Research Article
Inequality of Ostrowski Type for Mappings with Bounded Fourth Order Partial Derivatives

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A general Ostrowski’s type inequality for double integrals is given. We utilize function whose partial derivative of order four exists and is bounded.

1. Introduction

In 1938, Ostrowski [1] introduced the following integral inequality.

Theorem 1. Let \( f: [a, b] \rightarrow \mathbb{R} \) be continuous mapping on \([a, b]\) and differentiable on \((a, b)\), whose derivative \( f' : (a, b) \rightarrow \mathbb{R} \) is bounded on \((a, b)\), i.e., \( \|f'\|_{\infty} = \sup_{t \in [a, b]} |f'(t)| < \infty \), then for all \( x \in [a, b] \)

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[ \frac{1}{4} + \left( \frac{x - (a+b)/2}{(b-a)^2} \right)^2 \right] (b-a) \|f'\|_{\infty}.
\] (1)

The constant 1/4 is sharp in the sense that it cannot be replaced by a smaller one.

In 1975, Milovanović [2] proposed the following generalization of (1) for a function \( f \) of several variables whose first order partial derivatives are bounded.

Theorem 2. Let \( f: \mathbb{R}^m \rightarrow \mathbb{R} \) be a differentiable function defined on \( D \) and let \( |\partial f/\partial x_i| \leq M_i \) (\( M_i > 0; i = 1, \ldots, m \)) in \( D \). Then, for every \( X = (x_1, \ldots, x_m) \in D \),

\[
\left| f(x_1, \ldots, x_m) - \frac{1}{\prod_{i=1}^m (b_i - a_i)} \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} f(y_1, \ldots, y_m) \, dy_1 \cdots dy_m \right|
\leq \sum_{i=1}^m \left[ \frac{1}{4} + \left( \frac{x_i - (a_i + b_i)/2}{(b_i - a_i)^2} \right)^2 \right] (b_i - a_i) M_i.
\] (2)

In 1998, Barnett and Dragomir [3] proved the following Ostrowski type inequality for mappings of two variables with bounded second order partial derivatives.

Theorem 3. Let \( f: [a, b] \times [c, d] \rightarrow \mathbb{R} \) continuous on \([a, b] \times [c, d] \), \( f''_{x,y} = \partial^2 f/\partial x \partial y \) exists on \((a, b) \times (c, d)\) and is bounded, i.e.,

\[
\|f''_{x,y}\|_{\infty} = \sup_{(x,y) \in [a,b] \times [c,d]} \left| \frac{\partial^2 f(x,y)}{\partial x \partial y} \right| < \infty,
\] (3)

Then we have the inequality

\[
\left| \int_a^b \int_c^d f(s, t) \, ds \, dt - \left[ (b-a) \int_c^d f(x, t) \, dt \right. \right. + \left. \left. (d-c) \int_a^b f(s, y) \, ds - (d-c)(b-a)f(x, y) \right] \right|
\leq \left[ \frac{1}{4} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right] \left[ \frac{1}{4} (d-c)^2 + \left( y - \frac{c+d}{2} \right)^2 \right] \|f''_{x,y}\|_{\infty},
\] (4)

for all \((x, y) \in [a,b] \times [c,d] \).
In [4], Xue et al. derive the following inequality of Ostrowski type.

**Theorem 4.** Let \( f : [a, b] \times [c, d] \rightarrow \mathbb{R} \) be an absolutely continuous function such that the partial derivatives of order two exist and suppose that there exist constants \( \gamma, \Gamma \in \mathbb{R} \) with \( \gamma \leq \frac{\partial^2 f(t,s)}{\partial t \partial s} \leq \Gamma \) for all \((t,s) \in [a,b] \times [c,d] \). Then we have

\[
\left| \int_{[a,b]} \int_{[c,d]} \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|_{\infty} \leq \frac{m_2(a,b)m_2(c,d)}{4} \left( \frac{x - a + b}{2} \right)^2 \left( \frac{y - c + d}{2} \right)^2
\]

for all \((t,s) \in [a,b] \times [c,d] \).

To prove this theorem, we use the following inequalities:

\[
\left| \int_{[a,b]} \int_{[c,d]} \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|_{\infty} \leq \frac{m_2(a,b)m_2(c,d)}{4} \left( \frac{x - a + b}{2} \right)^2 \left( \frac{y - c + d}{2} \right)^2
\]

More recently, Sarikaya et al. [5] establish weighted Ostrowski type inequalities considering function whose second order partial derivatives are bounded as follows.

