Research Article

Least-Norm of the General Solution to Some System of Quaternion Matrix Equations and Its Determinantal Representations

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We constitute some necessary and sufficient conditions for the system

\[\begin{align*}
A_1 X_1 & = C_1, \\
X_1 B_1 & = C_2, \\
A_2 X_2 & = C_3, \\
X_2 B_2 & = C_4, \\
A_3 X_1 B_3 & + A_4 X_2 B_4 & = C_5,
\end{align*}\]

to have a solution over the quaternions skew field in this paper. An novel expression of general solution to this system is also established when it has a solution. The least norm of the solution to this system is also researched in this article. Some former consequences can be regarded as particular cases of this article. Finally, we give determinantal representations (analogs of Cramer’s rule) of the least norm solution to the system using row-column noncommutative determinants. An algorithm and numerical examples are given to elaborate our results.

1. Introduction

In the whole article, the notation \( \mathbb{R} \) is reserved for the real number field and \( \mathbb{H}^{m \times n} \) stands for the set of all \( m \times n \) matrices over the quaternion skew field

\[\mathbb{H} = \{ b_0 + b_1 i + b_2 j + b_3 k \mid i^2 = j^2 = k^2 = ijk \} , \]

\[\mathbb{H}^{m \times n}_r \]

specifies its subset of matrices with a rank \( r \). For \( A \in \mathbb{H}^{m \times n}_r \), let \( A^* \), \( \mathcal{R}(A) \) and \( \mathcal{N}(A) \) designate the conjugate transpose, the column right space and the left row space of \( A \). dim \( \mathcal{R}(A) \) illustrates the size of \( \mathcal{R}(A) \) and dim \( \mathcal{N}(A) = \text{dim} \mathcal{N}(A) \) by [1], which is known as the rank of \( A \) denoted by \( r(A) \).

Definition 1. The Moore-Penrose inverse of \( A \in \mathbb{H}^{m \times n}_r \), denoted by \( A^+ \), is defined to be the unique solution \( X \) to the following four matrix equations

\[\begin{align*}
(1) \quad AXA & = A, \\
(2) \quad XAX & = X,
\end{align*}\]

\[\begin{align*}
(3) \quad (AX)^* & = AX, \\
(4) \quad (XA)^* & = XA.
\end{align*}\]

Matrices satisfying (1) and (2) are known as reflexive inverses.

Note that the reflexive inverse is denoted most often by \( A^r \) but sometimes by \( A^+ \) (see, e.g., [2]) that is different from the denotation of the Moore-Penrose by \( A^+ \). We will use the denotation \( A^+ \) for the reflexive inverse.

Suppose \( I \) refers an identity matrix with feasible size. In addition, \( R_A = I - AA^+ \), \( L_A = I - A^+A \) represent a pair of orthogonal projectors induced by \( A \), respectively, and \( R_A^2 = R_A, R_A^* = R_A, L_A^2 = L_A, L_A^* = L_A \), and \( R_A^* = L_A \).

Quaternions were invented by Hamilton in 1843. Zhang presented a detail survey on quaternion matrices in [3]. Quaternions provide a concise mathematical method for representing the automorphisms of three- and four-dimensional spaces. The representations by quaternions are more compact and quicker to compute than the representations by matrices [4]. For this reason, an increasing number of applications
based on quaternions are found in various fields, such as color imaging, geometry, mechanics, linear adaptive filter, altitude control and computer science, signal processing, in particular as quaternion-valued neural networks, etc. [5–10].

The research of matrix equations have both applied and theoretical importance. In particular, the Sylvester-type matrix equations have far reaching applications in singular system control [11], system design [12], robust control [13], feedback [14], perturbation theory [15], linear descriptor systems [16], neural networks [17], and theory of orbits [18].

Some recent work on generalized Sylvester matrix equations and their systems can be observed in [19–31]. In 2014, Bao [32] examined the least-norm and extremal ranks of the least square solution to the quaternion matrix equations

\[ A_1 X = C_1, \]
\[ X B_1 = C_2, \]
\[ A_3 X B_3 = C_3. \]

Wang et al. [33] examined the expression of the general solution to the system

\[ A_1 X_1 = C_1, \]
\[ A_2 X_2 = C_3, \]
\[ A_3 X_1 B_3 + A_4 X_2 B_4 = C_c. \]

And, as an application, the $P$-symmetric and $P$-skew-symmetric solution to

\[ A_a X = C_a, \]
\[ A_b X B_b = C_b \]

has been established. Li et al. [34] established a novel expression to the general solution of system (4) and they computed the least-norm of general solution to (4). In 2009, Wang et al. [35] constituted the expression of the general solution to

\[ A_1 X_1 = C_1, \]
\[ X_1 B_1 = C_2, \]
\[ A_2 X_2 = C_3, \]
\[ X_2 B_2 = C_4, \]
\[ A_3 X_1 B_3 + A_4 X_2 B_4 = C_c. \]

and as an application they explored the $(P, Q)$-symmetric solution to the system

\[ A_a X = C_a, \]
\[ X B_b = C_b, \]
\[ A_c X B_c = C_c. \]

Some latest findings on the least-norm of matrix equations and $(P, Q)$-symmetric matrices can be consulted in [36–40]. Furthermore, our main system (6) is a special case of the following system:

\[ A_1 X_1 = C_1, \]
\[ X_2 B_1 = C_2, \]
\[ A_2 X_3 = C_2, \]
\[ X_3 B_2 = D_2, \]
\[ A_3 X_4 = C_3, \]
\[ X_4 B_3 = D_3, \]
\[ A_4 X_1 + X_2 B_4 + C_4 X_3 D_4 + C_5 X_4 D_5 = C_c, \]

which has been investigated by Zhang in 2014. But the expressions provided for the $X_1, X_2, X_3, \text{ and } X_4$ in [41], we are in position to calculate the least-norm of the solutions with its determinantal representations. When some given matrices are zero in (8), then it becomes our system and we will give such kind of expressions in which the least-norm of the solutions can also be computed with its determinantal representations. It is worthy to note that Zhang examined (8) with complex settings and we will consider our system (6) with quaternion settings.

According to our best of knowledge, the least-norm of the general solution to system (6) is not investigated by any one. Motivated by the vast application of quaternion matrices and the latest interest of least-norm of matrix equations, we construct a novel expression of the general solution to system (6) and apply this to investigate the least-norm of the general solution to system (6) over $\mathbb{H}$ in this paper. Observing that systems (3) and (4) are particular cases of our system (6), solving system (6) will encourage the least-norm to a wide class of problems in the collected work.

Since the general solutions of considered systems are expressed in term of generalized inverses, another goal of the paper is to give determinantal representations of the least-norm of the general solution to system (6) based on determinantal representations of generalized inverses.

Determinantal representation of a solution gives a direct method of its finding analogous to the classical Cramer’s rule that has important theoretical and practical significance. Through looking for their more applicable explicit expressions, there are various determinantal representations of generalized inverses even with the complex or real entries, in particular for the Moore-Penrose inverse (see, e.g., [42–44]). By virtue of noncommutativity of quaternions, the problem for determinantal representation of generalized quaternion inverses is more complicated, and only now it can be solved due to the theory of column-row determinants introduced in [45, 46]. Within the framework of the theory of noncommutative row-column determinants, determinantal representations of various generalized quaternion inverses and generalized inverse solutions to quaternion matrix equations have been derived by one of our authors (see, e.g., [47–54]) and by other researchers (see, e.g., [55–57]). Moreover,
Song et al. [58] have just recently considered determinantal representations of general solution to the two-sided coupled generalized Sylvester matrix equation over \( \mathbb{H} \) obtained using the theory of row-column determinants as well. But their proposed approach differs from our proposed. In [58], for determinantal representations of the general solution to the equation supplementary matrices have been used that not always easy to get. While, by proposed method only coefficient matrices of the equations are used. More detailed Cramer’s rule to solutions and (skew-)Hermitian solutions of some systems of matrix equations and generalized Sylvester matrix equation over \( \mathbb{H} \) are recently explored in [59, 60] and [61, 62], respectively.

The remainder of our article is directed as follows. In Section 2, we commence with some needed known results about systems of matrix equations and determinantal representations of the Moore-Penrose inverse as well. Ithas taken place in ([64], Lemma 2.4). Since Moore-Penrose inverses are reflexive stand for the projectors \( L_{\Phi^*} = I - \Phi^*_1 \Phi_1, R_{\Omega^*_1} = I - \Omega^*_1 \Omega_1 \) induced by \( \Phi_1, \Omega_1 \), respectively.

