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Research Article

On One Method of Studying Spectral Properties of Non-selfadjoint Operators

Maksim V. Kukushkin [1,2]

[1] *Moscow State University of Civil Engineering, 129337, Moscow, Russia*
[2] *Kabardino-Balkarian Scientific Center, RAS, 360051, Nalchik, Russia*

Correspondence should be addressed to Maksim V. Kukushkin; kukushkinmv@rambler.ru

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In this paper, we explore a certain class of Non-selfadjoint operators acting on a complex separable Hilbert space. We consider a perturbation of a nonselfadjoint operator by an operator that is also nonselfadjoint. Our consideration is based on known spectral properties of the real component of a nonselfadjoint compact operator. Using a technique of the sesquilinear forms theory, we establish the compactness property of the resolvent and obtain the asymptotic equivalence between the real component of the resolvent and the resolvent of the real component for some class of nonselfadjoint operators. We obtain a classification of nonselfadjoint operators in accordance with belonging their resolvent to the Schatten-von Neumann class and formulate a sufficient condition of completeness of the root vector system. Finally, we obtain an asymptotic formula for the eigenvalues.

1. Introduction

It is remarkable that initially, the perturbation theory of selfadjoint operators was born in the works of Keldysh [1–3] and had been motivated by the works of famous scientists such as Carleman [4] and Tamarkin [5]. Many papers were published within the framework of this theory over time, for instance Browder [6], Livshits [7], Mukminov [8], Glazman [9], Krein [10], Lidsky [11], Marcus [12, 13], Matsaev [14, 15], Agmon [16], Katznelson [17], and Okazawa [18]. Nowadays, there exists a huge amount of theoretical results formulated in the work of Shkalikov [19]. However, for applying these results for a concrete operator W, we must have a representation of it by a sum of operators $W = T + A$. It is essential that T must be an operator of a special type either a selfadjoint or normal operator. If we consider a case where in the representation the operator T is neither selfadjoint nor normal and we cannot approach the required representation in an obvious way, then it is possible to use another technique based on the properties of the real component of the initial operator. Note that in this case, the made assumptions related to the initial operator W allow us to consider a m-accretive operator class which was thoroughly studied by mathematicians such as Kato [20] and Okazawa [21, 22]. This is a subject to consider in the second section. In the third section, we demonstrate the significance of the obtained abstract results and consider concrete operators. Note that the relevance of such consideration is based on the following. The eigenvalue problem is still relevant for the second-order fractional differential operators. Many papers were devoted to this question, for instance the papers [23–27]. The singular number problem for the resolvent of the second-order differential operator with the Riemann-Liouville fractional derivative in the final term is considered in the paper [23]. It is proved that the resolvent belongs to the Hilbert-Schmidt class. The problem of root functions system completeness is researched in the paper [24], also a similar problem is considered in the paper [25]. We would like to study the spectral properties of some class of nonselfadjoint operators in the abstract case. Via obtained results, we study a multidimensional case corresponding to the second-order fractional differential operator; this case can be reduced to the cases considered in the papers listed above. We consider a Kipriyanov fractional differential operator, considered in detail in the papers [28–30], which presents itself as a fractional derivative in a weaker sense with respect to the approach classically known with the name of the Riemann-Liouville derivative. More precisely, in the one-dimensional case, the Kipriaynov operator coincides with the Marchaud operator, in which relationship with the Weyl and Riemann-Liouville operators is well known [31, 32].

2. Preliminaries

Let C, C_i, $i \in \mathbb{N}_0$ be positive real constants. We assume that the values of C can be different in various formulas but the values of C_i, $i \in \mathbb{N}_0$ are certain. Everywhere further, we consider linear densely defined operators acting on a separable complex Hilbert space $\mathfrak{H}$. Denote by $\mathscr{B}(\mathfrak{H})$, the set of linear bounded operators acting in $\mathfrak{H}$. Denote by $\mathrm{D}(L), \mathrm{R}(L), \mathrm{N}(L)$, the domain of definition, the range, and the inverse image of zero of the operator L accordingly. The deficiency (codimension) of $\mathrm{R}(L)$ is denoted by def L. Let $\mathrm{P}(L)$ be a resolvent set of the operator L. Denote by $R_L(\zeta), \zeta \in \mathrm{P}(L), [R_L := R_L(0)]$, the resolvent of the operator L. Let $\lambda_i(L), i \in \mathbb{N}$, denote the eigenvalues of the operator L. Suppose L is a compact operator and $|L| := (L^* L)^{1/2}, r(|L|) := \dim \mathrm{R}(|L|)$, then the eigenvalues of the operator $|L|$ are called *singular numbers* (*s-numbers*) of the operator L and are denoted by $s_i(L), i = 1, 2, \cdots, r(|L|)$. If $r(|L|) < \infty$, then we put by definition $s_i = 0, i = r(|L|) + 1, 2, \cdots$. According to the terminology of the monograph [33], the dimension of the root vector subspace corresponding to a certain eigenvalue λ_k is called an *algebraic multiplicity* of the eigenvalue λ_k. Let $\nu(L)$ denote the sum of all algebraic multiplicities of the operator L. Denote by $\mathfrak{S}_p(\mathfrak{H}), 0 < p < \infty$ the Schatten-von Neumann class, and let $\mathfrak{S}_\infty(\mathfrak{H})$ denote the set of compact operators. By definition, put

$$\mathfrak{S}_p(\mathfrak{H}) := \left\{ L : \mathfrak{H} \longrightarrow \mathfrak{H}, \sum_{i=1}^{\infty} s_i^p(L) < \infty, 0 < p < \infty \right\}. \quad (1)$$

