamma\xi}\psi(\xi)\right.$$
$$\left.+e^{-\gamma x}\int_0^x d\xi e^{\gamma\xi}\psi(\xi)-e^{-\gamma x}\int_0^\infty d\xi e^{-\gamma\xi}\psi(\xi)\right]. \quad (24)$$

On the other hand, the Helmholtz equation on the half-line

$$v_{xx}(x)-\omega^2 v(x)=\varphi(x),\quad x\geq 0,\ v(0)=0 \quad (25)$$

has a solution

$$v(x)=-\frac{1}{2\omega}e^{\omega x}\int_x^\infty d\xi e^{-\omega\xi}\varphi(\xi)$$
$$-\frac{1}{2\omega}e^{-\omega x}\int_0^x d\xi e^{\omega\xi}\varphi(\xi) \quad (26)$$
$$+\frac{1}{2\omega}e^{-\omega x}\int_0^\infty d\xi e^{-\omega\xi}\varphi(\xi).$$

Now let us understand the behavior of $u(x,t)$ for large t, pointwise with respect to x, assuming $\operatorname{supp}\psi\subset[0,M]$. If $t>x+M$, the Bromwich integral that expresses $u(x,t)$ via (24) can be closed on the left, yielding that $u(x,t)$ is simply a sum of residues:

$$u(x,t)=e^{\gamma t}\frac{1}{2(-i\lambda)}\left[\left[\int_x^\infty d\xi\psi(\xi)+\int_0^x d\xi\psi(\xi)\right.\right.$$
$$\left.-\int_0^\infty d\xi\psi(\xi)\right]+e^{i\lambda t}\frac{1}{2i\lambda}\left[e^{i\lambda x}\int_x^\infty d\xi e^{-i\lambda\xi}\psi(\xi)\right. \quad (27)$$
$$\left.+e^{-i\lambda x}\int_0^x d\xi e^{i\lambda\xi}\psi(\xi)-e^{-i\lambda x}\int_0^\infty d\xi e^{-i\lambda\xi}\psi(\xi)\right].$$

Looking at (26) we recognize that

$$u(x,t)=e^{i\lambda t}v(x),\quad t>x+M, \quad (28)$$

where $v(x)$ is a solution of the Helmholtz equation above with $\omega=i\lambda$ and $\varphi=-\psi$:

$$v_{xx}(x)+\lambda^2 v(x)=-\psi(x). \quad (29)$$

4. Case $\beta\leq\alpha$

This section contains the proof of Theorem 1. We begin by constructing a solution $u(x,t)$ of the inhomogeneous fractional wave equation using the method of Laplace transform. Later on $b(\gamma)$ in Lemma 5 will be specialized to the Laplace transform of a Mittag-Leffler function.

Lemma 5. *Let*

(a) $f(x)$ be a continuous function with compact support

(b) $b(\gamma)$ be an analytic function defined in $\mathbb{C}\setminus[D(0,R)\cup(-\infty,0]]$ estimated as $O(|\gamma|^{-1})$ as $|\gamma|\longrightarrow\infty$ in that region

(c) $\int_{c-i\infty}^{c+i\infty}e^{\gamma t}b(\gamma)d\gamma$ be defined for $t=0$ in the sense of principal value

Then the function

$$u(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{t\gamma}\mathcal{F}(x,\gamma)\,d\gamma,\quad t\geq 0, \quad (30)$$

where

$$\mathcal{F}(x,\gamma)=b(\gamma)\int_0^x d\xi f(\xi)$$
$$\cdot\left[(x-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^\beta(x-\xi)^\alpha\right)\right.$$
$$\left.-e^{-\xi\gamma^{\beta/\alpha}}x^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^\beta x^\alpha\right)\right]-b(\gamma) \quad (31)$$
$$\cdot\int_x^\infty d\xi e^{-\xi\gamma^{\beta/\alpha}}x^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^\beta x^\alpha\right)f(\xi)$$

for $x\geq 0$ solves the FDE

$$D_x^\alpha u(x,t)-{}^C D_t^\beta u(x,t)$$
$$=f(x)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\gamma t}b(\gamma)\,d\gamma. \quad (32)$$

Moreover, $u(x,t)$ satisfies the boundary condition

$$D_x^{\alpha-2}u(0+,t)=0 \quad (33)$$

and

$$u(x,0)=u_t(x,0)=0. \quad (34)$$

Remark that since $xE_{2,2}(x^2)=\sinh x$, the function (31) is a fractional generalization of (22). While the statement of the lemma is quite natural on the algebraic level, various analytic justifications need to be carried out.