Remark 5. Since Moore-Penrose inverses are reflexive inverses, this lemma can be used for Moore-Penrose inverses without any changes. It has taken place in ([64], Lemma 2.4). But for more credibility, we prove this lemma below for the Moore-Penrose inverses as well.

Lemma 6 (see [66]). Suppose that

\[
B_1 X C_1 + B_2 Y C_2 = A
\]

is consistent linear matrix equation, where \( B_1 \in \mathbb{H}^{m \times p}, C_1 \in \mathbb{H}^{p \times n}, B_2 \in \mathbb{H}^{m \times s}, C_2 \in \mathbb{H}^{s \times n} \) and \( A \in \mathbb{H}^{m \times n} \), respectively. Then

\[
1) \text{The general solution of the homogeneous equation, } \\
B_1 X C_1 + B_2 Y C_2 = 0,
\]

\[
2) \text{can be expressed by } \\
X = X_1 X_2 + X_3, \\
Y = Y_1 Y_2 + Y_3,
\]

where \( X_1 - X_3 \) and \( Y_1 - Y_3 \) are general solution of the following four homogeneous matrix expressions

\[
B_1 X_1 = -B_2 Y_1, \\
X_1 C_1 = Y_2 C_2, \\
B_1 X_1 C_1 = 0, \\
B_2 Y_2 C_2 = 0.
\]

By computing the value of unknowns in the above equations and using them in \( X \) and \( Y \), we have

\[
X = S_1 L C U R T_1 + L R V_1 + V_2 R C_2, \\
Y = S_2 L C U R T_2 + L R W_1 + W_2 R C_2,
\]

where \( S_1 = [I_p, 0], S_2 = [0, I_p], T_1 = [I_q], T_2 = [I_q], G = [B_1, B_2], \) and \( H = [C_1, C_2] \); the matrices \( U, V_1, V_2, W_1 \) and \( W_2 \) are free to vary over \( \mathbb{H} \).
Lemma 8 (see [35]). Let \( A_1 \in \mathbb{H}^{m \times n_1}, B_1 \in \mathbb{H}^{p \times q_1}, C_1 \in \mathbb{H}^{m \times n_2}, \) \( A_2 \in \mathbb{H}^{m \times n_1}, B_2 \in \mathbb{H}^{p \times q_2}, C_2 \in \mathbb{H}^{m \times n_2}, \) and \( C_3 \in \mathbb{H}^{n_1 \times n_2} \) be given and \( X_1 \in \mathbb{H}^{p \times q_2} \) to be determined. Denote

\[
\begin{align*}
A &= A_2 L_{A_1}, \\
B &= R_{B_1} B_3, \\
C &= A_4 L_{A_2}, \\
D &= R_{B_1} B_4, \\
N &= DL_{B_1}, \\
M &= R_{C_1}, \\
S &= CL_{M}, \\
E &= C_3 - A_3 A_2^1 C_1 B_3 - AC_2 B_1^1 B_3 - A_4 A_2^1 C_1 B_4 \\
&\quad - CC_4 B_1^1 B_4.
\end{align*}
\]

Then the following conditions are tantamount:

1. System (6) is resolvable.
2. The conditions in (19) are met and

\[
\begin{align*}
R_{A_2} &= C_3 = 0, \\
C_4 L_{B_1} &= 0, \\
A_2 C_4 &= C_3 B_2, \\
R_M R_{A_4} E &= 0, \\
R_A E L_D &= 0, \\
E L_B L_N &= 0, \\
R_C E L_B &= 0.
\end{align*}
\]

3. The equalities in (19) and (22) are satisfied and

\[
\begin{align*}
M M^† R_{A_4} D^† D &= R_A E, \\
CC^† E L_B N^† N &= EL_B.
\end{align*}
\]

In these conditions, the general solution to system (6) can be written as

\[
\begin{align*}
X_1 &= A_1^† C_1 + L_{A_1} C_2 B_1^† + L_{A_1} U_1 R_{B_1}, \\
&\quad - L_{A_1} A^† C M^† E B^† R_{B_1}, \\
&\quad - L_{A_1} A^† C S E N^† D B^† R_{B_1}, \\
&\quad - L_{A_1} A^† S V_1 R_{N} D B^† R_{B_1}, \\
&\quad + L_{A_1} (U_A U_1 + Z_1 R_B) R_{B_1}, \\
X_2 &= A_2^† C_3 + L_{A_4} C_4 B_2^† + L_{A_4} M^† R_{A_4} E D^† R_{B_1}, \\
&\quad + L_{A_2} L_{M_2} S E C^† S N^† R_{B_1}, \\
&\quad + L_{A_2} L_{M} (V_1 - S^† S V_1 N N^† ) R_{B_1}, \\
&\quad + L_{A_2} W_1 R_{B_1} R_{B_2},
\end{align*}
\]

where \( U_1, V_1, W_1 \) and \( Z_1 \) are free matrices over \( \mathbb{H} \) with assignable dimensions.

2.2. Determinantal Representations of Solutions to the Quaternion Matrix Equations. Due to noncommutativity of quaternions there is a problem of a determinant of matrices with noncommutative entries (which are also defined as noncommutative determinants). There are several versions of defining of noncommutative determinants (e.g., see [68–70]). But any of the previous noncommutative determinants has not fully retained those properties which it owned for matrices with real settings. Moreover, if functional properties of a noncommutative determinant over a ring are satisfied, then it takes on a value in its commutative subset. This dilemma can be avoided due to the theory of row-column determinants.
For $A \in \mathbb{H}_r^{m \times n}$, we define $n$ row determinants and $n$ column determinants. Suppose $S_n$ is the symmetric group on the set $I_n = \{1, \ldots, n\}$.

**Definition 9** (see [45]). The $i$th row determinant of $A = (a_{ij}) \in \mathbb{H}_r^{m \times n}$ is defined for all $i = 1, \ldots, n$ by putting

$$\text{rdet}_i A = \sum_{\sigma \in S_n} (-1)^{\sigma - i} \left( a_{i_1, i_{\sigma(1)}}, \ldots, a_{i_n, i_{\sigma(n)}} \right),$$

and the order of disjoint cycles (except for the first one) is strictly conditioned by increase from right to left of their first elements, $i_k < i_{k+1} < \cdots < i_t$.

**Definition 10** (see [45]). The $j$th column determinant of $A = (a_{ij}) \in \mathbb{H}_r^{m \times n}$ is defined for all $j = 1, \ldots, n$ by putting

$$\text{cdet}_j A = \sum_{\tau \in S_n} (-1)^{\tau - j} \left( a_{\tau(1), i_1}, \ldots, a_{\tau(n), i_n} \right),$$

and the order of disjoint cycles (except for the first one) is strictly conditioned by increase from left to right of their first elements, $j_k < j_{k+1} < \cdots < j_t$.

Since $[45]$ for Hermitian $A$ we have

$$\text{rdet}_i A = \cdet_i A = \cdet_i A \in \mathbb{R}, \quad (31)$$

the determinant of a Hermitian matrix is defined by putting

$$\det A = \text{rdet}_i A = \cdet_i A \quad \text{for all } i = 1, \ldots, n. \quad (32)$$

Its properties are similar to the properties of an usual (commutative) determinant and they have been completely explored in [46] by using row and column determinants that are so defined only by construction.

For determinantal representations of the Moore-Penrose inverse, we shall use the following notations. Let $\alpha = \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta = \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min(m, n)$. Let $A^\beta_\alpha$ be a submatrix of $A$ whose rows are indexed by $\alpha$ and the columns by $\beta$. Similarly, let $A_{\alpha, \beta}$ be a principal submatrix of $A$ whose rows and columns indexed by $\alpha$. If $A \in \mathbb{H}_r^{m \times n}$ is Hermitian, then $\det A_{\alpha, \beta}$ is the corresponding principal minor of $\det A$. For $1 \leq k \leq n$, the collection of strictly increasing sequences of $k$ integers chosen from $\{1, \ldots, n\}$ is denoted by $I_k = \{\alpha : \alpha = \{\alpha_1, \ldots, \alpha_k\}, 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let $J_{r, m}[i] = \{\alpha : \alpha \in \{r, m\} \cap \alpha\}$ denotes the collection of subsequences of row indexes that contain the index $i$, and $J_{r, n}[j] = \{\beta : \beta \in \{r, n\} \cap \beta\}$ denotes the collection of subsequences of column indexes that contain $j$.

Let $a_{ij}$ be the $i$th column and $a_{ij}$ the $i$th row of $A$, respectively. Suppose $A_{\alpha, \beta}(b)$ denotes the matrix obtained from $A$ by replacing its $\beta$th column with the column-vector $b$, and $A_{\alpha, \beta}(b)$ denotes the matrix obtained from $A$ by replacing its $\beta$th row with the row-vector $b$. We denote the $\beta$th row and the $\alpha$th column of $A^\beta_\alpha$ by $a_{\alpha, \beta}^\alpha$ and $a_{\alpha, \beta}^\alpha$, respectively.