**Theorem 5.** Let \( f : [a, b] \times [c, d] \rightarrow \mathbb{R} \) be an absolutely continuous function such that the partial derivatives of order two exist and are bounded, i.e.,

\[
\left| \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|_{\infty} < \infty,
\]

for all \((t,s) \in [a,b] \times [c,d] \). Then we have

\[
m_1(a,b)m_1(c,d) \left( x - \mu(a,b) \right) \left( y - \mu(c,d) \right) f(x,y)
\]

\[
- m_2(a,b) \left( y - \mu(c,d) \right) \left( x - \mu(a,b) \right) f(x,y)
\]

\[
+ \int_{x}^{y} \left( \int_{a}^{b} \omega(u) \, du \right) f(t,y) \, dt
\]

\[
- m_2(a,b) \left( y - \mu(c,d) \right) \left( x - \mu(a,b) \right) f(x,y)
\]

\[
+ \int_{x}^{y} \left( \int_{a}^{b} \omega(u) \, du \right) f(t,y) \, dt
\]

\[
- m_2(a,b)m_2(c,d)
\]

\[
\cdot \left| \int_{[a,b]} \int_{[c,d]} \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|_{\infty}
\]

where

\[
m_1(a,b) = \int_{a}^{b} r \omega(t) \, dt, \quad i = 0, 1, \ldots,
\]

\[
\mu(a,b) = \frac{m_1(a,b)}{m_2(a,b)}
\]

\[
\sigma^2(a,b) = \frac{m_2(a,b)}{m_2(a,b)} - \mu^2(a,b)
\]

For other related work, we refer the reader to [6–15].

In this paper, motivated by the ideas in both [4, 5], we shall derive a new inequality of Ostrowski’s type similar to the inequalities (5) and (7), involving functions of two independent variables.

### 2. Main Results

In order to introduce our main results, we commence with the following lemma.

**Lemma 6.** Let \( f : [a, b] \times [c, d] \rightarrow \mathbb{R} \) be an absolutely continuous function such that the partial derivative of order 4 exists for all \((x,y) \in [a + h((b-a)/2), b - h((b-a)/2)] \times [c + h((d-c)/2), d - h((d-c)/2)]\) and \(h \in [0,1] \). Then for any two mappings \( K_1(t,x) : [a, b] \times [a, b] \rightarrow \mathbb{R} \) and \( K_2(s,y) : [c, d] \times [c, d] \rightarrow \mathbb{R} \), where

\[
K_1(t,x) = \begin{cases} \frac{1}{2} \left( t - a + h \frac{b-a}{2} \right)^2, & t \in [a,x] \\ \frac{1}{2} \left( t - b + h \frac{b-a}{2} \right)^2, & t \in (x,b] \end{cases}
\]

\[
K_2(s,y) = \begin{cases} \frac{1}{2} \left( s - c + h \frac{c-d}{2} \right)^2, & s \in [c,y] \\ \frac{1}{2} \left( s - d + h \frac{c-d}{2} \right)^2, & s \in (y,d] \end{cases}
\]

Then we have

\[
K_1(t,x)K_2(s,y) \leq m_1(a,b)m_2(c,d)
\]

\[
\cdot \left( x - \mu(a,b) \right) \left( y - \mu(c,d) \right) f(x,y)
\]

\[
- m_2(a,b) \left( y - \mu(c,d) \right) \left( x - \mu(a,b) \right) f(x,y)
\]

\[
+ \int_{x}^{y} \left( \int_{a}^{b} \omega(u) \, du \right) f(t,y) \, dt
\]

\[
- m_2(a,b) \left( y - \mu(c,d) \right) \left( x - \mu(a,b) \right) f(x,y)
\]

\[
+ \int_{x}^{y} \left( \int_{a}^{b} \omega(u) \, du \right) f(t,y) \, dt
\]

\[
- m_2(a,b)m_2(c,d)
\]

\[
\cdot \left| \int_{[a,b]} \int_{[c,d]} \frac{\partial^2 f(t,s)}{\partial t \partial s} \right|_{\infty}
\]

where

\[
m_1(a,b) = \int_{a}^{b} r \omega(t) \, dt, \quad i = 0, 1, \ldots,
\]

\[
\mu(a,b) = \frac{m_1(a,b)}{m_2(a,b)}
\]

\[
\sigma^2(a,b) = \frac{m_2(a,b)}{m_2(a,b)} - \mu^2(a,b)
\]
and
\[ K_2(s, y) = \begin{cases} \frac{1}{2} \left[ s - \left( c + h \frac{d - c}{2} \right) \right]^2, & t \in [c, y] \\ \frac{1}{2} \left[ s - \left( d - h \frac{d - c}{2} \right) \right]^2, & t \in (y, d], \end{cases} \]

the identity
\[
E(f; h) = \int_a^b \int_c^d K_1(t; x) K_2(s; y) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt = \frac{1}{4} \int_a^b \int_c^d K_1(t; x) K_2(s; y) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt.
\]