Proof of the Lemma. Denote

$$\mathcal{F}_1(x,\gamma)=\int_0^x d\xi f(\xi)\left[(x-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^\beta(x-\xi)^\alpha\right)\right.$$
$$\left.-e^{-\xi\gamma^{\beta/\alpha}}x^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^\beta x^\alpha\right)\right], \quad (35)$$

$$\mathcal{F}_2(x,\gamma)=\int_x^\infty d\xi e^{-\xi\gamma^{\beta/\alpha}}x^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^\beta x^\alpha\right)f(\xi). \quad (36)$$

Let us first show that the integral (30) converges. According to [8, Th.1.3], for $|\gamma|\longrightarrow\infty$, $\Re\gamma\geq 0$,

$$(x-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^\beta(x-\xi)^\alpha\right)$$
$$\sim\frac{1}{\alpha}\gamma^{\beta(1-\alpha)/\alpha}e^{\gamma^{\beta/\alpha}(x-\xi)}-\sum_{k=2}^\infty\frac{\gamma^{-\beta k}(x-\xi)^{(1-k)\alpha-1}}{\Gamma(\alpha-\alpha k)};$$
$$e^{-\xi\gamma^{\beta/\alpha}}x^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^\beta x^\alpha\right) \quad (37)$$
$$\sim e^{-\xi\gamma^{\beta/\alpha}}\frac{1}{\alpha}\gamma^{\beta(1-\alpha)/\alpha}e^{\gamma^{\beta/\alpha}x}-e^{-\xi\gamma^{\beta/\alpha}}\sum_{k=2}^\infty\frac{\gamma^{-k\beta}x^{(1-k)\alpha-1}}{\Gamma(\alpha-\alpha k)}$$

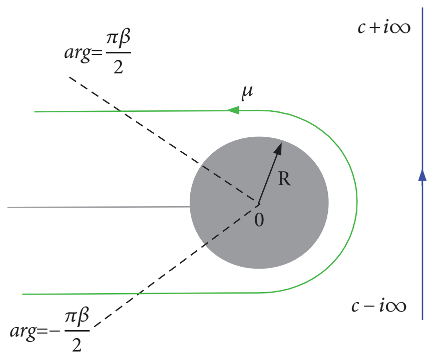

Figure 1: The contour μ in the γ-plane.

and so the bracket in (35) has an asymptotic expansion

$$\left[e^{-\xi\gamma^{\beta/\alpha}}\sum_{k=2}^{\infty}\frac{\gamma^{-k\beta}x^{(1-k)\alpha-1}}{\Gamma(\alpha-\alpha k)}-\sum_{k=2}^{\infty}\frac{\gamma^{-\beta k}(x-\xi)^{(1-k)\alpha-1}}{\Gamma(\alpha-\alpha k)}\right]. \quad (38)$$

Using that $\Re\gamma^{\beta/\alpha}>0$ for $\Re\gamma\geq 0$ and that $\xi\geq 0$, we see that (38) is $O(|\gamma|^{-2\beta})$ as $|\gamma|\longrightarrow\infty$, $\Re\gamma\geq 0$. Thus the integral $\int_{c-i\infty}^{c+i\infty}e^{\gamma t}b(\gamma)\mathscr{F}_1(x,\gamma)d\gamma$ converges.

Concerning the $\mathscr{F}_2$ term, integrate (36) by parts to get

$$\mathscr{F}_2(x,\gamma)=\gamma^{-\beta/\alpha}E_{\alpha,\alpha}\left(\gamma^{\beta}x\right)\cdot\left\{-e^{-x\gamma^{\beta/\alpha}}f(x)+\int_x^{\infty}e^{-\xi\gamma^{\beta/\alpha}}f'(\xi)\,d\xi\right\}. \quad (39)$$

Analogously to (37), we see that the right-hand side of (39) is $O(\gamma^{-\beta/\alpha}\cdot\gamma^{\beta(1-\alpha)/\alpha})=O(\gamma^{-\beta})$ for $|\gamma|\longrightarrow\infty$, $\Re\gamma\geq 0$. As $b(\gamma)=O(|\gamma|^{-1})$ by assumptions of the lemma, the integral $\int_{c-i\infty}^{c+i\infty}e^{\gamma t}b(\gamma)\mathscr{F}_2(x,\gamma)d\gamma$ also converges.

Thus the definition of $u(x,t)$ by (30) makes sense.

If $t=0$, we can close the integration contour in (30) on the right and obtain $u(x,0)=0$. The decay of the integrand in (30) for $|\gamma|\longrightarrow\infty$ is sufficiently fast to allow the differentiation under the integral sign; putting $t=0$ in the integral

$$u_t(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{t\gamma}\mathscr{F}(x,\gamma)\cdot\gamma\,d\gamma \quad (40)$$

and again closing the contour on the right, we obtain $u_t(x,0)=0$.