**Lemma 11** (see [47]). If $A \in \mathbb{H}_r^{m \times n}$, then the Moore-Penrose inverse $A^\dagger = (a_{ij}^\dagger) \in H_r^{m \times n}$ have the following determinantal representations,

\[
a_{ij}^\dagger = \frac{\sum_{\beta \in \{r, m\}} \text{cdet}_j (A^\dagger A)_{\alpha, \beta} (a_{ij}^\dagger)^\beta}{\sum_{\beta \in \{r, m\}} |A^\dagger A|_\alpha^\beta}, \quad (33)
\]

and

\[
a_{ij}^\dagger = \frac{\sum_{\alpha \in \{r, m\}} \text{rdet}_i (A A^\dagger)_{\alpha, \beta} (a_{ij}^\dagger)^\alpha}{\sum_{\alpha \in \{r, m\}} |A A^\dagger|_\alpha^\alpha}. \quad (34)
\]

**Remark 12.** For an arbitrary full-rank matrix $A \in \mathbb{H}_r^{m \times n}$, a row-vector $b \in \mathbb{H}_r^{m \times 1}$, and a column-vector $c \in \mathbb{H}_r^{1 \times n}$, we put

\[ (i) \quad \text{rdet}_i \left((A A^\dagger)_\alpha \right) (b) = \sum_{\alpha \in \{r, m\}} \text{rdet}_i \left((A A^\dagger)_\alpha \right)^\alpha, \]

\[ (ii) \quad \text{cdet}_j \left((A^\dagger A)_j \right) (c) = \sum_{\beta \in \{r, m\}} \text{cdet}_j \left((A^\dagger A)_j \right)^\beta, \]

\[ \text{det} (A A^\dagger) = \sum_{\alpha \in \{r, m\}} |A A^\dagger|_\alpha^\alpha, \quad \text{when } r = m, \]

\[ \text{det} (A^\dagger A) = \sum_{\beta \in \{r, m\}} |A^\dagger A|_\beta^\beta, \quad \text{when } r = n. \]

**Corollary 13.** If $A \in \mathbb{H}_r^{m \times n}$, then the projection matrix $A^\dagger A =: Q_A = (q_{ij}^\dagger)_{m \times n}$ has the determinantal representation

\[
q_{ij}^\dagger = \frac{\sum_{\beta \in \{r, m\}} \text{cdet}_j \left((A^\dagger A)_{\alpha, \beta} \right) (a_{ij}^\dagger)^\beta}{\sum_{\beta \in \{r, m\}} |A^\dagger A|_\beta^\beta}, \quad (36)
\]

where $a_{ij}^\dagger$ is the $i$th column of $A^\dagger A \in \mathbb{H}_r^{m \times n}$. 

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Lemma 15. If \( A \in \mathbb{H}^{mxn} \), then the projection matrix \( AA^\dagger =: P_A = (p_{ij})_{mn} \) has the determinantal representation

\[
P_{ij} = \frac{\sum_{\alpha \in I_{2,\alpha}} rdet_j \left( (AA^*)_j \right)_{ij}}{\sum_{\alpha \in I_{2,\alpha}} |AA^*|_{ij}^\alpha},
\]

where \( \alpha \) is the \( \alpha \)-th row of \( AA^* \) in \( \mathbb{H}^{m*n} \).

Lemma 16 (see [2]). Let \( A \in \mathbb{H}^{mxn}, B \in \mathbb{H}^{nxs}, C \in \mathbb{H}^{ms} \) be known and \( X \in \mathbb{H}^{n*r} \) be unknown. Then the matrix equation

\[
AXB = C
\]

is consistent if and only if \( AA^\dagger CB^\dagger = C \). In this case, its general solution can be expressed as

\[
X = A^\dagger CB^\dagger + L_AV + WR_B,
\]

where \( V,W \) are arbitrary matrices over \( \mathbb{H} \) with appropriate dimensions.

In [71], it’s proved that (39) is the least squares solution to (38), and its minimum norm least squares solution is \( X_{LS} = A^\dagger CB^\dagger \).

Lemma 17. Let \( A \in \mathbb{H}^{mxn}, B \in \mathbb{H}^{nxs}, C \in \mathbb{H}^{ms} \) be known and \( X \in \mathbb{H}^{n*r} \) be unknown. Then the matrix equation

\[
AXB = C
\]

is consistent if and only if \( AA^\dagger CB^\dagger = C \). In this case, its general solution can be expressed as

\[
X = A^\dagger CB^\dagger + L_AV + WR_B,
\]

where \( V,W \) are arbitrary matrices over \( \mathbb{H} \) with appropriate dimensions.

Corollary 14. If \( A \in \mathbb{H}^{mxn} \), then the projection matrix \( AA^\dagger =: P_A = (p_{ij})_{mn} \) has the determinantal representation

\[
P_{ij} = \frac{\sum_{\alpha \in I_{2,\alpha}} rdet_j \left( (AA^*)_j \right)_{ij}}{\sum_{\alpha \in I_{2,\alpha}} |AA^*|_{ij}^\alpha},
\]

where \( \alpha \) is the \( \alpha \)-th row of \( AA^* \) in \( \mathbb{H}^{m*n} \).

Lemma 18 (see [48]). Let \( A \in \mathbb{H}^{mxn}, B \in \mathbb{H}^{nxs}, C \in \mathbb{H}^{ms} \) be known and \( X \in \mathbb{H}^{n*r} \) be unknown. Then the equation \( XB = C \) is solvable if and only if \( C = CB^\dagger + WR_B \), where \( W \) is any matrix with conformable dimension. Moreover, its minimum norm least squares solution is \( X = CB^\dagger \) with the following determinantal representation,

\[
x_{ij} = \frac{\sum_{\alpha \in I_{2,\alpha}} rdet_j \left( (BB^*)_j \right)_{ij}}{\sum_{\alpha \in I_{2,\alpha}} |BB^*|_{ij}^\alpha},
\]

where \( \alpha \) is the \( \alpha \)-th row of \( \mathbb{C} = CB^* \).

3. A New Expression of the General Solution to System (6)

First, we show that Lemma 4 is true for the Moore-Penrose inverses.

Lemma 19. Let \( \Phi, \Omega \) be matrices over \( \mathbb{H} \) and

\[
\Phi = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix},
\]

\[
\Omega = \begin{bmatrix} \Omega_1 \\ \Omega_2 \end{bmatrix},
\]

\[
F = \Phi_2 L_{\Phi_1},
\]

\[
T = R_{\Omega_1} T_{\Omega_2}.
\]

Then

\[
L_\Phi = L_{\Phi_1} L_F,
\]

\[
L_\Omega = \begin{bmatrix} L_{\Omega_1} & -\Omega_1 \Omega_2 L_T \\ 0 & L_T \end{bmatrix},
\]

\[
R_\Omega = R_1 R_{\Omega_1},
\]

\[
R_\Phi = \begin{bmatrix} R_{\Phi_1} & 0 \\ -R_2 \Phi_1 R_{\Phi_1} & R_F \end{bmatrix}.
\]

where \( \Omega_1^\dagger, \Phi_1^\dagger \) are the Moore-Penrose inverses of \( \Omega_1 \) and \( \Phi_1 \), and \( L_{\Phi_1}, R_{\Omega_1}, L_T, R_F, L_{\Omega_1}, \) and \( R_\Phi \) are projectors with respect to the corresponding Moore-Penrose inverses.

Proof. In ([65], Lemma 2.4), it is proved that for fixed reflexive inverses \( \Omega_1^\dagger \) and \( T^\dagger \), the reflexive inverse \( \Omega_1^\dagger \) can be expressed as follows,

\[
\Omega_1^\dagger = \begin{bmatrix} \Omega_1^\dagger - \Omega_1^\dagger \Omega_2 T^\dagger R_{\Omega_1} \\ T^\dagger R_{\Omega_1} \end{bmatrix}.
\]
We choose \( \Omega^\dagger \) as the Moore-Penrose inverses, and \( R_{\Omega^1} \) as the projector with respect to the Moore-Penrose inverse \( \Omega^\dagger \) and show that the obtained matrix

\[
\Omega^\dagger = \begin{bmatrix} \Omega_1^\dagger - \Omega_1^\dagger \Omega_2 T^\dagger R_{\Omega_1} \\ T^\dagger R_{\Omega_1} \end{bmatrix}
\]  

(49)

is the Moore-Penrose inverse of \( \Omega \). For this, it is enough to prove that \( \Omega^\dagger \) satisfies the conditions (3) and (4) in Definition 1.