Suppose L is an operator that has a compact resolvent and $s_n(R_L) \le C n^{-\mu}, n \in \mathbb{N}, 0 \le \mu < \infty$; then, we denote by $\mu(L)$ the order of the operator L in accordance with the definition given in the paper [19]. Denote by $L_{\mathfrak{R}} := (L + L^*)/2$, $L_{\mathfrak{I}} := (L - L^*)/2i$ the real and the imaginary component of the operator L accordingly, and let $\tilde{L}$ denote the closure of the operator L. In accordance with the terminology of the monograph [34], the set $\Theta(L) := \{z \in \mathbb{C} : z = (Lf, f)_{\mathfrak{H}}, f \in \mathrm{D}(L), \|f\|_{\mathfrak{H}} = 1\}$ is called *numerical range* of the operator L. We use the definition of the sectorial property given in [34], p.280. An operator L is called *sectorial*, if its numerical range belongs to a closed sector $\mathfrak{L}_\gamma(\theta) := \{\zeta : |\arg(\zeta - \gamma)| \le \theta < \pi/2\}$, where γ is the vertex and θ is the semiangle of the sector $\mathfrak{L}_\gamma(\theta)$. We shall say that the operator L has a positive sector if $\mathrm{Im}\,\gamma = 0, \gamma > 0$. According to the terminology of the monograph [34], an operator L is called *strictly accretive* if the following relation holds $\mathrm{Re}\,(Lf, f)_{\mathfrak{H}} \ge C\|f\|^2_{\mathfrak{H}}, f \in \mathrm{D}(L)$. In accordance with the definition [34], p.279, an operator L is called *m -accretive* if the next relation holds $(A + \zeta)^{-1} \in \mathscr{B}(\mathfrak{H}), \|(A + \zeta)^{-1}\| \le (\mathrm{Re}\,\zeta)^{-1}, \mathrm{Re}\,\zeta > 0$. An operator L is called *m -sectorial* if L is sectorial and $L + \beta$ is *m* -accretive for some constant β. An operator L is called *symmetric* if one is densely defined and the next equality holds $(Lf, g)_{\mathfrak{H}} = (f, Lg)_{\mathfrak{H}}, f, g \in \mathrm{D}(L)$. A symmetric operator is called *positive* if the values of its quadratic form are nonnegative. Denote by $\mathfrak{H}_L, \|\cdot\|_L$ the energetic space generated by the operator L and the norm on this space, respectively (see [35, 36]). In accordance with the denotation of the paper [34], we consider a sesquilinear form $\mathfrak{t}[\cdot, \cdot]$ defined on a linear manifold of the Hilbert space $\mathfrak{H}$ (further we use the term *form*). Denote by $\mathfrak{t}[\cdot]$ the quadratic form corresponding to the sesquilinear form $\mathfrak{t}[\cdot, \cdot]$. Let $\mathfrak{Re}\,\mathfrak{t} = (\mathfrak{t} + \mathfrak{t}^*)/2, \mathfrak{Im}\,\mathfrak{t} = (\mathfrak{t} - \mathfrak{t}^*)/2i$ be the real and imaginary component of the form $\mathfrak{t}$, respectively, where $\mathfrak{t}^*[u, v] = \overline{\mathfrak{t}[v, u]}$, $\mathrm{D}(\mathfrak{t}^*) = \mathrm{D}(\mathfrak{t})$. According to these definitions, we have $\mathfrak{Re}\,\mathfrak{t}[\cdot] = \mathrm{Re}\ \mathfrak{t}[\cdot], \mathfrak{Im}\,\mathfrak{t}[\cdot] = \mathrm{Im}\ \mathfrak{t}[\cdot]$. Denote by $\tilde{\mathfrak{t}}$ the closure of the form $\mathfrak{t}$. The range of a quadratic form $\mathfrak{t}[f], f \in \mathrm{D}(\mathfrak{t}), \|f\|_{\mathfrak{H}} = 1$ is called *range* of the sesquilinear form $\mathfrak{t}$ and is denoted by $\Theta(\mathfrak{t})$. A form $\mathfrak{t}$ is called *sectorial* if its range belongs to a sector having the vertex γ situated at the real axis and the semiangle $0 \le \theta < \pi/2$. Suppose $\mathfrak{l}$ is a closed sectorial form; then, a linear manifold $\mathrm{D}' \subset \mathrm{D}(\mathfrak{l})$ is called *core* of $\mathfrak{l}$ if the restriction of $\mathfrak{l}$ to D' has the closure $\mathfrak{l}$. Due to Theorem 2.7 [34], p.323, there exist unique *m*-sectorial operators $L_{\mathfrak{l}}, L_{\mathfrak{Re}\,\mathfrak{l}}$, associated with the closed sectorial forms $\mathfrak{l}, \mathfrak{Re}\,\mathfrak{l}$, respectively. The operator $L_{\mathfrak{Re}\,\mathfrak{l}}$ is called a *real part* of the operator $L_{\mathfrak{l}}$ and is denoted by $\mathrm{Re}\ L_{\mathfrak{l}}$. Suppose L is a sectorial densely defined operator and $\mathfrak{k}[u, v] := (Lu, v)_{\mathfrak{H}}$, $\mathrm{D}(\mathfrak{k}) = \mathrm{D}(L)$; then, due to Theorem 1.27 [34], p.318, the form $\mathfrak{k}$ is closable, due to Theorem 2.7 [34], p.323 there exists a unique *m*-sectorial operator $T_{\tilde{\mathfrak{k}}}$ associated with the form $\tilde{\mathfrak{k}}$. In accordance with the definition [34], p.325 the operator $T_{\tilde{\mathfrak{k}}}$ is called a *Friedrichs extension* of the operator L.

Further, if it is not stated otherwise we use the notations of the monographs [32–34]. Consider a pair of complex Hilbert spaces $\mathfrak{H}, \mathfrak{H}_+$ such that

$$\mathfrak{H}_+ \subset\subset \mathfrak{H}. \quad (2)$$

This denotation implies that we have a bounded embedding provided by the inequality

$$\|f\|_{\mathfrak{H}} \le \|f\|_{\mathfrak{H}_+}, f \in \mathfrak{H}_+, \quad (3)$$

moreover any bounded set in the space $\mathfrak{H}_+$ is a compact set in the space $\mathfrak{H}$. We also assume that $\mathfrak{H}_+$ is a dense set in $\mathfrak{H}$. We consider nonselfadjoint operators that can be represented by a sum $W = T + A$, where the operators T and A act on $\mathfrak{H}$. We assume that: there exists a linear manifold $\mathfrak{M} \subset \mathfrak{H}_+$ that is dense in $\mathfrak{H}_+$, the operators T, A and their adjoint operators are defined on $\mathfrak{M}$. Further, we assume that $\mathrm{D}(W) = \mathfrak{M}$. These give us the opportunity to claim that $\mathrm{D}(W) \subset \mathrm{D}(W^*)$ thus, by virtue of this fact, the real component of W, is defined on $\mathfrak{M}$. Suppose the operator W^+ is the restriction of W^* to $\mathrm{D}(W)$; then, the operator W^+ is called a *formal* *adjoint* operator with respect to W. Denote by $\tilde{W}^{+}$ the closure of the operator W^{+}. Further, we assume that the following conditions are fulfilled:

$$\text{i) } \mathrm{Re}\,(Tf,f)_{\mathfrak{H}} \geq C_0\|f\|^2_{\mathfrak{H}_+},\ \text{ii)} \left|(Tf,g)_{\mathfrak{H}}\right| \leq C_1\|f\|_{\mathfrak{H}_+}\|g\|_{\mathfrak{H}_+},$$
$$\text{iii) } \mathrm{Re}\,(Af,f)_{\mathfrak{H}} \geq C_2\|f\|^2_{\mathfrak{H}},\ \text{iv)} |(Af,g)_{\mathfrak{H}}| \leq C_3\|f\|_{\mathfrak{H}_+}\|g\|_{\mathfrak{H}}, f,g \in \mathfrak{M}. \tag{4}$$

Due to these conditions, it is easy to prove that the operators $W, W_{\mathfrak{R}}$ are closeable (see Theorem 4 [34], p.268). Denote by $\tilde{W}_{\mathfrak{R}}$ the closure of the operator $W_{\mathfrak{R}}$. To make some formulas readable, we also use the following form of notation:

$$V := (R_{\tilde{W}})_{\mathfrak{R}}, H := \tilde{W}_{\mathfrak{R}}. \tag{5}$$

3. Main Results

In this section, we formulate abstract theorems that are generalizations of some particular results obtained by the author. First, we generalize Theorem 4.2 [37] establishing the sectorial property of the second-order fractional differential operator.