For $t>0$ the Bromwich integration contour in (30) can be replaced with the integration contour μ shown on Figure 1:

$$u(x,t)=\frac{1}{2\pi i}\int_{\mu}e^{t\gamma}\mathscr{F}(x,\gamma)\,d\gamma \quad (41)$$

where now

$$\mathscr{F}(x,\gamma)=b(\gamma)\cdot\left[\int_0^x f(\xi)(x-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^{\beta}(x-\xi)^{\alpha}\right)d\xi+x^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^{\beta}x^{\alpha}\right)\int_0^{\infty}e^{-\xi\gamma^{\beta/\alpha}}f(\xi)\,d\xi\right]. \quad (42)$$

By definition (11), since $1<\alpha<2$,

$$D_x^{\alpha}u(x,t)=\frac{d^2}{dx^2}\left[\frac{1}{\Gamma(2-\alpha)}\cdot\int_0^x u(\eta,t)(x-\eta)^{1-\alpha}d\eta\right]=\frac{d^2}{dx^2}\left[\frac{1}{\Gamma(2-\alpha)}\cdot\int_0^x d\eta(x-\eta)^{1-\alpha}\frac{1}{2\pi i}\int_{\mu}d\gamma e^{\gamma t}\mathscr{F}(\eta,\gamma)\right]. \quad (43)$$

The assumptions of Fubini's theorem are clearly satisfied for interchanging the order of integration with respect to $d\eta$ and $d\gamma$ (while keeping x fixed); thus

$$=\frac{d^2}{dx^2}\left[\frac{1}{\Gamma(2-\alpha)}\frac{1}{2\pi i}\int_{\mu}d\gamma e^{\gamma t}\cdot\int_0^x d\eta(x-\eta)^{1-\alpha}\mathscr{F}(\eta,\gamma)\right]. \quad (44)$$

Claim

$$(44)=\frac{1}{\Gamma(2-\alpha)}\frac{1}{2\pi i}\int_{\mu}d\gamma e^{\gamma t}\frac{d^2}{dx^2}\cdot\int_0^x d\eta(x-\eta)^{1-\alpha}\mathscr{F}(\eta,\gamma). \quad (45)$$

Proof of the Claim. We have to work around the obstacle mentioned in Remark 3. We need to show that the first and the second x-derivatives in the integrand of (44) are integrable with respect to $d\gamma e^{\gamma t}$.

Step A. The function

$$\frac{d^2}{dx^2}\frac{1}{\Gamma(2-\alpha)}\int_0^x d\eta(x-\eta)^{1-\alpha}\mathscr{F}(\eta,\gamma)=D_x^{\alpha}\mathscr{F}(\eta,\gamma) \quad (46)$$

equals by Theorem 4

$$\gamma^{\beta}\mathscr{F}(x,\gamma)+f(x) \quad (47)$$

which is integrable along μ with respect to the measure $e^{\gamma t}d\gamma$, uniformly for x in compact sets.

Step B. Using the beta-integral, for $x>\xi$ we find

$$\int_{\eta=\xi}^{\eta=x}(x-\eta)^{1-\alpha}(\eta-\xi)^{k\alpha-1}d\eta=\frac{\Gamma(2-\alpha)\Gamma(k\alpha)}{\Gamma(2+(k-1)\alpha)}(x-\xi)^{(k-1)\alpha+1}. \quad (48)$$

Therefore, for each fixed value of $\gamma \in \mathbb{C}$ and $x \geq 0$,

$$\begin{aligned} &\int_0^x d\eta\,(x-\eta)^{1-\alpha} \\ &\quad\cdot \int_0^\eta f(\xi)\,(\eta-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(\gamma^\beta(\eta-\xi)\right) d\xi \\ &= \int_0^x d\eta\,(x-\eta)^{1-\alpha} \\ &\quad\cdot \int_0^\eta f(\xi) \sum_{k=0}^\infty \frac{\gamma^{\beta k}(\eta-\xi)^{(k+1)\alpha-1}}{\Gamma(k\alpha+\alpha)} d\xi \end{aligned} \tag{49}$$

(since the series converges uniformly with respect to the integration variables)

$$\begin{aligned} &= \int_0^x d\xi f(\xi) \sum_{k=0}^\infty \frac{\Gamma(2-\alpha)(x-\xi)^{k\alpha+1}\gamma^{\beta k}}{\Gamma(2+k\alpha)} \\ &= \Gamma(2-\alpha)\int_0^x d\xi f(\xi)\cdot(x-\xi) E_{\alpha,2}\left(\gamma^\beta(x-\xi)^\alpha\right). \end{aligned} \tag{50}$$

The integrand is regular enough to differentiate under the integral sign:

$$\begin{aligned} \frac{\partial}{\partial x}(50) &= \int_0^x d\xi f(\xi) \sum_{k=0}^\infty \frac{\Gamma(2-\alpha)(x-\xi)^{k\alpha}\gamma^{\beta k}}{\Gamma(1+k\alpha)} \\ &= \Gamma(2-\alpha)\int_0^x d\xi f(\xi) E_\alpha\left(\gamma^\beta(x-\xi)^\alpha\right), \end{aligned} \tag{51}$$

which is integrable along μ with respect to $d\gamma b(\gamma)e^{\gamma t}$.