Since by Lemma 3, \( T^\dagger R_{\Omega^1} = (R_{\Omega^1} \Omega_2)^\dagger R_{\Omega^1} = T^\perp \), then \( \Omega^\dagger \) can be expressed as

\[
\Omega^\dagger = \begin{bmatrix} \Omega_1^\dagger - \Omega_1^\dagger \Omega_2 T^\dagger \\ T^\dagger \end{bmatrix}.
\]  

(50)

So,

\[
\Omega \Omega^\dagger = [\Omega_1 \ \Omega_2] \begin{bmatrix} \Omega_1^\dagger - \Omega_1^\dagger \Omega_2 T^\dagger \\ T^\dagger \end{bmatrix}
= [\Omega_1 \Omega_1^\dagger - \Omega_1^\dagger \Omega_2 T^\dagger + \Omega_2 T^\dagger]
= [\Omega_1 \Omega_1^\dagger + R_{\Omega_1} \Omega_2 T^\dagger]
= [\Omega_1 \Omega_1^\dagger + R_{\Omega_1} \Omega_2 (R_{\Omega_1} \Omega_2)^\dagger]
\]

(51)

Since condition (3) is satisfied by components, namely,

\[
(\Omega \Omega^\dagger)^* = \Omega \Omega^\dagger,
\]

(52)

\[
(R_{\Omega_1} \Omega_2 (R_{\Omega_1} \Omega_2)^\dagger) = R_{\Omega_1} \Omega_2 (R_{\Omega_1} \Omega_2)^\dagger
\]

(53)

it follows that \( \Omega^\dagger \) satisfies condition (3) as well; i.e., \((\Omega^\dagger)^* = \Omega^\dagger \).

Similarly, it can be shown that \( \Omega^\dagger \) satisfies condition (4). Hence, the Moore-Penrose inverse of \( \Omega \) can be expressed by (49). From this (46) immediately follow.

The equations (47) can be proved similarly. \( \square \)

Now we demonstrate the principal theorem of this section.

**Theorem 20.** Assume that \( S_1 = [I, 0], S_2 = [0, I], T_1 = \begin{bmatrix} I_0 \\ 0 \end{bmatrix}, T_2 = \begin{bmatrix} I_0 \\ 0 \end{bmatrix}, G = [A \ C], H = \begin{bmatrix} \beta_3 \\ \beta_4 \end{bmatrix} \), \( H_1 = L_A L_A^\perp \), \( H_2 = L_A S_L \), \( H_3 = L_A S_L H_4 = L_A S_L H_5 = L_A S_L \), \( H_6 = R_{\beta_4}^\perp T_2 R_{\beta_2} \), and system (6) is solvable, then the general solution to our system can be formed as

\[
X_1 = A_1^\perp C_1 + L_A A_1 C_2 B_1^\perp + L_A A_1 E B_1 R_{\beta_1}
- L_A A_1 C M^\perp E B_1 R_{\beta_1}
\]

and

\[
X_2 = A_2^\perp C_2 + L_A A_2 C_4 B_2^\perp + L_A A_2 M^\perp R_{\beta_1}
+ L_A A_2 M^\perp S^\perp C^\perp E^\perp R_{\beta_2}
\]

\[
+ L_A A_2 M^\perp S^\perp C^\perp E^\perp H_2 W_2 R_{\beta_2}
+ H_2 W_2 R_{\beta_2}
\]

where \( U, V_1, V_2, W_1, \) and \( W_2 \) are free matrices over \( \mathbb{H} \) with allowable dimensions.

**Proof.** Our proof contains three parts. At the first step, we show that the matrices \( X_1 \) and \( X_2 \) have the forms

\[
X_1 = \phi_0 + H_1 V_1 R_{\beta_1} + L_A A_1 V_2^\perp R_{\beta_1} + H_2 U H_3,
\]

(56)

\[
X_2 = \psi_0 + H_2 W_2 R_{\beta_2} + L_A A_2 W_2^\perp R_{\beta_2} + H_2 U H_6,
\]

(57)

where \( \phi_0 \) and \( \psi_0 \) are any pair of particular solution to system (6), \( V_1, V_2, W_1, W_2 \) and \( U \) are free matrices of allowable shape over \( \mathbb{H} \), are solutions to system (6). At the second step, we display that any couple of solutions \( \phi_0 \) and \( \psi_0 \) to system (6) can be established as (56) and (57), respectively. At the end, we confirm that

\[
\mu = A_1^\perp C_1 + L_A A_1 C_2 B_1^\perp + A_1^\perp E B_1 - A_1^\perp C M^\perp E B_1
- A_1^\perp S C^\perp E^\perp R_{\beta_2}
\]

\[
= A_1^\perp C_1 + L_A A_1 C_2 B_1^\perp + L_A A_2 M^\perp R_{\beta_1} + L_A A_2 M^\perp S^\perp C^\perp E^\perp R_{\beta_2}
\]

\[
+ L_A A_2 M^\perp S^\perp C^\perp E^\perp H_2 W_2 R_{\beta_2}
+ L_A A_2 M^\perp S^\perp C^\perp E^\perp H_2 W_2 R_{\beta_2}
\]

(58)

(59)

are a couple of particular solutions to system (6).

Now we prove that a couple of matrices \( X_1 \) and \( X_2 \) having the shape of (56) and (57), respectively, are solutions to system (6). Observe that

\[
A_1^\perp C_1 B_1 + L_A A_1 C_2 B_1^\perp = A_1^\perp A_1 C_2 + L_A A_1 C_2 = C_2,
\]

(60)

\[
A_2^\perp C_4 B_2 + L_A A_2 C_4 B_2^\perp = A_2^\perp A_2 C_4 + L_A A_2 C_4 = C_4.
\]

It is evident that \( X_1 \) having the form (56) is a solution of \( A_1 X_1 = C_1 \) and \( X_2 B_1 = C_2 \) and \( X_2 \) having the form (57) is a solution to \( A_2 B_1 = C_3 \), \( X_2 B_2 = C_4 \). Now we are left to show that \( A_2 X_1 B_1 + A_1 X_2 B_2 = C_0 \) is satisfied by \( X_1 \) and \( X_2 \) given in (56) and (57). By Lemma 4, we have

\[
A S_L = A [I, 0] \begin{bmatrix} L_A & -A^\perp C M \end{bmatrix}
\]

\[
= A \begin{bmatrix} L_A & -A^\perp C M \end{bmatrix} \begin{bmatrix} 0 & -A A^\perp C M \end{bmatrix}
\]

\[
= \begin{bmatrix} 0 & -A A^\perp C M \end{bmatrix}
\]

(61)

\[
= -[0 \ S] = -C S_2 L_G.
\]
It is manifest that

$R_H T_1 B = \begin{bmatrix} R_B & 0 \\ R_N D B^\dagger & 0 \end{bmatrix} \left[ \begin{array}{c} I_{n_1} \\ 0 \end{array} \right] = \begin{bmatrix} R_B \\ R_N D (I - L_B) \end{bmatrix} B$

\[ = \begin{bmatrix} 0 \\ R_N D \end{bmatrix} = R_H T_2 D. \tag{62} \]

Observe that $AL_A = 0$ and by using (61) and (62), we arrive that

$$A_3 X_1 B_3 + A_4 X_2 B_4 = C_c. \tag{63}$$

Conversely, assume that $\mu_0$ and $\nu_0$ are any couple of solutions to our system (6). By Lemma 7, we have

$$A_1 A_1^\dagger C_1 = C_1,$$
$$C_c B_1^\dagger B_1 = C_2,$$
$$A_2 A_2^\dagger C_3 = C_3,$$
$$C_c B_2^\dagger B_2 = C_4,$$
$$A_1 C_2 = C_1 B_1,$$
$$A_2 C_4 = C_3 B_2. \tag{64}$$

Observe that

$$L_A, \mu_0 R_{B_1} = (I - A_1^\dagger A_1) \mu_0 (I - B_1 B_1^\dagger)$$
$$= \mu_0 - \mu_0 B_1 B_1^\dagger - A_1^\dagger A_1 \mu_0 + A_1^\dagger A_1 \mu_0 B_1 B_1^\dagger$$
$$= \mu_0 - C_2 B_1^\dagger - A_1^\dagger C_1 + A_1^\dagger A_1 C_2 B_1^\dagger$$
$$= \mu_0 - L_A, C_2 B_1^\dagger - A_1^\dagger C_1 \tag{65}$$

produces

$$\mu_0 = L_A, C_2 B_1^\dagger + A_1^\dagger C_1 + L_A, \mu_0 R_{B_1}. \tag{66}$$