Lemma 1. *The operators $\tilde{W}, \tilde{W}^{+}$ have a positive sector.*

Proof. Due to inequalities (3) and (4), we conclude that the operator W is strictly accretive, i.e.

$$\mathrm{Re}\,(Wf,f)_{\mathfrak{H}} \geq C_0\|f\|^2_{\mathfrak{H}}, f \in \mathrm{D}(W). \tag{6}$$

Let us prove that the operator $\tilde{W}$ is canonical sectorial. Combining (4, ii) and (4, iii), we get

$$\begin{aligned}\mathrm{Re}\,(Wf,f)_{\mathfrak{H}} &= \mathrm{Re}\,(Tf,f)_{\mathfrak{H}} + \mathrm{Re}\,(Af,f)_{\mathfrak{H}} \\ &\geq C_0\|f\|_{\mathfrak{H}_+} + C_2\|f\|_{\mathfrak{H}}, f \in \mathrm{D}(W).\end{aligned} \tag{7}$$

Obviously, we can extend the previous inequality to

$$\mathrm{Re}\,(\tilde{W}f,f)_{\mathfrak{H}} \geq C_0\|f\|_{\mathfrak{H}_+} + C_2\|f\|_{\mathfrak{H}}, f \in \mathrm{D}(\tilde{W}). \tag{8}$$

By virtue of (8), we obtain $\mathrm{D}(\tilde{W}) \subset \mathfrak{H}_+$. Note that we have the estimate

$$\begin{aligned}\left|\mathrm{Im}\,(Wf,f)_{\mathfrak{H}}\right| &\leq \left|\mathrm{Im}\,(Tf,f)_{\mathfrak{H}}\right| + \left|\mathrm{Im}\,(Af,f)_{\mathfrak{H}}\right| \\ &= I_1 + I_2, f \in \mathrm{D}(W).\end{aligned} \tag{9}$$

Using inequality (4, ii), the Young inequality, we get

$$\begin{aligned}I_1 &= \left|(Tv,u)_{\mathfrak{H}} - (Tu,v)_{\mathfrak{H}}\right| \\ &\leq \left|(Tv,u)_{\mathfrak{H}}\right| + \left|(Tu,v)_{\mathfrak{H}}\right| \\ &\leq 2C_1\|u\|_{\mathfrak{H}_+}\|v\|_{\mathfrak{H}_+} \leq C_1\|f\|^2_{\mathfrak{H}_+},\end{aligned} \tag{10}$$

where $f = u + iv$. Consider I_2. Applying the Cauchy Schwartz inequality and inequality (4, iv), we obtain for arbitrary positive ε:

$$\begin{aligned}&\left|(Av,u)_{\mathfrak{H}}\right| \leq C_3\|v\|_{\mathfrak{H}_+}\|u\|_{\mathfrak{H}} \leq \frac{C_3}{2}\left\{\frac{1}{\varepsilon}\|u\|^2_{\mathfrak{H}} + \varepsilon\|v\|^2_{\mathfrak{H}_+}\right\}; \\ &\left|(Au,v)_{\mathfrak{H}}\right| \leq \frac{C_3}{2}\left\{\frac{1}{\varepsilon}\|v\|^2_{\mathfrak{H}} + \varepsilon\|u\|^2_{\mathfrak{H}_+}\right\}.\end{aligned} \tag{11}$$

Hence

$$\begin{aligned}I_2 &= \left|(Av,u)_{\mathfrak{H}} - (Au,v)_{\mathfrak{H}}\right| \\ &\leq \left|(Av,u)_{\mathfrak{H}}\right| + \left|(Au,v)_{\mathfrak{H}}\right| \\ &\leq \frac{C_3}{2}\left\{\frac{1}{\varepsilon}\|f\|^2_{\mathfrak{H}} + \varepsilon\|f\|^2_{\mathfrak{H}_+}\right\}.\end{aligned} \tag{12}$$

Finally, we have the following estimate

$$\left|\mathrm{Im}\,(Wf,f)_{\mathfrak{H}}\right| \leq \frac{C_3}{2}\,\varepsilon^{-1}\|f\|^2_{\mathfrak{H}} + \left(\frac{C_3}{2}\,\varepsilon + C_1\right)\|f\|^2_{\mathfrak{H}_+}, f \in \mathrm{D}(W). \tag{13}$$

Thus, we conclude that the next inequality holds for arbitrary $k > 0$:

$$\begin{aligned}&\mathrm{Re}(Wf,f)_{\mathfrak{H}} - k\left|\mathrm{Im}(Wf,f)_{\mathfrak{H}}\right| \geq \\ &\geq \left[C_0 - k\left(\frac{C_3}{2}\,\varepsilon + C_1\right)\right]\|f\|^2_{\mathfrak{H}_+} + \left(C_2 - k\,\frac{C_3}{2}\,\varepsilon^{-1}\right)\|f\|^2_{\mathfrak{H}}, f \in \mathrm{D}(W).\end{aligned} \tag{14}$$

Using the continuity property of the inner product, we can extend the previous inequality to the set $\mathrm{D}(\tilde{W})$. It follows easily that

$$\begin{aligned}&\left|Im([\tilde{W} - \gamma(\varepsilon)]f,f)_{\mathfrak{H}}\right| \leq \frac{1}{k(\varepsilon)}\mathrm{Re}([\tilde{W} - \gamma(\varepsilon)]f,f)_{\mathfrak{H}}, f \in \mathrm{D}(\tilde{W}), \\ &k(\varepsilon) = C_0\left(\frac{C_3}{2}\,\varepsilon + C_1\right)^{-1}, \ \gamma(\varepsilon) = C_2 - k(\varepsilon)\,\frac{C_3}{2}\,\varepsilon^{-1}.\end{aligned} \tag{15}$$

The previous inequality implies that the numerical range of the operator $\tilde{W}$ belongs to the sector $\mathfrak{L}_{\gamma}(\theta)$ with the vertex situated at the point γ and the semiangle $\theta = \arctan(1/k)$. Solving system of equation (15) relative to ε, we obtain the positive root ξ corresponding to the value $\gamma = 0$ and the following description for the coordinates of the sector vertex γ:

$$\gamma := \begin{cases} \gamma < 0, \ \varepsilon \in (0,\xi), \\ \gamma \geq 0, \ \varepsilon \in [\xi,\infty), \xi = \sqrt{\left(\frac{C_1}{C_3}\right)^2 + \frac{C_0}{C_2}} - \frac{C_1}{C_3}. \end{cases} \tag{16}$$

It follows that the operator $\tilde{W}$ has a positive sector. The proof corresponding to the operator $\tilde{W}^{+}$ follows from the reasoning given above if we note that W^{+} is formal adjoint with respect to W.

Lemma 2. *The operators $\tilde{W}, \tilde{W}^{+}$ are m-accretive; their resolvent sets contain the half-plane $\{\zeta : \zeta \in \mathbb{C}, \operatorname{Re}\ \zeta < C_0\}$.*

Proof. Due to Lemma 1, we know that the operator $\tilde{W}$ has a positive sector, i.e., the numerical range of $\tilde{W}$ belongs to the sector $\mathfrak{L}_{\gamma}(\theta), \gamma > 0$. In consequence to Theorem 3.2 [34], p.268, we have $\forall \zeta \in \mathbb{C} \setminus \mathfrak{L}_{\gamma}(\theta)$, the set $\mathrm{R}(\tilde{W} - \zeta)$ is a closed space, and the next relation holds:

$$\operatorname{def}\left(\tilde{W} - \zeta\right) = \eta,\ \eta = \text{const.} \tag{17}$$

Due to Theorem 3.2 [34], p.268, the inverse operator $(\tilde{W} + \zeta)^{-1}$ is defined on the subspace $\mathrm{R}(\tilde{W} + \zeta), \operatorname{Re}\zeta > 0$. In accordance with the definition of m-accretive operator given in the monograph [34], p.279, we need to show that

$$\operatorname{def}\left(\tilde{W} + \zeta\right) = 0, \left\| \left(\tilde{W} + \zeta\right)^{-1} \right\| \le (\operatorname{Re}\zeta)^{-1}, \operatorname{Re}\zeta > 0. \tag{18}$$