By definitions of \( K_1(t; x) \) and \( K_2(s; y) \) in both (9) and (10), we have
\[
E(f; h) = \int_a^b \int_c^d K_1(t; x) K_2(s; y) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt = \frac{1}{4} \int_a^b \int_c^d K_1(t; x) K_2(s; y) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt.
\]

Proof. By definitions of \( K_1(t; x) \) and \( K_2(s; y) \) in both (9) and (10), we have
\[
E(f; h) = \int_a^b \int_c^d K_1(t; x) K_2(s; y) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt = \frac{1}{4} \int_a^b \int_c^d K_1(t; x) K_2(s; y) \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt.
\]

holds.

For integration by parts yields
\[
I_1 = \frac{1}{4} \int_a^b \int_c^d \left[ -\left( a + h \frac{b - a}{2} \right) \right]^2.
\]

\[
\cdot \left[ s - \left( c + h \frac{d - c}{2} \right) \right]^2 \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt.
\]

\[
= I_1 + I_2 + I_3 + I_4.
\]

\[
I_1 = \frac{1}{4} \int_a^b \int_c^d \left[ -\left( a + h \frac{b - a}{2} \right) \right]^2.
\]

\[
\cdot \left[ s - \left( c + h \frac{d - c}{2} \right) \right]^2 \frac{\partial^4 f(t, s)}{\partial t^2 \partial s^2} ds dt.
\]

\[
= I_1 + I_2 + I_3 + I_4.
\]
\[ E(f; h) = \int_{[a, b] \times [c, d]} \left[ y - \left( c + h \frac{d - c}{2} \right) \right]^2 f(t, s) \, dt \, ds + \int_a^b \int_c^d \left[ t - \left( b - h \frac{b - a}{2} \right) \right] f(t, s) \, ds \, dt. \]

By further algebraic manipulations and assuming result by [4], the proof of Lemma 6 is completed. \[ \square \]

**Theorem 7.** Let \( f : [a, b] \times [c, d] \to \mathbb{R} \) such that \( f \in C^4([a, b] \times [c, d]) \) be an absolutely continuous function such that the partial derivative of order 4 exists and is bounded; i.e.,

\[
\left\| \frac{\partial^4 f(t, s)}{\partial t^4 \partial s^2} \right\|_{\infty} = \sup_{(x, y) \in [a, b] \times [c, d]} \left\| \frac{\partial^4 f(t, s)}{\partial t^4 \partial s^2} \right\| < \infty,
\]

for all \((t, s) \in [a, b] \times [c, d]\). Then for all \((x, y) \in [a + h((b - a)/2), b - h((b - a)/2)] \times [c + h((d - c)/2), d - h((d - c)/2)]\) and \(h \in [0, 1]\), we have

\[
|E(f; h)| \leq \frac{h^3(b - a)^3}{24} + \frac{(1 - h)(b - a)}{2} \left( \frac{(1 - h)^2(b - a)^2}{12} \right) + \left( x - \frac{a + b}{2} \right)^2 \left( \frac{h^2(d - c)^3}{24} \right)
\]

\[
+ \left( y - \frac{c + d}{2} \right)^2 \left( \frac{(1 - h)^2(d - c)^2}{12} \right)
\]

\[
+ \left( x - \frac{a + b}{2} \right)^2 \left( \frac{(1 - h)^2(b - a)^2}{12} \right)
\]

where the functional \(E(f; h)\) is given by (II). \[ \square \]

**Proof.** By considering (II), we have

\[
|E(f; h)| = \int_a^b \int_c^d K_1(t; x) K_2(s; y) \partial^4 f(t, s) \frac{\partial^4 f(t, s)}{\partial t^4 \partial s^2} \, dt \, ds\]

\[
\leq \int_a^b \int_c^d |K_1(t; x)||K_2(s; y)| \frac{|\partial^4 f(t, s)|}{\partial t^4 \partial s^2} \, dt \, ds \leq \left\| \frac{\partial^4 f(t, s)}{\partial t^4 \partial s^2} \right\|_{\infty} \int_a^b \int_c^d K_1(t; x) \, dt \cdot \int_c^d K_2(s; y) \, ds.
\]