On the same lines, we can get

$$\nu_0 = L_A, C_3 B_1^\dagger + A_1^\dagger C_3 + L_A, \nu_0 R_{B_1}. \tag{67}$$

It is manifest that $\mu_0$ and $\nu_0$ defined in (66)-(67) are also solution pair of

$$AX_1 B + CX_2 D = E. \tag{68}$$

Since

$$AX_1 B + CX_2 D = A_3 L_A, \mu_0 R_{B_1} B_3 + A_4 L_A, \nu_0 R_{B_2} B_4$$

$$= A_3 \left( \mu_0 - L_A, C_2 B_1^\dagger - A_1^\dagger C_1 \right) B_3$$

$$= A_3 \mu_0 B_3 - A_3 L_A, C_2 B_1^\dagger B_3$$
$$- A_1^\dagger C_1 B_3 + A_4 \nu_0 B_4$$
$$- A_4 L_A, C_4 B_1^\dagger B_4 - A_4 A_1^\dagger C_3 B_4$$

$$= A_3 \mu_0 B_3 + A_4 \nu_0 B_4 - AC_2 B_1^\dagger B_3$$
$$- A_1^\dagger C_1 B_3 - CC_4 B_1^\dagger B_4$$
$$- A_4 A_1^\dagger C_3 B_4$$

$$= C_c - AC_2 B_1^\dagger B_3 - A_1^\dagger C_1 B_3$$
$$- CC_4 B_1^\dagger B_4 - A_4 A_1^\dagger C_3 B_4 = E. \tag{69}$$

Hence by Lemma 6, $\mu_0$ and $\nu_0$ can be written as

$$\mu_0 = X_{01} + S_1 L_G U R_H T_1 + L_A V_1 + V_2 R_B, \tag{70}$$
$$\nu_0 = X_{02} + S_2 L_G U R_H T_2 + L_C V_1 + W_2 R_D, \tag{71}$$

where $X_{01}$ and $X_{02}$ are a couple of special solutions to (68) and $U, V_1, V_2, W_1$ and $W_2$ are free matrices with agreeable dimensions. Using (70) and (71) in (66) and (67), respectively, we get

$$\mu_0 = X_{10} + H_3 U H_4 + H_1 V_1 R_B + L_A, V_2 R_R B_1,$$
$$\nu_0 = X_{20} + H_3 U H_6 + H_4 W_1 R_B + L_A, W_2 R_D R_B, \tag{72}$$

where $X_{10} = A_1^\dagger C_1 + L_A, C_2 B_1^\dagger + L_A, X_{01} R_B$ and $X_{20} = A_1^\dagger C_3 + L_A, C_4 B_2^\dagger + L_A, X_{02} R_B$. It is evident that $X_{10}$ and $X_{20}$ are a couple of solutions to system (6). It is clear that $\mu_0$ and $\nu_0$ can be represented by (56) and (57), respectively. Lastly, by putting $U_1, V_1, W_1$, and $Z_1$ equal to zero in (24) and (25), we conclude that $\mu$ and $\nu$ are special solutions to system (6). Hence the expressions (54) and (55) represent the general solution to system (6) and the theorem is completed. \qed

Remark 21. Due to Lemma 3 and taking into account $L_A L_M = L_M L_A$, we have the following simplification of the solution pair to system (6) that is identical for (24)-(25) and (54)-(55) when $U, V_1, V_2, Z_1, W_1$, and $W_2$ disappear,

$$X_1 = A_1^\dagger C_1 + L_A, C_2 B_1^\dagger + A_1^\dagger E B_1^\dagger - A_1^\dagger A_4 M^* E B_1^\dagger$$
$$- A_1^\dagger S C^* E N^* B_4 B_1^\dagger, \tag{73}$$

$$X_2 = A_1^\dagger C_3 + L_A, C_4 B_2^\dagger + M^* E D^\dagger + S^* E C^* E N^\dagger.$$

Comment 1. We have established a novel expression of the general solution to system (6) in Theorem 20 which is different from one created in [35]. With the help of this novel expression, we can explore the least-norm of the general solution which can not be studied with the help of the
expression given in [35], which is one of the advantage of our new expression.

Now we discuss some special cases of our system.

If \( B_1, B_2, C_2 \) and \( C_4 \) disappear in Theorem 20, then we gain the following conclusion.

**Corollary 22.** Denote \( S_1 = [I_p, 0], S_2 = [0, I_p], T_1 = \left[ \begin{array}{c} \small{[I_0]} \end{array} \right], \)
\( T_2 = \left[ \begin{array}{c} \small{[I_0]} \end{array} \right], G = [A, C], H = \left[ \begin{array}{c} \small{[B_1, B_2]} \end{array} \right], H_1 = L_A L_A H_2 = L_A S_1 H_3, H_4 = L_A L_A H_5 = L_A S_2 H_6, \)
then we get the least-norm of the solution of (8) with the help of Theorem 20 quite smoothly. This is one of the advantage of our expressions over the expression given in [35], which is one of the advantage of our expressions over the expressions given in [35].

**Algorithm 24.** (1) Input \( A_1, B_1, C_1, A_2, B_2, C_2, D_2, A_3, B_3, C_3, \)
\( D_3, A_4, B_4, C_4 \) with viable dimensions over \( \mathbb{H} \).
(2) Evaluate \( X_1 \) and \( X_2 \) by (54)-(55).

**Example 25.** For given matrices

\[
A_1 = \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix},
B_1 = \begin{bmatrix}
-j & k \\
-i & -k
\end{bmatrix},
C_1 = \begin{bmatrix}
i & k \\
0 & 1
\end{bmatrix},
C_2 = \begin{bmatrix}
i & j \\
-k & 1
\end{bmatrix},
A_2 = \begin{bmatrix}
i & 0 \\
0 & j
\end{bmatrix},
A_3 = \begin{bmatrix}
i & -1 \\
k & j
\end{bmatrix},
A_4 = \begin{bmatrix}
0 & j \\
0 & 0
\end{bmatrix},
B_2 = \begin{bmatrix}
j & k \\
0 & 1
\end{bmatrix},
B_3 = \begin{bmatrix}
i & 1 \\
k & -j
\end{bmatrix},
B_4 = \begin{bmatrix}
j & -k \\
k & -j
\end{bmatrix},
C_3 = \begin{bmatrix}
i & -k \\
0 & 1
\end{bmatrix},
C_4 = \begin{bmatrix}
i & -1 \\
j & 0
\end{bmatrix},
C_5 = \begin{bmatrix}
-1 & -j + k \\
2 + i - j + k
\end{bmatrix},
C_6 = \begin{bmatrix}
-1 & -j + k \\
-1 & -j + k
\end{bmatrix}.
\]

By these given matrices, the consistency conditions of (6) from Lemma 3 are fulfilled. So, system (6) is resolvable.

Now we compute the partial solution to system (6) when the consistency conditions of (6) are fulfilled. So, system (6) is resolvable. Using determinantal representations (33)-(34) for computing Moore-Penrose inverses, we find that

\[
A_1^{-1} = \begin{bmatrix}
1 & -j \\
i & -k
\end{bmatrix},
L_{A_1} = \frac{1}{2} \begin{bmatrix}
1 & i \\
-i & 1
\end{bmatrix}.
\]
\[ B_1^* = \frac{1}{6} \begin{bmatrix} j & 1 \\ -k & -i \\ i & k \end{bmatrix}, \]
\[ R_{B_1} = \frac{1}{2} \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}, \]
\[ B_2^* = \frac{1}{2} \begin{bmatrix} -i & -k \\ k & -j \end{bmatrix}, \]
\[ A_2^* = \frac{1}{3} \begin{bmatrix} -i & k & -j \end{bmatrix}, \]
\[ A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k & j \end{bmatrix}, \]
\[ B = \begin{bmatrix} i & 1 \\ k & -j \end{bmatrix}, \]
\[ A^* = \frac{1}{2} \begin{bmatrix} 0 & 0 & -k \\ 0 & 0 & -j \end{bmatrix}, \]
\[ B^* = \frac{1}{4} \begin{bmatrix} -1 & -k \\ 1 & j \end{bmatrix}, \]
\[ E = \begin{bmatrix} k & -j \\ -1 & i \\ j & k \end{bmatrix}, \]
\[ R_{B_2} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \]

Since \( L_{A_2} = 0 \) and \( D = 0 \), then \( C, S, M, N \) are zero-matrices. Hence the general solution to our system (6) is
\[ X_1 = A_1^* C_1 + L_{A_1} C_2 B_1^* + A^1 E B^1 \]
\[ = \frac{1}{12} \begin{bmatrix} 5 + i - 2j - k & -2 - i + 7j + 5k \\ -5 + i - j + 2k & -1 + 2i - j - k \end{bmatrix}, \]
\[ X_2 = A_2^* C_3 = \frac{1}{3} \begin{bmatrix} 2 + k & -1 - 2j \end{bmatrix}. \]
where
\[ J = 2 \text{ Re} \left[ \text{tr} \left( (H_1 V_1 R_B) \right) \right] \]

\[ + H_2 U H_3 + L_{A_1} V_2 R_B R_B^* \left( A_1 C_1 + L_{A_1} C_2 B_1^* \right)^* \left( A_1 C_1 + L_{A_1} C_2 B_1^* \right) + A_1^{\dagger} E B^\dagger - A_1^{\dagger} A_4 M^i E B^i - A_1^{\dagger} S C^i E N^i B_4^i \right]. \]

Now we want to show that \( J = 0 \). Applying Lemmas 3, 4, and 26, we have

By using (84)-(86) in (83) produces \( J = 0 \). Since \( X_1 \) is arbitrary, we get (79) from (82). On the same way, we can prove that (80) hold.