Step C. Similarly, using (48),

$$\begin{aligned} &\int_0^x d\eta\,(x-\eta)^{1-\alpha}\eta^{\alpha-1}E_{\alpha,\alpha}\left(\gamma^\beta\eta^\alpha\right) \\ &= \int_0^x d\eta\,(x-\eta)^{1-\alpha}\sum_{k=0}^\infty \frac{\gamma^{\beta k}\eta^{(k+1)\alpha-1}}{\Gamma(k\alpha+\alpha)} \\ &= \sum_{k=0}^\infty \frac{\Gamma(2-\alpha)x^{k\alpha+1}\gamma^{\beta k}}{\Gamma(2+k\alpha)} = \Gamma(2-\alpha)\,xE_{\alpha,2}\left(\gamma^\beta x^\alpha\right), \end{aligned} \tag{52}$$

and $(\partial/\partial x)$ (52) is also integrable along μ with respect to $d\gamma b(\gamma)e^{\gamma t}$.

Step D. Adding the results of Steps B and C we conclude that also

$$\frac{d}{dx}\int_0^x d\eta\,(x-\eta)^{1-\alpha}\mathcal{F}(\eta,\gamma) \tag{53}$$

is integrable along μ with respect to $e^{\gamma t}d\gamma$, uniformly with respect to x on compact sets. Therefore two consecutive differentiations with respect to x can be carried out under the integral sign in (44) and the claim is proven.

Resuming the Proof of Lemma 5. By the result of the claim and by Step A in its proof, we have

$$D_x^\alpha u(x,t) = \frac{1}{2\pi i}\int_\mu d\gamma e^{\gamma t}\left(\gamma^\beta\mathcal{F}(x,\gamma)+f(x)\right). \tag{54}$$

By definition (13),

$$\begin{aligned} &{}^C D_t^\beta u(x,t) \\ &\quad= \frac{1}{\Gamma(2-\beta)}\int_0^t\left[\frac{d^2}{d\tau^2}u(x,\tau)(t-\tau)^{1-\beta}\right]d\tau \end{aligned} \tag{55}$$

and similarly to [5, (5.3.3)]

$${}^C D_t^\beta \int_\mu d\gamma e^{\gamma t}\mathcal{F}(x,\gamma) = \int_\mu d\gamma\gamma^\beta e^{\gamma t}\mathcal{F}(x,\gamma). \tag{56}$$

Here we used the fact that convolutions interact with the Laplace integral in the usual way even if the contour of the Laplace integral is not rectilinear; see [11, Pré I.5].

Collecting the terms from (54) and (56), we obtain (32).

Finally, let us compute $[D_x^{\alpha-2}u(x,t)]_{x=0+}$. With the same analytical details as above,

$$\begin{aligned} \left[D_x^{\alpha-2}u(x,t)\right]_{x=0+} &= \int_\mu e^{\gamma t}\left[D_x^{\alpha-2}\mathcal{F}(x,\gamma)\right]_{x=0+}d\gamma \\ &= 0, \end{aligned} \tag{57}$$

using Theorem 4. □

Let us now subtract from the solution of the form (30) the solution of the type (7) and estimate the difference. Since we want $\varepsilon(t) = E_\beta(\Omega t^\beta)$ in (3), we take in (32)

$$b(\gamma) = L.T.\left(E_\beta\left(\Omega t^\beta\right)\right) = \frac{\gamma^{\beta-1}}{\gamma^\beta-\Omega}. \tag{58}$$

We will treat Ω as belonging to the cut complex plane $\mathbb{C}\setminus(-\infty,0]$; and let, for definiteness, $\Omega^{1/\alpha}$ be the principal branch, i.e., $\Omega^{1/\alpha}>0$ if $\Omega>0$. If $\arg\Omega\in(-(2-\beta)\pi,(2-\beta)\pi)$ then $s^\beta-\Omega$ has only one root $s=r_1$ in the cut complex plane $\mathbb{C}\setminus(-\infty;0]$; otherwise it has two roots $s=r_1$ and $s=r_2$. If $\arg\Omega\in[-\pi\beta/2,\pi\beta/2]$, one of the roots r_κ is located in the left half-plane $\Re s>0$.

As a solution of the Helmholtz equation (5) satisfying $D^{\alpha-2}v(0+)=0$ we take

$$\begin{aligned} v(x) &= \int_0^x d\xi f(\xi)\left[(x-\xi)^{\alpha-1}E_{\alpha,\alpha}\left(\Omega(x-\xi)^\alpha\right)\right. \\ &\quad\left.- e^{-\xi\Omega^{1/\alpha}}x^{\alpha-1}E_{\alpha,\alpha}\left(\Omega x^\alpha\right)\right] \\ &\quad- \int_x^\infty d\xi e^{-\xi\Omega^{1/\alpha}}x^{\alpha-1}E_{\alpha,\alpha}\left(\Omega x^\alpha\right)f(\xi). \end{aligned} \tag{59}$$