For this purpose, assume that $\zeta_0 \in \mathbb{C} \setminus \mathfrak{L}_{\gamma}(\theta),\ \operatorname{Re}\zeta_0 < 0$. Using (6), we get

$$\operatorname{Re}\left(f, \left[\tilde{W} - \zeta_0\right] f\right)_{\mathfrak{H}} \ge (C_0 - \operatorname{Re}\zeta_0)\|f\|_{\mathfrak{H}}^2, f \in \mathrm{D}\left(\tilde{W}\right). \tag{19}$$

Since the operator $\tilde{W} - \zeta_0$ has the closed range $\mathrm{R}(\tilde{W} - \zeta_0)$, it follows that

$$\mathfrak{H} = \mathrm{R}\left(\tilde{W} - \zeta_0\right) \oplus \mathrm{R}\left(\tilde{W} - \zeta_0\right)^{\perp}. \tag{20}$$

Note that the intersection of the sets $\mathfrak{M}$ and $\mathrm{R}(\tilde{W} - \zeta_0)^{\perp}$ is zero. If we assume otherwise, then applying inequality (19) for any element $u \in \mathfrak{M} \cap \mathrm{R}(\tilde{W} - \zeta_0)^{\perp}$, we get

$$(C_0 - \operatorname{Re}\zeta_0)\|u\|_{\mathfrak{H}}^2 \le \operatorname{Re}\left(u, \left[\tilde{W} - \zeta_0\right] u\right)_{\mathfrak{H}} = 0. \tag{21}$$

Hence, $u = 0$. Thus, the intersection of the sets $\mathfrak{M}$ and $\mathrm{R}(\tilde{W} - \zeta_0)^{\perp}$ is zero. It implies that

$$(g, v)_{\mathfrak{H}} = 0,\ \forall g \in \mathrm{R}\left(\tilde{W} - \zeta_0\right)^{\perp},\ \forall v \in \mathfrak{M}. \tag{22}$$

Since $\mathfrak{M}$ is a dense set in $\mathfrak{H}_{+}$, then taking into account (3), we obtain that $\mathfrak{M}$ is a dense set in $\mathfrak{H}$. Hence, $\mathrm{R}(\tilde{W} - \zeta_0)^{\perp} = 0, \operatorname{def}(\tilde{W} - \zeta_0) = 0$. Combining this fact with Theorem 3.2 [34], p.268, we get $\operatorname{def}(\tilde{W} - \zeta) = 0,\ \zeta \in \mathbb{C} \setminus \mathfrak{L}_{\gamma}(\theta)$. It is clear that $\operatorname{def}(\tilde{W} + \zeta) = 0, \forall\zeta, \operatorname{Re}\zeta > 0$. Let us prove that $\|(\tilde{W} + \zeta)^{-1}\| \le (\operatorname{Re}\zeta)^{-1}, \forall\zeta, \operatorname{Re}\zeta > 0$. We must notice that

$$\begin{aligned}(C_0 + \operatorname{Re}\zeta)\|f\|_{\mathfrak{H}}^2 &\le \operatorname{Re}\left(f, \left[\tilde{W} + \zeta\right] f\right)_{\mathfrak{H}} \\ &\le \|f\|_{\mathfrak{H}} \left\| \left(\tilde{W} + \zeta\right) f \right\|_{\mathfrak{H}}, f \\ &\in \mathrm{D}\left(\tilde{W}\right),\ \operatorname{Re}\zeta > 0.\end{aligned} \tag{23}$$

By virtue of the fact $\operatorname{def}(\tilde{W} + \zeta) = 0, \forall\zeta, \operatorname{Re}\zeta > 0$, we know that the resolvent is defined. Therefore

$$\left\| \left(\tilde{W} + \zeta\right)^{-1} f \right\|_{\mathfrak{H}} \le (C_0 + \operatorname{Re}\ \zeta)^{-1}\|f\|_{\mathfrak{H}} \le (\operatorname{Re}\ \zeta)^{-1}\|f\|_{\mathfrak{H}}, f \in \mathfrak{H}. \tag{24}$$

This implies that

$$\left\| \left(\tilde{W} + \zeta\right)^{-1} \right\| \le (\operatorname{Re}\ \zeta)^{-1},\ \forall\zeta, \operatorname{Re}\zeta > 0. \tag{25}$$

If we combine inequality (8) with Theorem 3.2 [34], p.268, we get $P(\tilde{W}) \supset \{\zeta : \zeta \in \mathbb{C}, \operatorname{Re}\ \zeta < C_0\}$. The proof corresponding to the operator $\tilde{W}^{+}$ is absolutely analogous.

Lemma 3. *The operator $\tilde{W}_{\mathfrak{R}}$ is strictly accretive, m-accretive, and selfadjoint.*

Proof. It is obvious that $W_{\mathfrak{R}}$ is a symmetric operator. Due to the continuity property of the inner product, we can conclude that $\tilde{W}_{\mathfrak{R}}$ is symmetric, too. Hence, $\Theta(\tilde{W}_{\mathfrak{R}}) \subset \mathbb{R}$. By virtue of (7), we have

$$(W_{\mathfrak{R}} f, f)_{\mathfrak{H}} \ge C_0\|f\|_{\mathfrak{H}_{+}}^2, f \in \mathrm{D}(W). \tag{26}$$

Using inequality (3) and the continuity property of the inner product, we obtain

$$\left(\tilde{W}_{\mathfrak{R}} f, f\right)_{\mathfrak{H}} \ge C_0\|f\|_{\mathfrak{H}_{+}}^2 \ge C_0\|f\|_{\mathfrak{H}}^2, f \in \mathrm{D}\left(\tilde{W}_{\mathfrak{R}}\right). \tag{27}$$

It implies that $\tilde{W}_{\mathfrak{R}}$ is strictly accretive. In the same way as in the proof of Lemma 2, we come to conclusion that $\tilde{W}_{\mathfrak{R}}$ is m-accretive. Moreover, we obtain the relation $\operatorname{def}(\tilde{W}_{\mathfrak{R}} - \zeta) = 0, \operatorname{Im}\zeta \ne 0$. Hence, by virtue of Theorem 3.16 [34], p.271, the operator $\tilde{W}_{\mathfrak{R}}$ is selfadjoint.

Theorem 4. *The operators $\tilde{W}_{\mathfrak{R}}, \tilde{W}, \tilde{W}^{+}$ have compact resolvents.*

Proof. First, note that due to Lemma 3, the operator $\tilde{W}_{\mathfrak{R}}$ is selfadjoint. Using (27), we obtain the estimates

$$\|f\|_{H} \ge \sqrt{C_0}\|f\|_{\mathfrak{H}_{+}} \ge \sqrt{C_0}\|f\|_{\mathfrak{H}}, f \in \mathfrak{H}_{H}, \tag{28}$$

where $H := \tilde{W}_{\mathfrak{R}}$. Since $\mathfrak{H}_{+} \subset\subset \mathfrak{H}$, then we conclude that each set bounded with respect to the energetic norm generated by the operator $\tilde{W}_{\mathfrak{R}}$ is compact with respect to the norm $\|\cdot\|_{\mathfrak{H}}$. Hence, in accordance with the theorem in [35], p.216, we conclude that $\tilde{W}_{\mathfrak{R}}$ has a discrete spectrum. Note that in consequence to Theorem 5 [35], p.222, we conclude that a selfadjoint strictly accretive operator with discrete spectrum has a compact inverse operator. Thus, using Theorem 6.29 [34], p.187, we obtain that $\tilde{W}_{\mathfrak{R}}$ has a compact resolvent.