A special cases of our system (6) are given below.

If \( B_1, B_2, C_2 \) and \( C_4 \) become zero matrices in Theorem 27, then again we get the principal result of [30].

**Corollary 28.** Assume that system (4) is solvable, then the least-norm of the solution pair \( X_1 \) and \( X_2 \) to system (4) can be furnished as

\[ \| X_1 \|_{\text{min}} = A_1^{\dagger} C_1 + A_1^{\dagger} E B^\dagger - A_1^{\dagger} A_4 M^i E B^i - A_1^{\dagger} S C^i E N^i B_4^i, \]

If \( A_2, B_2, C_3, A_4, B_4, \) and \( C_4 \) vanish in our system, then we get the next consequence.

**Corollary 29.** Suppose that \( A_1, B_1, C_1, C_2, A_3, B_3, \) and \( C_3 \) are given. Then the least-norm of the least square solution to system (3) is launched by

\[ \| X_1 \|_{\text{min}} = A_1^{\dagger} C_1 + L_{A_1} C_2 B_1^* + \left( A_3 L_{A_1} \right)^* \cdot \left( C_c - A_3^{\dagger} A_1^{\dagger} C_1 B_1 - A_3 L_{A_1} C_2 B_1^* B_3^* \right) \left( R_B B_3^* \right)^*. \]

**Comment 5.** Corollary 29 is the key result of [32].
5. Determinantal Representations of the Least-Norm Solution to System (6)

In this section, we give determinantal representations of the least-norm solution to system (6). Let $A_1 \in \mathbb{H}^{p \times a}$, $B_1 \in \mathbb{H}^{r \times a}$, $A_2 \in \mathbb{H}^{k \times p}$, $B_2 \in \mathbb{H}^{q \times d}$, $A_3 \in \mathbb{H}^{p \times a}$, $B_3 \in \mathbb{H}^{r \times q}$, $A_4 \in \mathbb{H}^{p \times k}$, $B_4 \in \mathbb{H}^{q \times d}$, $r(A) = r_0$, $r(B) = r_1$, $r(C) = r_11$, $r(D) = r_12$, $r(M) = r_13$, $r(N) = r_14$, and $r(S) = r_15$.

First, consider each term of (79) separately.

\[
X_{ij}^{(12)} = \frac{\sum_{\alpha \in I_2(i)} \text{r.det}_f \left( \left( B_i B_i^* \right)_{j} \left( c_{ij}^{(12)} \right) \right) \alpha}{\sum_{\alpha \in I_2} |B_i B_i^*|_a^\alpha} - \frac{\sum_{\beta \in I_1(i)} \text{c.det} \left( \left( A_i A_i^* \right)_{j} \left( c_{ij}^{(12)} \right) \right) \beta}{\sum_{\beta \in I_1} |A_i A_i^*|_a^\beta},
\]

where $c_{ij}^{(12)}$ is the $i$th row of $C_{12} = A_i C_{12}$ and $d_{ij}^{(1)}$ is the $j$th column of $A_i^* A_1$.

Construct the matrix $\Psi_1 = (\psi_{ij}^{(1)})$, where

\[
\psi_{ij}^{(1)} = \sum_{\beta \in I_n(i)} \text{c.det} \left( \left( A_i^* A_1 \right)_{j} \left( d_{ij}^{(1)} \right) \right) \beta,
\]

and denote $\tilde{\Psi}_1 = \Psi_1 C_{22}^*$. Then, from (90), it follows that

\[
X_{ij}^{(12)} = \frac{\sum_{\alpha \in I_2(i)} \text{r.det}_f \left( \left( B_i B_i^* \right)_{j} \left( c_{ij}^{(12)} \right) \right) \alpha}{\sum_{\alpha \in I_2} |B_i B_i^*|_a^\alpha} - \frac{\sum_{\beta \in I_1(i)} \text{c.det} \left( \left( A_i^* A_1 \right)_{j} \left( c_{ij}^{(12)} \right) \right) \beta}{\sum_{\beta \in I_1} |A_i A_i^*|_a^\beta},
\]

where $\psi_{ij}^{(1)}$ is the $i$th row of the matrix $\Psi_1$.

If we construct the matrix $\Psi_2 = (\psi_{ij}^{(2)})$, where

\[
\psi_{ij}^{(2)} = \sum_{\alpha \in I_2(i)} \text{r.det}_f \left( \left( B_i B_i^* \right)_{j} \left( c_{ij}^{(12)} \right) \right) \alpha,
\]

and denote $\tilde{\Psi}_2 = A_i^* A_1 \Psi_2$, and then, from (90), we obtain

\[
X_{ij}^{(12)} = \frac{\sum_{\alpha \in I_2(i)} \text{r.det}_f \left( \left( B_i B_i^* \right)_{j} \left( c_{ij}^{(12)} \right) \right) \alpha}{\sum_{\alpha \in I_2} |B_i B_i^*|_a^\alpha} - \frac{\sum_{\beta \in I_1(i)} \text{c.det} \left( \left( A_i^* A_1 \right)_{j} \left( \psi_{ij}^{(2)} \right) \right) \beta}{\sum_{\beta \in I_1} |A_i A_i^*|_a^\beta},
\]

where $\psi_{ij}^{(2)}$ is the $j$th column of the matrix $\Psi_2$.

(i) Denote $C_{11} := A_i^* C_{11}$. Due to Corollary 17 for the first term of (79), $X_{11} = A_i^* C_{11} = (x_{ij}^{(11)})$, we have

\[
x_{ij}^{(11)} = \frac{\sum_{\beta \in I_n(i)} \text{c.det}_f \left( \left( A_i^* A_1 \right)_{j} \left( c_{ij}^{(11)} \right) \right) \beta}{\sum_{\beta \in I_n} |A_i^* A_1|_a^\beta} \tag{89}
\]

where $c_{ij}^{(11)}$ is the $j$th column of $C_{11}$.

(ii) For the second term of (79) we have, $X_{12} = (x_{ij}^{(12)}) = L_{A_i} C_{22}^1 = C_{22}^1 - Q_{A_i} C_{22}^1$. So, due to Corollaries 18 and 13,

\[
x_{ij}^{(13)} = \frac{\sum_{\alpha \in I_2(i)} \text{r.det}_f \left( \left( B_i^* B_i \right)_{j} \left( d_{ij}^{(1)} \right) \right) \alpha}{\sum_{\alpha \in I_2} |B_i^* B_i|_a^\alpha} \tag{90}
\]

or

\[
x_{ij}^{(13)} = \frac{\sum_{\alpha \in I_2(i)} \text{r.det}_f \left( \left( B_i^* B_i \right)_{j} \left( d_{ij}^{(1)} \right) \right) \alpha}{\sum_{\alpha \in I_2} |B_i^* B_i|_a^\alpha} \tag{91}
\]

where

\[
\psi_{ij}^{(1)} = \sum_{\beta \in I_n(i)} \text{r.det}_f \left( \left( B_i^* B_i \right)_{j} \left( d_{ij}^{(1)} \right) \right) \alpha, \quad u = 1, \ldots, m,
\]

\[
\psi_{ij}^{(2)} = \sum_{\alpha \in I_2(i)} \text{r.det}_f \left( \left( B_i^* B_i \right)_{j} \left( d_{ij}^{(1)} \right) \right) \alpha, \quad v = 1, \ldots, r,
\]

are the column vector and the row vector, respectively, $a_{ij}^{(1)}$ and $c_{ij}^{(1)}$ are the $i$th row and the $v$th column of $E_1 = A^* B^*$. (iv) For the fourth term of (79), $A_i^* A_4 M^* B_1^* = X_{14} = (x_{ij}^{(4)})$, using the determinantal representation (33) for $A_i^*$ and by Theorem 2.15, we have

\[
x_{ij}^{(14)} = \frac{\sum_{\alpha \in I_2(i)} \text{c.det} \left( \left( A_i^* A_1 \right)_{j} \left( \phi_{ij}^{(4)} \right) \right) \beta}{\sum_{\alpha \in I_2} |A_i A_i^*|_a^\alpha} \tag{92}
\]
and

$$\phi_{ij}^B = \left[ \sum_{\alpha \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (BB^*)_{_{j \cdot l}} \left( \epsilon^{(2)}_{y} \right) \right) \right]^\alpha \in \mathbb{H}_1^{\alpha \cdot 1},$$

$$u = 1, \ldots, p,$$

$$v = 1, \ldots, r,$$

are the column vector and the row vector, respectively, $a_{f_{(1)}}$ is the $f_{th}$ column of $A_{14} := A^* A_{4}, e_{u_{_{(2)}}}$ and $e_{v_{_{(3)}}}$ are the $u_{th}$ row and the $v_{th}$ column of $E_2 := M^* E B^*$, respectively.