The fact that $v(x)$ is indeed a solution follows from Theorem 4; the form of $v(x)$ was derived similarly to (31) by imitating (26) in the case of the wave equation. As $u(x,t)$ we take the solution of the fractional wave equation constructed in Lemma 5. With these choices,

$$\begin{aligned} u(x,t) - E_\beta\left(\Omega t^\beta\right) v(x) = \frac{1}{2\pi i}\int_\mu d\gamma e^{\gamma t}\frac{\gamma^{\beta-1}}{\gamma^\beta-\Omega} \\ \times\left[\int_0^x (x-\xi)^{\alpha-1}\right. \\ \cdot\left\{E_{\alpha,\alpha}\left(\gamma^\beta (x-\xi)^\alpha\right) - E_{\alpha,\alpha}\left(\Omega(x-\xi)^\alpha\right)\right\} \\ \cdot f(\xi)\, d\xi \\ - \int_0^\infty \left\{e^{-\xi\gamma^{\beta/\alpha}} E_{\alpha,\alpha}\left(\gamma^\beta x^\alpha\right) - e^{-\xi\Omega^{1/\alpha}} E_{\alpha,\alpha}\left(\Omega x^\alpha\right)\right\} \\ \left.\cdot f(\xi)\, x^{\alpha-1} d\xi\right]. \end{aligned} \tag{60}$$

The integrand of (60) is analytic at r_κ, $\kappa = 1$ or $\kappa = 1, 2$, as the case may be. Therefore the only contribution to the integral comes from the discontinuity along the cut $(-\infty, 0]$. To the jump of the integrand along that cut we apply the following.

Lemma 6 (generalized Watson's lemma, [12, p.22]). *Consider the integral*

$$G(z) = \int_0^{\infty e^{i\gamma}} g(t)\, e^{-zt} dt \tag{61}$$

in the complex domain, where $\gamma \in \mathbb{R}$, and the path of integration is the straight line joining $t = 0$ to $t = \infty e^{i\gamma}$. Suppose that the integral $G(z)$ exists for some fixed $z = z_0$ and that, as $t \longrightarrow 0$ along $\arg t = \gamma$,

$$g(t) \sim \sum_{n=0}^\infty a_n t^{\lambda_n - 1}, \tag{62}$$

where $\lambda_0 > 0$ and $\lambda_{n+1} > \lambda_n$. Then

$$G(z) \sim \sum_{n=0}^\infty a_n \Gamma(\lambda_n)\, z^{-\lambda_n} \tag{63}$$

as $z \longrightarrow \infty$ in $|\arg(ze^{i\gamma})| \le \pi/2 - \Delta$ for any Δ in the interval $0 < \Delta \le \pi/2$.

An elementary calculation with power series shows that

$$\begin{aligned} \frac{\gamma^{\beta-1}}{\gamma^\beta - \Omega}\left[\int_0^x (x-\xi)^{\alpha-1}\right. \\ \cdot\left\{E_{\alpha,\alpha}\left(\gamma^\beta (x-\xi)^\alpha\right) - E_{\alpha,\alpha}\left(\Omega(x-\xi)^\alpha\right)\right\} \\ \cdot f(\xi)\, d\xi \\ - \int_0^\infty \left\{e^{-\xi\gamma^{\beta/\alpha}} E_{\alpha,\alpha}\left(\gamma^\beta x^\alpha\right) - e^{-\xi\Omega^{1/\alpha}} E_{\alpha,\alpha}\left(\Omega x^\alpha\right)\right\} \\ \left.\cdot f(\xi)\, x^{\alpha-1} d\xi\right] = C_{\beta-1}(x)\,\gamma^{\beta-1} + C_{\beta-1+\beta/\alpha}(x) \\ \cdot\gamma^{\beta-1+\beta/\alpha} + C_{2\beta-1}\gamma^{2\beta-1} + O\left(\gamma^{\beta-1+2\beta/\alpha}\right), \end{aligned} \tag{64}$$

where in the ordering of the terms we remembered that $1/2 < \beta/\alpha \le 1$ and $1 < \beta < 2$ and where

$$C_{\beta-1}(x) = \int_0^x (x-\xi)^{2\alpha-1} E_{\alpha,2\alpha}\left(\Omega(x-\xi)^\alpha\right) f(\xi)\, d\xi \tag{65}$$

$$+ \frac{x^{\alpha-1}}{\Omega}\int_0^\infty \left\{\frac{1}{\Gamma(\alpha)} - e^{-\xi\Omega^{1/\alpha}} E_{\alpha,\alpha}\left(\Omega x^\alpha\right)\right\} f(\xi)\, d\xi; \tag{66}$$

$$C_{\beta-1+\beta/\alpha}(x) = -\frac{x^{\alpha-1}}{\Omega\Gamma(\alpha)}\int_0^\infty \xi f(\xi)\, d\xi; \tag{67}$$

$$\begin{aligned} C_{2\beta-1}(x) = \frac{1}{\Omega} C_{\beta-1} \\ - \frac{1}{\Omega\Gamma(2\alpha)}\int_0^x \left[(x-\xi)^{2\alpha-1} - x^{2\alpha-1}\right] f(\xi)\, d\xi. \end{aligned} \tag{68}$$

Now (8) follows from Lemma 6 and the formula

$$\frac{1}{2\pi i}\int_\mu \gamma^{q-1} e^{\gamma t} d\gamma = \frac{\sin q\pi}{\pi}\Gamma(q)\cdot\frac{1}{t^q}. \tag{69}$$

The proof of Theorem 1 is complete.