Further, we need the technique of the sesquilinear forms theory stated in [34]. Consider the sesquilinear forms:

$$\mathfrak{t}[f,g]=\left(\tilde{W}f,g\right)_{\mathfrak{H}}, f,g\in \mathrm{D}\left(\tilde{W}\right), \mathfrak{h}[f,g] = \left(\tilde{W}_{\mathfrak{R}}f,g\right)_{\mathfrak{H}}, f,g\in \mathrm{D}\left(\tilde{W}_{\mathfrak{R}}\right). \tag{29}$$

Recall that due to inequality (8), we came to the conclusion that $\mathrm{D}(\tilde{W})\subset\mathfrak{H}_+$. In the same way, we can deduce that $\mathrm{D}(\tilde{W}_{\mathfrak{R}})\subset\mathfrak{H}_+$. By virtue of Lemmas 1 and 3, it is easy to prove that the sesquilinear forms $\mathfrak{t}$, $\mathfrak{h}$ are sectorial. Applying Theorem 1.27 [34], p.318, we conclude that these forms are closable. Now, note that $\mathfrak{Re}\,\tilde{\mathfrak{t}}$ is a sum of two closed sectorial forms. Hence, in consequence to Theorem 1.31 [34], p.319, we conclude that $\mathfrak{Re}\,\tilde{\mathfrak{t}}$ is a closed form. Let us show that $\mathfrak{Re}\,\tilde{\mathfrak{t}}=\tilde{\mathfrak{h}}$. First, note that this equality is true on the elements of the linear manifold $\mathfrak{M}\subset\mathfrak{H}_+$. This fact can be obtained from the following obvious relations:

$$\tilde{\mathfrak{t}}[f,g]=(Wf,g)_{\mathfrak{H}},\ \overline{\tilde{\mathfrak{t}}[g,f]}=(W^{+}f,g)_{\mathfrak{H}}, f,g\in\mathfrak{M}. \tag{30}$$

On the other hand

$$\tilde{\mathfrak{h}}[f,g]=\left(\tilde{W}_{\mathfrak{R}}f,g\right)_{\mathfrak{H}}=(W_{\mathfrak{R}}f,g)_{\mathfrak{H}}, f,g\in\mathfrak{M}. \tag{31}$$

Hence

$$\mathfrak{Re}\,\tilde{\mathfrak{t}}[f,g]=\tilde{\mathfrak{h}}[f,g],\ f,g\in\mathfrak{M}. \tag{32}$$

Using (4), we get

$$C_0\|f\|^2_{\mathfrak{H}_+}\le \mathrm{Re}\ \tilde{\mathfrak{t}}[f]\le C_4\|f\|^2_{\mathfrak{H}_+}, C_0\|f\|^2_{\mathfrak{H}_+}\le\tilde{\mathfrak{h}}(f)\le C_4\|f\|^2_{\mathfrak{H}_+}, f\in\mathfrak{M}, \tag{33}$$

where $C_4=C_1+C_3$. Since $\mathfrak{Re}\,\tilde{\mathfrak{t}}[f]=\mathrm{Re}\ \tilde{\mathfrak{t}}[f], f\in\mathfrak{M}$, the sesquilinear forms $\mathfrak{Re}\,\tilde{\mathfrak{t}}$, $\tilde{\mathfrak{h}}$ are closed forms; then, using (33), it is easy to prove that $\mathrm{D}(\mathfrak{Re}\,\tilde{\mathfrak{t}})=\mathrm{D}(\tilde{\mathfrak{h}})=\mathfrak{H}_+$. Using estimates (33), it is not hard to prove that $\mathfrak{M}$ is a core of the forms $\mathfrak{Re}\,\tilde{\mathfrak{t}}$, $\tilde{\mathfrak{h}}$. Hence, using (32), we obtain $\mathfrak{Re}\,\tilde{\mathfrak{t}}[f]=\tilde{\mathfrak{h}}[f], f\in\mathfrak{H}_+$. In accordance with the polarization principle (see (1.1) [34], p.309), we have $\mathfrak{Re}\,\tilde{\mathfrak{t}}=\tilde{\mathfrak{h}}$. Now, recall that the forms $\tilde{\mathfrak{t}}$, $\tilde{\mathfrak{h}}$ are generated by the operators $\tilde{W}$, $\tilde{W}_{\mathfrak{R}}$, respectively. Note that in consequence of Lemmas 1–3, these operators are m-sectorial. Hence, by virtue of Theorem 2.9 [34], p.326, we get $T_{\tilde{\mathfrak{t}}}=\tilde{W}$, $T_{\tilde{\mathfrak{h}}}=\tilde{W}_{\mathfrak{R}}$. Since we have proved that $\mathfrak{Re}\,\tilde{\mathfrak{t}}=\tilde{\mathfrak{h}}$, then $T_{\mathfrak{Re}\,\tilde{\mathfrak{t}}}=\tilde{W}_{\mathfrak{R}}$. Therefore, by definition, we conclude that the operator $\tilde{W}_{\mathfrak{R}}$ is the real part of the m-sectorial operator $\tilde{W}$, by symbol $\tilde{W}_{\mathfrak{R}}=\mathrm{Re}\ \tilde{W}$. Since we proved above that $\tilde{W}_{\mathfrak{R}}$ has a compact resolvent, then using Theorem 3.3 [34], p.337, we conclude that the operator $\tilde{W}$ has a compact resolvent. The proof corresponding to the operator $\tilde{W}^{+}$ is absolutely analogous.

Theorem 5. *The following relation holds:*

$$\lambda_i(R_H)\asymp\lambda_i(V). \tag{34}$$

Proof. It was shown in the proof of Theorem 4 that $H=\mathrm{Re}\ \tilde{W}$. Hence, in consequence to Lemmas 1, 2, and Theorem 3.2 [34], p.337, there exist the selfadjoint operators $B_i:=\{B_i\in\mathscr{B}(\mathfrak{H}),\|B_i\|\le\tan\theta\}, i=1,2$ (where θ is the semiangle of the sector $\mathfrak{L}_0(\theta)\supset\Theta(\tilde{W})$) such that

$$\tilde{W}=H^{1/2}(I+iB_1)H^{1/2},\quad \tilde{W}^{+}=H^{1/2}(I+iB_2)H^{1/2}. \tag{35}$$

Since the set of linear operators generates ring, it follows that

$$\begin{aligned}Hf&=\frac{1}{2}\left[H^{1/2}(I+iB_1)+H^{1/2}(I+iB_2)\right]H^{1/2}=\\&=\frac{1}{2}\left\{H^{1/2}[(I+iB_1)+(I+iB_2)]\right\}H^{1/2}=\\&=Hf+\frac{i}{2}H^{1/2}(B_1+B_2)H^{1/2}f,\ f\in\mathfrak{M}.\end{aligned} \tag{36}$$

Consequently

$$H^{1/2}(B_1+B_2)H^{1/2}f=0,\ f\in\mathfrak{M}. \tag{37}$$

Let us show that $B_1=-B_2$. In accordance with Lemma 3, the operator H is m-accretive; hence, we have $(H+\zeta)^{-1}\in\mathscr{B}(\mathfrak{H})$, $\mathrm{Re}\ \zeta>0$. Using this fact, we get

$$\begin{aligned}\mathrm{Re}\left([H+\zeta]^{-1}Hf,f\right)_{\mathfrak{H}}&=\mathrm{Re}\left([H+\zeta]^{-1}[H+\zeta]f,f\right)_{\mathfrak{H}}-\mathrm{Re}\left(\zeta\,[H+\zeta]^{-1}f,f\right)_{\mathfrak{H}}\ge\\&\ge\|f\|^2_{\mathfrak{H}}-|\zeta|\cdot\left\|(H+\zeta)^{-1}\right\|\cdot\|f\|^2_{\mathfrak{H}}\\&=\|f\|^2_{\mathfrak{H}}\left(1-|\zeta|\cdot\left\|(H+\zeta)^{-1}\right\|\right),\\ \mathrm{Re}\,\zeta>0, f\in\mathrm{D}(H).\end{aligned} \tag{38}$$