But,

\[
\int_a^b K_1(t; x) \, dt = \frac{h^3(b - a)^3}{24} + \frac{(1 - h)(b - a)}{2} \left( \frac{(1 - h)^2(b - a)^2}{12} + \left( x - \frac{a + b}{2} \right)^2 \right),
\]

and

\[
\int_c^d K_2(s; y) \, ds = \frac{h^3(d - c)^3}{24} + \frac{(1 - h)(d - c)}{2} \left( \frac{(1 - h)^2(d - c)^2}{12} + \left( y - \frac{c + d}{2} \right)^2 \right).
\]
Now, substituting (18), (19) into (17) gives (16) and, hence, completes the proof.

**Corollary 8.** Under the assumption of Theorem 7 with \( h = 0 \), we have

\[
\begin{align*}
\int f(x, y) + \left( \frac{a + b}{2} - x \right) \left( \frac{c + d}{2} - y \right) f_1(x, y) \\
+ \left( a + b \right) f_2(x, y) \\
- \left[ \frac{1}{b-a} \left( \frac{c + d}{2} - y \right) \right] \int_a^b f_1(t, y) \, dt \\
+ \frac{1}{d-c} \left( \frac{a + b}{2} - x \right) \int_c^d f_2(x, s) \, ds \\
- \left[ \frac{1}{b-a} \right] \int_a^b f(t, y) \, dt + \frac{1}{d-c} \int_c^d f(x, s) \, ds \\
+ \frac{1}{b-a} (a + b) \int_a^b f(t, s) \, ds \, ds dt \leq \frac{1}{4} \left( \frac{b-a)^2}{12} \\
+ \left( x - \frac{a + b}{2} \right) \left( \frac{d-c)^2}{12} + \left( y - \frac{c + d}{2} \right)^2 \right) \\
\| \frac{\partial^2 f(t, s)}{\partial t^2 \partial s^2} \|_{\infty}.
\end{align*}
\]

**Corollary 9.** Under the assumption of Theorem 7 with \( h = 0 \), \( x = (a+b)/2 \), and \( y = (c+d)/2 \) we have

\[
\begin{align*}
\int f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) - \left[ \frac{1}{b-a} \right] \int_a^b f \left( t, \frac{c + d}{2} \right) \, dt \\
+ \frac{1}{d-c} \int_c^d f \left( \frac{a + b}{2}, s \right) \, ds + \frac{1}{b-a} (a + b) \int_a^b f(t, s) \, ds \, ds dt \leq \frac{(b-a)^2 (d-c)^2}{576} \| \frac{\partial^2 f(t, s)}{\partial t^2 \partial s^2} \|_{\infty}.
\end{align*}
\]

**Corollary 10.** Under the assumption of Theorem 7 with \( h = 0 \), \( x = (a+b)/2 \), and \( y = (c+d)/4 \) we have

\[
\begin{align*}
\int f \left( \frac{a + b}{4}, \frac{c + d}{4} \right) + \left( a + b \right) \left( c + d \right) f_3 \left( \frac{a + b}{4}, \frac{c + d}{4} \right) \\
+ \left( a + b \right) f_4 \left( \frac{a + b}{4}, \frac{c + d}{4} \right) + \left( c + d \right) f_5 \left( \frac{a + b}{4}, \frac{c + d}{4} \right) \\
\cdot f_6 \left( \frac{a + b}{4}, \frac{c + d}{4} \right) - \left[ \frac{1}{b-a} \right] \int_a^b f \left( t, \frac{c + d}{4} \right) \, dt \\
+ \frac{1}{d-c} \int_c^d f \left( \frac{a + b}{4}, s \right) \, ds \\
+ \frac{1}{b-a} (a + b) \int_a^b f(t, s) \, ds \, ds dt \\
+ \frac{1}{d-c} \int_c^d f \left( \frac{a + b}{4}, s \right) \, ds \leq \frac{1}{4} \left( \frac{b-a)^2}{12} \\
+ \left( \frac{(d-c)^2}{12} + \left( \frac{c + d}{2} \right)^2 \right) \\
\| \frac{\partial^2 f(t, s)}{\partial t^2 \partial s^2} \|_{\infty}.
\end{align*}
\]

**Remark 11.** In Corollaries 8, 9, and 10 we assume that the involved integrals can more easily be computed than the original double integral.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares no conflicts of interest.

**References**