5. Case $(3/2)\alpha > \beta > \alpha$

In this section we prove Theorem 2, following the model of the previous section. The condition $(3/2)\alpha > \beta$ is imposed to make the function $E_{\alpha,\alpha}(\gamma^\beta c)$, for $c > 0$, decrease as $|\gamma| \longrightarrow \infty$, $\pi \le |\arg\gamma| \le \pi/2$; see [8, Theorems 1.3, 1.4].

Lemma 7. *Let $f(x)$ be a continuous function with compact support.*

Let $b(\gamma)$ be an analytic function defined in $\mathbb{C} \setminus [D(0,R) \cup (-\infty, 0]]$ and bounded in that region; moreover $\int_{c-i\infty}^{c+i\infty} e^{\gamma t} b(\gamma) d\gamma$ is defined for $t = 0$ in the sense of principal value. Then the function

$$u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{t\gamma}\mathscr{H}(x,\gamma)\, d\gamma, \quad t \ge 0, \tag{70}$$

where

$$\begin{aligned} \mathscr{H}(x,\gamma) \\ = b(\gamma)\int_0^x d\xi f(\xi)(x-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(\gamma^\beta (x-\xi)^\alpha\right) \end{aligned} \tag{71}$$

for $x \ge 0$, solves the FDE

$$\begin{aligned} D_x^\alpha u(x,t) - {}^C D_t^\beta u(x,t) \\ = f(x)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\gamma t} b(\gamma)\, d\gamma. \end{aligned} \tag{72}$$

Moreover, $u(x,t)$ satisfies the boundary conditions

$$D_x^{\alpha-2} u(0+,t) = D_x^{\alpha-1} u(0+,t) = 0. \tag{73}$$

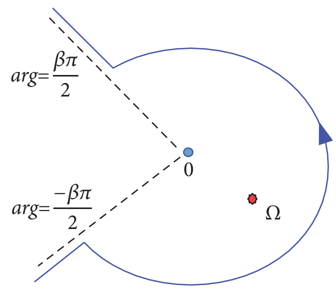

Figure 2: The contour L' in the s-plane.

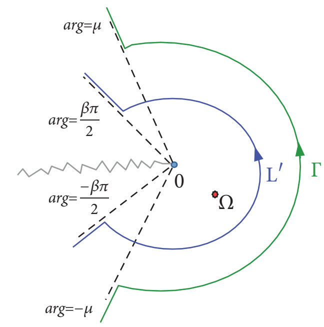

Figure 3: The contour Γ.

Proof. Since $E_{\alpha,\alpha}(\gamma^\beta(x-\xi)^\alpha) = O(|\gamma|^{-2\beta})$ as $\gamma \longrightarrow \pm i\infty$ and $\beta > 1$, in the integral

$$u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dt \cdot \int_0^x d\xi e^{t\gamma} b(\gamma)(x-\xi)^{\alpha-1} E_{\alpha,\alpha}\left(\gamma^\beta (x-\xi)^\alpha f(\xi)\right) \tag{74}$$

the integrations can be carried out in arbitrary order; also, similarly to the proof of Lemma 5, one can perform fractional differentiations under the integral sign even without modifying the integration contour. □

Next we assume that $b(\gamma) = L.T.(E_\beta(\Omega t^\beta)) = \gamma^{\beta-1}/(\gamma^\beta - \Omega)$ and work out the initial conditions of $u(x,t)$ from (70) for $t = 0$; namely, we compute

$$u(x,0) = \frac{1}{2\pi i}\int_0^x d\xi \int_{c-i\infty}^{c+i\infty} d\gamma (x-\xi)^{\alpha-1} \cdot E_{\alpha,\alpha}\left(\gamma^\beta (x-\xi)^\alpha\right)\left(\frac{\gamma^{\beta-1}}{\gamma^\beta-\Omega}\right) f(\xi) \tag{75}$$

and

$$u_t(x,0) = \frac{1}{2\pi i}\int_0^x d\xi \int_{c-i\infty}^{c+i\infty} d\gamma\gamma (x-\xi)^{\alpha-1} \cdot E_{\alpha,\alpha}\left(\gamma^\beta (x-\xi)^\alpha\right)\left(\frac{\gamma^{\beta-1}}{\gamma^\beta-\Omega}\right) f(\xi). \tag{76}$$

In the inner integral of (75) make the substitution $s = \gamma^\beta$ and obtain the integral

$$\frac{1}{2\pi i}\int_{L'} E_{\alpha,\alpha}\left(s(x-\xi)^\alpha\right)\frac{ds}{s-\Omega}, \tag{77}$$

where the integration contour L' is shown on Figure 2.