Applying inequality (27), we obtain

$$\begin{aligned}\|f\|_{\mathfrak{H}}\left\|(H+\zeta)^{-1}f\right\|_{\mathfrak{H}}&\ge\left|\left(f,[H+\zeta]^{-1}f\right)\right|\\&\ge(\mathrm{Re}\zeta+C_0)\left\|(H+\zeta)^{-1}f\right\|^2_{\mathfrak{H}}, f\in\mathfrak{H}.\end{aligned} \tag{39}$$

It implies that

$$\left\|(H+\zeta)^{-1}\right\|\le(\mathrm{Re}\,\zeta+C_0)^{-1},\ \ \mathrm{Re}\,\zeta>0. \tag{40}$$

Combining this estimate with (38), we have

$$\operatorname{Re}\left([H+\zeta]^{-1}Hf,f\right)_{\mathfrak{H}} \geq \|f\|_{\mathfrak{H}}^2\left(1-\frac{|\zeta|}{\operatorname{Re}\zeta+C_0}\right), \operatorname{Re}\zeta > 0, f\in \mathrm{D}(H). \tag{41}$$

Applying formula (3.45) [34], p.282, and taking into account that $H^{1/2}$ is selfadjoint, we get

$$\begin{aligned}\left(H^{1/2}f,f\right)_{\mathfrak{H}} &= \frac{1}{\pi}\int_0^\infty \zeta^{-1/2}\operatorname{Re}\left([H+\zeta]^{-1}Hf,f\right)_{\mathfrak{H}} d\zeta \geq \\ &\geq \|f\|_{\mathfrak{H}}^2\cdot\frac{C_0}{\pi}\int_0^\infty \frac{\zeta^{-1/2}}{\zeta+C_0}d\zeta = \sqrt{C_0}\|f\|_{\mathfrak{H}}^2, f\in\mathrm{D}(H).\end{aligned} \tag{42}$$

Since in accordance with Theorem 3.35 [34], p.281, the set $\mathrm{D}(H)$ is the core of the operator $H^{1/2}$, then we can extend (42) to

$$\left(H^{1/2}f,f\right)_{\mathfrak{H}} \geq \sqrt{C_0}\|f\|_{\mathfrak{H}}^2,\ f\in\mathrm{D}\left(H^{1/2}\right). \tag{43}$$

Hence, $\mathrm{N}(H^{1/2})=0$. Combining this fact and (37), we obtain

$$(B_1+B_2)H^{1/2}f=0,\ f\in\mathfrak{M}. \tag{44}$$

Let us show that the set $\mathfrak{M}$ is a core of the operator $H^{1/2}$. Note that due to Theorem 3.35 [34], p.281, the operator $H^{1/2}$ is selfadjoint, and $\mathrm{D}(H)$ is a core of the operator $H^{1/2}$. Hence, we have the representation

$$\left\|H^{1/2}f\right\|_{\mathfrak{H}}^2 = (Hf,f)_{\mathfrak{H}}, f\in\mathrm{D}(H). \tag{45}$$

To achieve our aim, it is sufficient to show the following:

$$\forall f_0\in\mathrm{D}\left(H^{1/2}\right), \exists\{f_n\}_1^\infty\subset\mathfrak{M}:\quad f_n\xrightarrow{\mathfrak{H}}f_0,\ H^{1/2}f_n\xrightarrow{\mathfrak{H}}H^{1/2}f_0. \tag{46}$$

Since in accordance with the definition the set $\mathfrak{M}$ is a core of H, then we can extend second relation (33) to $\sqrt{C_0}\|f\|_{\mathfrak{H}_+}\leq (Hf,f)_{\mathfrak{H}}\leq\sqrt{C_4}\|f\|_{\mathfrak{H}_+}, f\in\mathrm{D}(H)$. Applying (45), we can write

$$\sqrt{C_0}\|f\|_{\mathfrak{H}_+}\leq\left\|H^{1/2}f\right\|_{\mathfrak{H}}\leq\sqrt{C_4}\|f\|_{\mathfrak{H}_+}, f\in\mathrm{D}(H). \tag{47}$$

Using lower estimate (47) and the fact that $\mathrm{D}(H)$ is a core of $H^{1/2}$, it is not hard to prove that $\mathrm{D}(H^{1/2})\subset\mathfrak{H}_+$. Taking into account this fact and using upper estimate (47), we obtain (46). It implies that $\mathfrak{M}$ is a core of $H^{1/2}$. Note that in accordance with Theorem 3.35 [34], p.281, the operator $H^{1/2}$ is m-accretive. Hence, combining Theorem 3.2 [34], p.268, with (43), we obtain $\mathrm{R}(H^{1/2})=\mathfrak{H}$. Taking into account that $\mathfrak{M}$ is a core of the operator $H^{1/2}$, we conclude that $\mathrm{R}(H^{1/2})$ is dense in $\mathfrak{H}$, where $H^{1/2}$ is the restriction of the operator $H^{1/2}$ to $\mathfrak{M}$. Finally, by virtue of (44), we conclude that the sum B_1+B_2 equal to zero on the dense subset of $\mathfrak{H}$. Since these operators are defined on $\mathfrak{H}$ and bounded, then $B_1=-B_2$. Further, we use the denotation $B_1:=B$.

Note that due to Lemma 2, there exist the operators $R_{\tilde{W}}$, $R_{\tilde{W}^+}$. Using the properties of the operator B, we get $\|(I\pm iB)f\|_{\mathfrak{H}}\|f\|_{\mathfrak{H}}\geq\operatorname{Re}\left([I\pm iB]f,f\right)_{\mathfrak{H}}=\|f\|_{\mathfrak{H}}^2, f\in\mathfrak{H}$. Hence

$$\|(I\pm iB)f\|_{\mathfrak{H}}\geq\|f\|_{\mathfrak{H}}, f\in\mathfrak{H}. \tag{48}$$

It implies that the operators $I\pm iB$ are invertible. Since it was proved above that $\mathrm{R}(H^{1/2})=\mathfrak{H}$, $\mathrm{N}(H^{1/2})=0$, then there exists an operator $H^{-1/2}$ defined on $\mathfrak{H}$. Using representation (35) and taking into account the reasonings given above, we obtain

$$\begin{aligned}R_{\tilde{W}} &= H^{-1/2}(I+iB)^{-1}H^{-1/2},\\ R_{\tilde{W}^+} &= H^{-1/2}(I-iB)^{-1}H^{-1/2}.\end{aligned} \tag{49}$$