Construct the matrix $\Phi = (\phi_{ij})$, where $\phi_{ij}$ is given by (99) and denote $A^* A \Phi =: \Phi = (\bar{\phi}_{ij})$. Then, from (98), we get the following final determinantal representation of the fourth term of (79),

$$x_{ij}^{(1)} = \frac{\sum_{\beta \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (A^* A)_{j \cdot l} \left( \bar{\phi}_{ij} \right) \right)^\beta}{\sum_{\beta \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (A^* A)_{j \cdot l} \right)^\beta} \in \mathbb{H}_1^{\alpha \cdot 1},$$

and denote $\bar{\phi}_{j}$ is the $j_{th}$ column of $\Phi$.

(v) For the fifth term of (79), $A^* SC^* E N^t B_4 B^t \equiv X_{15} = (x_{ij}^{(15)})$, due to Corollary 17 to $A^* S$, by Theorem 2.15 to $C^* E N^1$, and Corollary 18 to $B_4 B^t$, we obtain

$$x_{ij}^{(15)} = \sum_{\beta \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (A^* A)_{j \cdot l} \left( \phi_{ij}^{(15)} \right) \right)^\beta \sum_{\alpha \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (BB^*)_{j \cdot l} \left( \phi_{ij}^{(15)} \right) \right)^\alpha$$

where $x_{ij}^{(15)}$ is the $l_{th}$ column of $S_1 := A^* S, b_{ij}^{(15)}$ is the $f_{th}$ row of $B_{15} = B_4 B^*$,

$$\omega_{ij} = \sum_{\beta \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (C^* C)_{j \cdot l} \left( \phi_{ij} \right) \right)^\beta$$

$$= \sum_{\alpha \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (NN^*)_{j \cdot l} \left( \phi_{ij} \right) \right)^\alpha, (103)$$

and

$$\phi_{ij}^{(N)} = \left[ \sum_{\alpha \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (NN^*)_{j \cdot l} \left( \phi_{ij}^{(N)} \right) \right) \right]^\alpha \in \mathbb{H}_1^{\alpha \cdot 1},$$

$$u = 1, \ldots, p,$$

are the column vector and the row vector, respectively, $e_{u}$ and $e_{v}$ are the $u_{th}$ row and the $v_{th}$ column of $E_3 = C^* E N^*$. Construct the matrix $\Omega = (\omega_{ij})$, where $\omega_{ij}$ is determined by (103), and denote $\Omega = A^* S \Omega B_4 B^*$. Then, from (102), it follows that

$$x_{ij}^{(1)} = \sum_{\beta \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (A^* A)_{j \cdot l} \left( \omega_{ij} \right) \right)^\beta$$

$$\sum_{\alpha \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (BB^*)_{j \cdot l} \left( \omega_{ij} \right) \right)^\alpha$$

where $e_{f}$ and $e_{l}$ are the unit row-vector and the unit column-vector whose components are 0 except the $f_{th}$ or $l_{th}$ components which are 1, respectively.

If we denote

$$\omega_{ij}^{(1)} = \sum_{\beta \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (A^* A)_{j \cdot l} \left( e_{f} \right) \right)^\beta \omega_{ij},$$

$$\omega_{ij}^{(1)} = \sum_{\beta \in I_{r_{12} \cdot l_{(j)}}} \text{rdet}_f \left( (BB^*)_{j \cdot l} \left( \omega_{ij}^{(1)} \right) \right)^\alpha$$

then, from (102), it follows the determinantal representation
where \( \omega^{(1)}_l \) is the \( i \)th row of the matrix \( \Omega^{(1)} = (\omega^{(1)}_j) \) that is determined by (106).

If we denote
\[
\omega^{(2)}_{ij} = \sum_{f} \tilde{\omega}_f \sum_{\alpha \in I_{\tau_{1f}(j)}} \text{rdet}_j \left( (BB^*)_j, (e_f) \right)^\alpha_{\alpha},
\]
then, from (102), it follows the determinantal representation
\[
X_{ij}^{(15)} = \frac{\sum_{\beta \in I_{\tau_{1f}(j)}} \text{cdet}_j \left( (A^* A)_j, (\omega^{(2)}_j) \right)^\beta_{\beta}}{\sum_{\beta \in I_{\tau_{1f}(j)}} |A^* A|_\beta^2 \sum_{\beta \in I_{\tau_{1f}(j)}} |C^* C|_\beta^2 \sum_{\alpha \in I_{\tau_{1f}(j)}} |N N^*|_{\alpha}^2 \sum_{\beta \in I_{\tau_{1f}(j)}} |BB^*|_{\alpha}^2}.
\]

where \( \omega^{(2)}_j \) is the \( j \)th column of the matrix \( \Omega^{(2)} = (\omega^{(2)}_j) \) that is determined by (108).

Similarly, consider each term of (80) separately.

\[
x^{(22)}_{gf} = \frac{\sum_{\alpha \in I_{\tau_{1f}(f)}} \text{rdet}_f \left( (B_2 B_2^*)_f, (c^{(22)}_g) \right)^\alpha_{\alpha}}{\sum_{\alpha \in I_{\tau_{1f}(f)}} |B_2 B_2^*|_{\alpha}}
- \frac{\sum_{\beta \in I_{\tau_{1f}(g)}} \text{cdet}_g \left( (A^* A)_g, (d^{(2)}_f) \right)^\beta_{\beta} \sum_{\alpha \in I_{\tau_{1f}(f)}} \text{rdet}_f \left( (B_2 B_2^*)_f, (c^{(22)}_g) \right)^\alpha_{\alpha} \sum_{\beta \in I_{\tau_{1f}(g)}} |A^* A|_\beta |B_2 B_2^*|_\beta}{\sum_{\beta \in I_{\tau_{1f}(g)}} |A^* A|_\beta^2 \sum_{\beta \in I_{\tau_{1f}(g)}} |B_2 B_2^*|_{\alpha}}.
\]

where \( c^{(22)}_g \) is the \( g \)th row of \( C_{22} := C_2^* C_2 \) and \( d^{(2)}_f \) is the \( f \)th column of \( A^*_2 A_2 \).