Since $E_{\alpha,\alpha}(s(x-\xi)^\alpha) = O(s^{-2})$ as $s \longrightarrow \infty$, $|\arg s| > \pi\alpha/2$, we can close the contour L' and calculate the integral (77) using the residue at $s = \Omega$:

$$\frac{1}{2\pi i}\oint E_{\alpha,\alpha}\left(s(x-\xi)^\alpha\right)\frac{ds}{s-\omega} = E_{\alpha,\alpha}\left(\Omega(x-\xi)^\alpha\right). \tag{78}$$

Similarly, the inner integral of (76) becomes

$$\frac{1}{2\pi i}\int_{L'} s^{1/\beta} E_{\alpha,\alpha}\left(s(x-\xi)^\alpha\right)\frac{ds}{s-\Omega}, \tag{79}$$

where we now introduce a cut $(-\infty; 0]$ in the s-plane. Let $\pi\alpha/2 < \mu < \pi\beta/2$; then the integral representation [8, (1.126)] implies

$$(79) = \frac{1}{2\pi i}\int_{L'} ds s^{1/\beta}\frac{1}{2\pi i\alpha} \cdot \int_\Gamma d\zeta \frac{\exp\left(\zeta^{1/\alpha}\right)\zeta^{(1/\alpha)-1}}{\zeta - s(x-\xi)^\alpha}\frac{1}{s-\Omega}, \tag{80}$$

where the contour Γ is as on Figure 3 and avoids a large enough circle around the origin so as not to intersect the similarity image $\max\{1,(x-\xi)^\alpha\} \cdot L'$ of the contour L'.

Interchanging the order of integration in (80) (legal because $\beta > 1$), we obtain

$$\frac{1}{(2\pi i)^2\alpha}\int_\Gamma d\zeta \exp\left(\zeta^{1/\alpha}\right)\zeta^{(1/\alpha)-1} \cdot \int_{L'} ds \frac{s^{1/\beta}}{\left(\zeta - s(x-\xi)^\alpha\right)(s-\Omega)}. \tag{81}$$

Closing the contour L' on the right (which is possible for $\beta > 1$) we pick up the residue at $s = \zeta/(x-\xi)^\alpha$, and (76) becomes

$$-\frac{1}{(2\pi i)\alpha(x-\xi)^{\alpha/\beta}}\int_\Gamma d\zeta \frac{\exp\left(\zeta^{1/\alpha}\right)\zeta^{(1/\beta)+(1/\alpha)-1}}{\zeta - \Omega(x-\xi)^\alpha}. \tag{82}$$

Let $\delta = \alpha - \alpha/\beta$; then in the numerator $\zeta^{(1/\beta)+(1/\alpha)-1} = \zeta^{(1-\delta)/\alpha}$, and the integral equals (see [8, (1.130)])

$$-\frac{1}{(x-\xi)^{\alpha/\beta}}E_{\alpha,\delta}\left(\Omega(x-\xi)^\alpha\right). \tag{83}$$

Coming back to (76), we obtain

$$(76) = -\int_0^x d\xi (x-\xi)^{\delta-1} E_{\alpha,\delta}\left(\Omega(x-\xi)^\alpha\right) f(\xi). \tag{84}$$

Putting together (78) and (84), we have proven the following.

Lemma 8. *If $f(x)$ is a smooth function with compact support, the solution $u(x,t)$ from Lemma 7 with $\beta(\gamma) = \gamma^{\beta-1}/(\gamma^{\beta} - \Omega)$ satisfies*

$$\begin{aligned} u(x,0) &= \int_0^x (x-\xi)^{\alpha-1} E_{\alpha,\alpha}(\Omega(x-\xi)^{\alpha}) f(\xi)\, d\xi, \\ u_t(x,0) &= \int_0^x (x-\xi)^{\delta-1} E_{\alpha,\delta}(\Omega(x-\xi)^{\alpha}) f(\xi)\, d\xi, \end{aligned} \tag{85}$$

where $\delta = \alpha(1-\beta^{-1})$.

Let

$$v(x) = \int_0^x d\xi f(\xi)(x-\xi)^{\alpha-1} E_{\alpha,\alpha}(\Omega(x-\xi)^{\alpha}). \tag{86}$$

By Theorem 4, $v(x)$ satisfies the Helmholtz equation (5) and the boundary conditions $D^{\alpha-2}v(0+) = D^{\alpha-1}v(0+) = 0$. With the same choice of the contour μ as in the previous section,

$$\begin{aligned} u(x,t) - E_{\beta}(\Omega t^{\beta}) v(x) &= \frac{1}{2\pi i}\int_{\mu} d\gamma e^{\gamma t}\frac{\gamma^{\beta-1}}{\gamma^{\beta}-\Omega} \\ &\quad \times \int_0^x (x-\xi)^{\alpha-1} \\ &\quad \cdot \left\{E_{\alpha,\alpha}(\gamma^{\beta}(x-\xi)^{\alpha}) - E_{\alpha,\alpha}(\Omega(x-\xi)^{\alpha})\right\} \\ &\quad \cdot f(\xi)\, d\xi, \end{aligned} \tag{87}$$

and the only contribution to the integral on the right-hand side of (87) comes from the cut $(-\infty, 0]$. We compute that