Note that the following equality can be proved easily $R_{\tilde{W}}^* = R_{\tilde{W}^+}$. Hence, we have

$$V=\frac{1}{2}\left(R_{\tilde{W}}+R_{\tilde{W}^+}\right). \tag{50}$$

Combining (49) and (50), we get

$$V=\frac{1}{2}H^{-1/2}\left[(I+iB)^{-1}+(I-iB)^{-1}\right]H^{-1/2}. \tag{51}$$

Using the obvious identity $(I+B^2)=(I+iB)(I-iB)=(I-iB)(I+iB)$, by direct calculation, we get

$$(I+iB)^{-1}+(I-iB)^{-1}=\left(I+B^2\right)^{-1}. \tag{52}$$

Combining (51) and (52), we obtain

$$V=\frac{1}{2}H^{-1/2}\left(I+B^2\right)^{-1}H^{-1/2}. \tag{53}$$

Let us evaluate the form $(Vf,f)_{\mathfrak{H}}$. Note that there exists the operator R_H (see Lemma 3). Since H is selfadjoint (see Lemma 3), then due to Theorem 3 [38], p.136, R_H is selfadjoint. It is clear that R_H is positive because H is positive. Hence, by virtue of the well-known theorem (see [39], p.174) there exists a unique square root of the operator R_H, the selfadjoint operator $\widehat{R}$ such that $\widehat{R}\widehat{R}=R_H$. Using the decomposition $H=H^{1/2}H^{1/2}$, we get $H^{-1/2}H^{-1/2}H=I$. Hence, $R_H\subset H^{-1/2}H^{-1/2}$, but $\mathrm{D}(R_H)=\mathfrak{H}$. It implies that $R_H=H^{-1/2}H^{-1/2}$. Using the uniqueness property of the square root, we obtain $H^{-1/2}=\widehat{R}$. Let us use the shorthand notation $S:=I+B^2$. Note that due to the obvious inequality $(\|Sf\|_{\mathfrak{H}}\geq\|f\|_{\mathfrak{H}}, f\in\mathfrak{H})$, the operator S^{-1} is bounded on the set $\mathrm{R}(S)$. Taking into account the reasoning given above, we get

$$(Vf,f)_{\mathfrak{H}} = (H^{-1/2}S^{-1}H^{-1/2}f,f)_{\mathfrak{H}} = (S^{-1}H^{-1/2}f,H^{-1/2}f)_{\mathfrak{H}} \leq$$
$$\leq \|S^{-1}H^{-1/2}f\|_{\mathfrak{H}}\|H^{-1/2}f\|_{\mathfrak{H}} \leq \|S^{-1}\|\cdot\|H^{-1/2}f\|^2_{\mathfrak{H}} = \|S^{-1}\|\cdot(R_H f,f)_{\mathfrak{H}},\ f\in\mathfrak{H}. \tag{54}$$

On the other hand, it is easy to see that $(S^{-1}f,f)_{\mathfrak{H}} \geq \|S^{-1}f\|^2_{\mathfrak{H}}, f\in \mathrm{R}(S)$. At the same time, it is obvious that S is bounded, and we have $\|S^{-1}f\|_{\mathfrak{H}} \geq \|S\|^{-1}\|f\|_{\mathfrak{H}}, f\in \mathrm{R}(S)$. Using these estimates, we have

$$(Vf,f)_{\mathfrak{H}} = (S^{-1}H^{-1/2}f,H^{-1/2}f)_{\mathfrak{H}} \geq \left\|S^{-1}H^{-1/2}f\right\|^2_{\mathfrak{H}} \geq$$
$$\geq \|S\|^{-2}\cdot\left\|H^{-1/2}f\right\|^2_{\mathfrak{H}} = \|S\|^{-2}\cdot(R_H f,f)_{\mathfrak{H}},\ f\in\mathfrak{H}. \tag{55}$$

Note that due to Theorem 4, the operator R_H is compact. Combining (50) with Theorem 4, we conclude that the operator V is compact. Taking into account these facts and using Lemma 1.1 [33], p.45, we obtain (34).

Remark 6. Since it was proved above that R_H is selfadjoint and positive, then we have $\lambda_i(R_H)=s_i(R_H), i\in\mathbb{N}$. Note that in accordance with the facts established above, the operator $H := \tilde{W}_{\mathfrak{R}}$ has a discrete spectrum and a compact resolvent. Due to results represented in [40–42], we have an opportunity to obtain the order of the operator H in an easy way in most particular cases.

The following theorem is formulated in terms of order $\mu := \mu(H)$ and devoted to the Schatten-von Neumann classification of the operator $R_{\tilde{W}}$.

Theorem 7. *We have the following classification:*

$$R_{\tilde{W}} \in \mathfrak{S}_p,\ p = \begin{cases} l,\ l>2/\mu,\ \mu\leq 1, \\ 1,\ \mu>1 \end{cases}. \tag{56}$$

Moreover, under the assumption $\lambda_n(R_H)\geq C\,n^{-\mu}, n\in\mathbb{N}$, we have

$$R_{\tilde{W}}\in\mathfrak{S}_p \Rightarrow \mu p>1,\ 1\leq p<\infty, \tag{57}$$

where $\mu := \mu(H)$.

Proof. Consider the case $(\mu\leq 1)$. Since we already know that $R^*_{\tilde{W}} = R_{\tilde{W}^+}$, then it can easily be checked that the operator $R^*_{\tilde{W}}R_{\tilde{W}}$ is a selfadjoint positive compact operator. Due to the well-known fact [39], p.174, there exists the operator $|R_{\tilde{W}}|$. By virtue of Theorem 9.2 [39], p.178, the operator $|R_{\tilde{W}}|$ is compact. Since $\mathrm{N}(|R_{\tilde{W}}|^2)=0$, it follows that $\mathrm{N}(|R_{\tilde{W}}|)=0$, Hence applying Theorem [38], p.189, we conclude that the operator $|R_{\tilde{W}}|$ has an infinite set of the eigenvalues. Using condition (4, iii), we get

$$\mathrm{Re}\,(R_{\tilde{W}}f,f)_{\mathfrak{H}} \geq C_0\|R_{\tilde{W}}f\|^2_{\mathfrak{H}},\ f\in\mathfrak{H}. \tag{58}$$

Hence

$$\left(|R_{\tilde{W}}|^2 f,f\right)_{\mathfrak{H}} = \|R_{\tilde{W}}f\|^2_{\mathfrak{H}} \leq C_0^{-1}\mathrm{Re}(R_{\tilde{W}}f,f)_{\mathfrak{H}}$$
$$= C_0^{-1}(Vf,f)_{\mathfrak{H}},\ V := (R_{\tilde{W}})_{\mathfrak{R}}. \tag{59}$$

Since we already know that the operators $|R_{\tilde{W}}|^2, V$ are compact, then using Lemma 1.1 [33], p.45, and Theorem 5, we get

$$\lambda_i\left(|R_{\tilde{W}}|^2\right) \leq C_0^{-1}\,\lambda_i(V) \leq Ci^{-\mu},\ i\in\mathbb{N}. \tag{60}$$

Recall that by definition, we have $s_i(R_{\tilde{W}})=\lambda_i(|R_{\tilde{W}}|)$. Note that the operators $|R_{\tilde{W}}|, |R_{\tilde{W}}|^2$ have the same eigenvectors. This fact can be easily proved if we note the obvious relation $|R_{\tilde{W}}|^2 f_i = |\lambda_i(|R_{\tilde{W}}|)|^2 f_i, i\in\mathbb{N}$ and the spectral representation for the square root of a selfadjoint positive compact operator

$$|R_{\tilde{W}}|f = \sum_{i=1}^{\infty}\sqrt{\lambda_i\left(|R_{\tilde{W}}|^2\right)}(f,\varphi_i)\varphi_i,\ f\in\mathfrak{H}, \tag{61}$$

where f_i, φ_i are the eigenvectors of the operators $|R_{\tilde{W}}|, |R_{\tilde{W}}|^2$, respectively (see (10.25) [39], p.201). Hence, $\lambda_i(|R_{\tilde{W}}|) = \sqrt{\lambda_i(|R_{\tilde{W}}|^2)}, i\in\mathbb{N}$. Combining this fact with (60), we get

$$\sum_{i=1}^{\infty}s_i^p(R_{\tilde{W}}) = \sum_{i=1}^{\infty}\lambda_i^{p/2}\left(|R_{\tilde{W}}|^2\right) \leq C\sum_{i=1}^{\infty}i^{-\mu p/2}. \tag{62}$$

This completes the proof for the case $(\mu\leq 1)$.