Construct the matrix \( Y_1 = (v^{(1)}_{gf}) \), where
\[
v^{(1)}_{gf} = \sum_{\beta \in I_{\tau_{1f}(g)}} \text{cdet}_g \left( (A^*_2 A_2)_g, (d^{(2)}_f) \right)^\beta_{\beta},
\]
and denote \( \tilde{Y}_1 = Y_1 C_2^* B_2^* \). Then, from (111), it follows that
\[
x^{(22)}_{gf} = \frac{\sum_{\alpha \in I_{\tau_{1f}(f)}} \text{rdet}_f \left( (B_2 B_2^*)_f, (c^{(22)}_g) \right)^\alpha_{\alpha}}{\sum_{\alpha \in I_{\tau_{1f}(f)}} |B_2 B_2^*|_{\alpha}}
- \frac{\sum_{\beta \in I_{\tau_{1f}(g)}} \text{cdet}_g \left( (A^*_2 A_2)_g, (\tilde{e}^{(1)}_g) \right)^\beta_{\beta} \sum_{\alpha \in I_{\tau_{1f}(f)}} \text{rdet}_f \left( (B_2 B_2^*)_f, (\tilde{e}^{(22)}_g) \right)^\alpha_{\alpha} \sum_{\beta \in I_{\tau_{1f}(g)}} |A^*_2 A_2|_\alpha |B_2 B_2^*|_\alpha}{\sum_{\beta \in I_{\tau_{1f}(g)}} |A^*_2 A_2|_\alpha^2 \sum_{\beta \in I_{\tau_{1f}(g)}} |B_2 B_2^*|_{\alpha}}.
\]

where \( \tilde{e}^{(1)}_g \) is the \( g \)th row of \( \tilde{Y}_1 \),

If we construct the matrix \( Y_2 = (v^{(2)}_{gf}) \), where
\[
v^{(2)}_{gf} = \sum_{\alpha \in I_{\tau_{1f}(f)}} \text{rdet}_f \left( (B_2 B_2^*)_f, (c^{(22)}_g) \right)^\alpha_{\alpha},
\]
and denote \( \tilde{Y}_2 = A^*_2 A_2 Y_2 \), then, from (111), we obtain
\[
x^{(22)}_{gf} = \frac{\sum_{\alpha \in I_{\tau_{1f}(f)}} \text{rdet}_f \left( (B_2 B_2^*)_f, (c^{(22)}_g) \right)^\alpha_{\alpha}}{\sum_{\alpha \in I_{\tau_{1f}(f)}} |B_2 B_2^*|_{\alpha}}
- \frac{\sum_{\beta \in I_{\tau_{1f}(g)}} \text{cdet}_g \left( (A^*_2 A_2)_g, (\tilde{e}^{(2)}_g) \right)^\beta_{\beta} \sum_{\alpha \in I_{\tau_{1f}(f)}} \text{rdet}_f \left( (B_2 B_2^*)_f, (\tilde{e}^{(22)}_g) \right)^\alpha_{\alpha} \sum_{\beta \in I_{\tau_{1f}(g)}} |A^*_2 A_2|_\alpha |B_2 B_2^*|_\alpha}{\sum_{\beta \in I_{\tau_{1f}(g)}} |A^*_2 A_2|_\alpha^2 \sum_{\beta \in I_{\tau_{1f}(g)}} |B_2 B_2^*|_{\alpha}}.
\]

where \( \tilde{e}^{(2)}_g \) is the \( g \)th row of \( \tilde{Y}_2 \).
has determinantal representations, where the term \( x_{ij}^{(11)} \) is (89),
\( x_{ij}^{(12)} \) is (92) or (94), \( x_{ij}^{(13)} \) is (95) or (96), \( x_{ij}^{(14)} \) is (101), and \( x_{ij}^{(15)} \)
is (107) or (109); similarly, \( x_{ij}^{(21)} \) is (110), \( x_{ij}^{(22)} \) is (113) or (115),
\( x_{ij}^{(23)} \) is (116) or (117), and \( x_{ij}^{(24)} \) is (120).

A numerical example is provided to obtain the least norm of the general solution of (6) with the help of Theorem 30.

**Example 31.** We use the given matrices from the Example 25. Since \( r(A_1) = 1 \) and
\[ C_{11} = A_1^* C_1 = \begin{bmatrix} 1+i & j+k \\ -1+i & -j-k \end{bmatrix}, \] (122)
\[ A_1^* A_1 = \begin{bmatrix} 2 & -2i \\ 2i & 2 \end{bmatrix}, \] (123)
and then, by (89),
\[ x_{ij}^{(11)} = \frac{1}{4} + \frac{1}{4}i, \] (124)
\[ x_{ij}^{(12)} = \frac{1}{4} - \frac{1}{4}j + \frac{1}{4}k, \] (125)
\[ x_{ij}^{(13)} = -\frac{1}{4} + \frac{1}{4}i, \]
\[ x_{ij}^{(14)} = \frac{1}{4} - \frac{1}{4}j + \frac{1}{4}k. \]

Now, by (92), we find \( x_{ij}^{(12)} \) for all \( i, j = 1, 2 \). So,
\[ C_{12} = C_2 B_1^* = \begin{bmatrix} -2i - k & -i + 2k \\ i + 2k & 2i - k \end{bmatrix}, \] (124)
\[ B_1 B_1^* = \begin{bmatrix} 3 & -3j \\ 3j & 3 \end{bmatrix}. \] (125)

Similarly, by (91), \( \Psi_1 = A_1^* A_1 \). So,
\[ \Psi_1 = \Psi_1^* C_2 B_1^* = \begin{bmatrix} 2 & 4i + 4j - 2k & 4 - 2i - 2j + 4k \\ 4 + 2i + 2j + 4k & 2 + 4i - 4j - 2k \end{bmatrix}, \] (125)

Since \( r(B_j) = 1 \), then by (92),
\[ x_{ij}^{(12)} = \frac{1}{6}(-2i - k) - \frac{1}{24}(2 - 4i + 4j - 2k) \]
\[ = -\frac{1}{12} - \frac{1}{2}i - \frac{1}{6}j - \frac{1}{12}k, \]
\[ x_{ij}^{(12)} = \frac{1}{6}(-i + 2k) - \frac{1}{24}(4 - 2i - 2j + 4k) \]
\[ = -\frac{1}{12} - \frac{1}{4}i + \frac{1}{12}j + \frac{1}{6}k. \]
Further, due to Example 25, 

$$x_{21}^{(12)} = \frac{1}{6} (i + 2k) - \frac{1}{24} (4 + 2i + 2j + 4k)$$

$$= -\frac{1}{6} + \frac{1}{12} - \frac{1}{12} j + \frac{1}{6} k,$$

$$x_{22}^{(12)} = \frac{1}{6} (2i - k) - \frac{1}{24} (2 + 4i - 4j - 2k)$$

$$= -\frac{1}{12} + \frac{1}{6} i + \frac{1}{2} j - \frac{1}{12} k.$$  \hspace{1cm} (126)

Since \( r(A) = r(B) = 1 \) and

$$E_1 = A^* EB^* = \begin{bmatrix} 2 & 2j \\ -2i & -2k \end{bmatrix},$$

$$A^* A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix},$$

$$BB^* = \begin{bmatrix} 2 & 2j \\ -2j & 2 \end{bmatrix},$$  \hspace{1cm} (127)

and then, by (95),

$$d_1^B = \begin{bmatrix} 2 \\ -2i \end{bmatrix},$$

$$d_2^B = \begin{bmatrix} 2j \\ -2k \end{bmatrix},$$  \hspace{1cm} (128)

and

$$x_{11}^{(13)} = \frac{1}{4},$$

$$x_{12}^{(13)} = \frac{1}{4} j,$$

$$x_{21}^{(13)} = -\frac{1}{4} i,$$

$$x_{22}^{(13)} = -\frac{1}{4} k.$$

Further, due to Example 25, \( x_{ij}^{(14)} = x_{ij}^{(15)} = 0 \) for all \( i, j = 1, 2 \).

So,

$$x_{11}^{(1)} = \frac{5}{12} + \frac{1}{12} i - \frac{1}{6} j - \frac{1}{12} k,$$

$$x_{12}^{(1)} = -\frac{1}{6} + \frac{1}{12} i + \frac{7}{12} j + \frac{5}{12} k,$$

$$x_{21}^{(1)} = -\frac{5}{12} + \frac{1}{12} i - \frac{1}{12} j + \frac{1}{6} k,$$

$$x_{22}^{(1)} = -\frac{1}{12} + \frac{1}{6} i - \frac{1}{12} j - \frac{1}{12} k.$$  \hspace{1cm} (130)

Since, \( r(A_2) = 1 \) and

$$C_{21} = A_2^* C_3 = [2 + k - i - 2j],$$

$$A_2^* A_2 = [3],$$

and, due to Example 25, \( x_{ij}^{(25)} = x_{ij}^{(23)} = x_{ij}^{(24)} = 0 \) for all \( j = 1, 2 \); then by (110) and (121)

$$x_{11}^{(2)} = x_{11}^{(21)} = \frac{2}{3} + \frac{1}{3} k,$$

$$x_{12}^{(2)} = x_{12}^{(21)} = -\frac{1}{3} i - \frac{2}{3} j.$$  \hspace{1cm} (132)

Hence, the least norm solution of (6) obtained by Cramer’s Rule and the matrix method in Example 25 are the same as expected.

Note that we used Maple with the package CLIFFORD in the calculations.

6. Conclusion

We have constructed a novel expression of the general solution to system (6) over \( \mathbb{H} \) and used this result to explore the least-norm of the general solution to this system when it is solvable. Some particular cases of our system are also discussed. Our results carry the principal results of [32, 64]. Finally, we give determinantal representations (analogous of Cramer’s Rule) of the least norm solutions to the systems using row-column noncommutative determinants. Numerical examples are also provided to interpret the results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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