$$\begin{aligned} &\frac{\gamma^{\beta-1}}{\gamma^{\beta}-\Omega}\int_0^x (x-\xi)^{\alpha-1} \\ &\quad \cdot \left\{E_{\alpha,\alpha}(\gamma^{\beta}(x-\xi)^{\alpha}) - E_{\alpha,\alpha}(\Omega(x-\xi)^{\alpha})\right\} \\ &\quad \cdot f(\xi)\, d\xi = B_{\beta-1}(x)\gamma^{\beta-1} + B_{2\beta-1}(x)\gamma^{\beta-2} \\ &\quad + O(\gamma^{3\beta-3}), \end{aligned} \tag{88}$$

where

$$B_{\gamma-1} = \int_0^x (x-\xi)^{2\alpha-1} E_{\alpha,2\alpha}(\Omega(x-\xi)^{\alpha}) f(\xi)\, d\xi; \tag{89}$$

$$B_{2\gamma-1} = \frac{1}{\Omega}B_{\gamma-1} - \frac{1}{\Gamma(2\alpha)\Omega}\int_0^x (x-\xi)^{2\alpha-1} f(\xi)\, d\xi. \tag{90}$$

Using Lemma 6 and formula (69), we obtain (10). The proof of Theorem 2 is complete.

Data Availability

No data were used to support this study.

Disclosure

The research was performed as a part of Alexander Getmanenko's Employment and Mateo Dulce's Studies at the Universidad de Los Andes.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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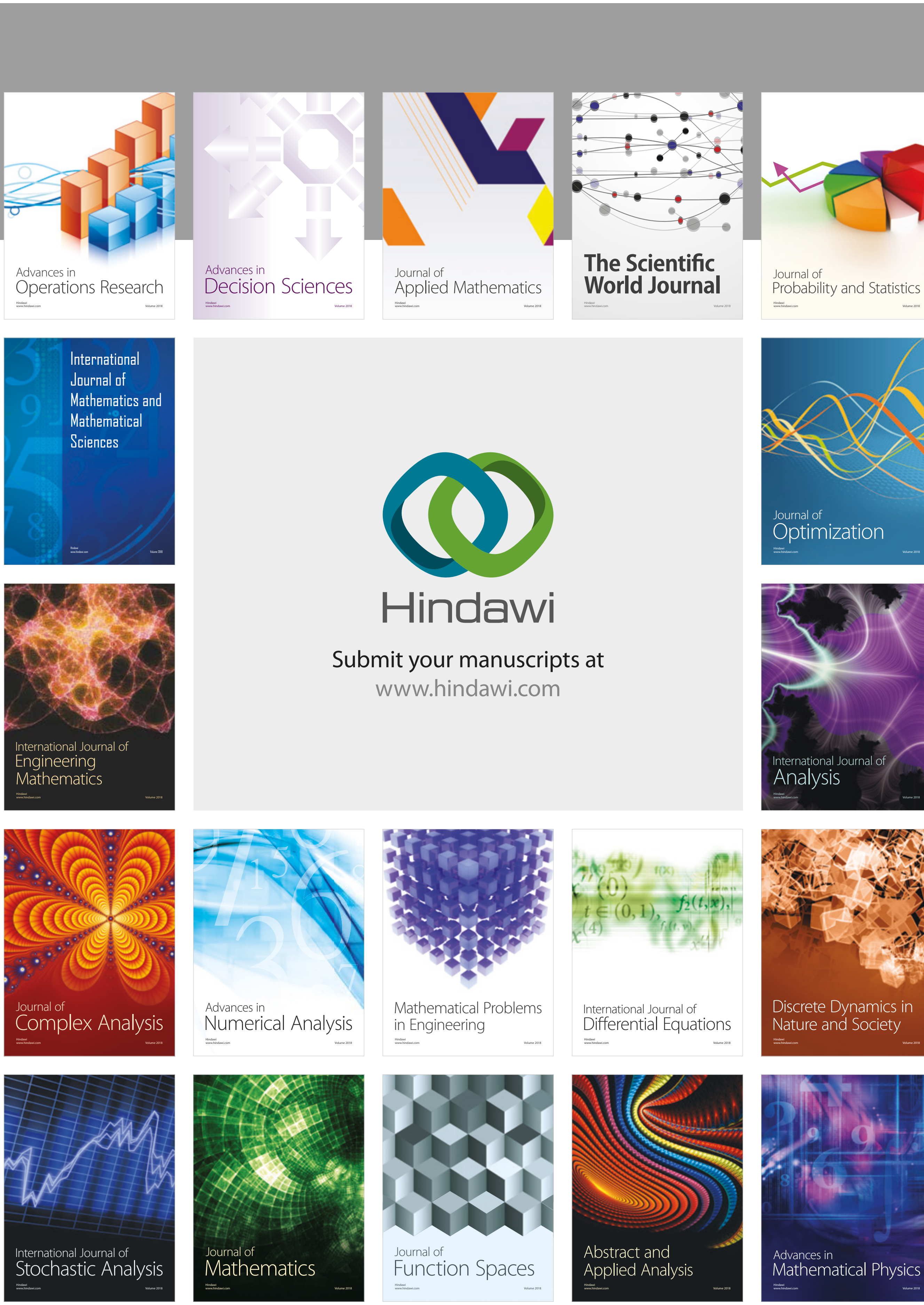

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