Consider the case $(\mu>1)$. It follows from (50) that the operator V is positive and bounded. Hence, by virtue of Lemma 8.1 [33], p.126, we conclude that for any orthonormal basis $\{\psi_i\}_1^{\infty}\subset\mathfrak{H}$, the following equalities hold

$$\sum_{i=1}^{\infty}\mathrm{Re}(R_{\tilde{W}}\psi_i,\psi_i)_{\mathfrak{H}} = \sum_{i=1}^{\infty}(V\psi_i,\psi_i)_{\mathfrak{H}} = \sum_{i=1}^{\infty}(V\varphi_i,\varphi_i)_{\mathfrak{H}}, \tag{63}$$

where $\{\varphi_i\}_1^{\infty}$ is the orthonormal basis of the eigenvectors of the operator V. Due to Theorem 5, we get

$$\sum_{i=1}^{\infty}(V\varphi_i,\varphi_i)_{\mathfrak{H}} = \sum_{i=1}^{\infty}s_i(V) \leq C\sum_{i=1}^{\infty}i^{-\mu}. \tag{64}$$

By virtue of Lemma 1, we get $|\mathrm{Im}\,(R_{\tilde{W}}\psi_i,\psi_i)_{\mathfrak{H}}| \leq k^{-1}(\xi)\ \mathrm{Re}\,(R_{\tilde{W}}\psi_i,\psi_i)_{\mathfrak{H}}$. Combining this fact with (63), we conclude that the following series is convergent

$$\sum_{i=1}^{\infty}(R_{\tilde{W}}\psi_i,\psi_i)_{\mathfrak{H}} < \infty. \tag{65}$$

Hence, by definition [33], p.125, the operator $R_{\tilde{W}}$ has a finite matrix trace. Using Theorem 8.1 [33], p.127, we get $R_{\tilde{W}}\in\mathfrak{S}_1$. This completes the proof for the case $(\mu>1)$.

Now, assume that $\lambda_n(R_H) \geq C\,n^{-\mu}, n \in \mathbb{N}, 0 \leq \mu < \infty$. Let us show that the operator V has the complete orthonormal system of the eigenvectors. Using formula (53), we get

$$V^{-1} = 2H^{1/2}\left(I + B^2\right)H^{1/2},\ \mathrm{D}\left(V^{-1}\right) = \mathrm{R}(V). \tag{66}$$

Let us prove that $\mathrm{D}(V^{-1}) \subset \mathrm{D}(H)$. Note that the set $\mathrm{D}(V^{-1})$consists of the elements $f + g$, where $f \in \mathrm{D}(\tilde{W}), g \in \mathrm{D}(\tilde{W}^{+})$. Using representation (35), it is easy to prove that $\mathrm{D}(\tilde{W}) \subset \mathrm{D}(H), \mathrm{D}(\tilde{W}^{+}) \subset \mathrm{D}(H)$. This gives the desired result. Taking into account the facts proven above, we get

$$\begin{aligned}\left(V^{-1}f, f\right)_{\mathfrak{H}} &= 2\left(SH^{1/2}f, H^{1/2}f\right)_{\mathfrak{H}} \geq 2\left\|H^{1/2}f\right\|_{\mathfrak{H}}^2 \\ &= 2(Hf, f)_{\mathfrak{H}}, f \in \mathrm{D}\left(V^{-1}\right),\end{aligned} \tag{67}$$

where $S = I + B^2$. Since V is selfadjoint, then due to Theorem 3 [38], p.136, the operator V^{-1} is selfadjoint. Combining (67) with Lemma 3, we conclude that V^{-1} is strictly accretive. Using these facts, we can write

$$\|f\|_{V^{-1}} \geq C\|f\|_{H}, f \in \mathfrak{H}_{V^{-1}}. \tag{68}$$

Since the operator H has a discrete spectrum (see Theorem 5.3 [37]), then any set bounded with respect to the norm $\mathfrak{H}_H$ is a compact set with respect to the norm $\mathfrak{H}$ (see Theorem 4 [35], p.220). Combining this fact with (68) and Theorem 3 [35], p.216, we conclude that the operator V^{-1} has a discrete spectrum, i.e., it has the infinite set of the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots, \lambda_i \longrightarrow \infty, i \longrightarrow \infty$, and the complete orthonormal system of the eigenvectors. Now note that the operators V, V^{-1} have the same eigenvectors. Therefore the operator V has the complete orthonormal system of the eigenvectors. Recall that any complete orthonormal system is a basis in separable Hilbert space. Hence, the complete orthonormal system of the eigenvectors of the operator V is a basis in the space $\mathfrak{H}$. Let $\{\varphi_i\}_1^{\infty}$ be the complete orthonormal system of the eigenvectors of the operator V, and suppose $R_{\tilde{W}} \in \mathfrak{S}_p$; then, by virtue of inequalities (7.9) [33], p.123, and Theorem 5, we get

$$\begin{aligned}\sum_{i=1}^{\infty}|s_i(R_{\tilde{W}})|^p &\geq \sum_{i=1}^{\infty}\left|(R_{\tilde{W}}\varphi_i, \varphi_i)_{\mathfrak{H}}\right|^p \geq \sum_{i=1}^{\infty}\left|\mathrm{Re}(R_{\tilde{W}}\varphi_i, \varphi_i)_{\mathfrak{H}}\right|^p = \\ &= \sum_{i=1}^{\infty}\left|(V\varphi_i, \varphi_i)_{\mathfrak{H}}\right|^p = \sum_{i=1}^{\infty}|\lambda_i(V)|^p \geq C\sum_{i=1}^{\infty} i^{-\mu p}.\end{aligned} \tag{69}$$

We claim that $\mu p > 1$. Assuming the converse in the previous inequality, we come to the contradiction with the condition $R_{\tilde{W}} \in \mathfrak{S}_p$. This completes the proof.

The following theorem establishes the completeness property of the system of root vectors of the operator $R_{\tilde{W}}$.

Theorem 8. *Suppose $\theta < \pi\mu/2$; then, the system of root vectors of the operator $R_{\tilde{W}}$ is complete, where θ is the semiangle of the sector $\mathfrak{L}_0(\theta) \supset \Theta(\tilde{W}), \mu := \mu(H)$.*

Proof. Using Lemma 1, we have

$$|\ \mathrm{Im}\ (R_{\tilde{W}}f, f)_{\mathfrak{H}}| \leq k^{-1}(\xi)\ \mathrm{Re}\ (R_{\tilde{W}}f, f)_{\mathfrak{H}}, f \in \mathfrak{H}. \tag{70}$$

Therefore, $\Theta(\bar{R}_{\tilde{W}}) \subset \mathfrak{L}_0(\theta)$. Note that the map $z : \mathbb{C} \longrightarrow \mathbb{C}, z = 1/\zeta$ takes each eigenvalue of the operator $R_{\tilde{W}}$ to the eigenvalue of the operator $\tilde{W}$. It is also clear that $z : \mathfrak{L}_0(\theta) \longrightarrow \mathfrak{L}_0(\theta)$. Using the definition [33], p.302, let us consider the following set:

$$\mathfrak{P} := \left\{z : z = t\,\xi, \xi \in \Theta(\bar{R}_{\tilde{W}}), 0 \leq t < \infty\right\}. \tag{71}$$

It is easy to see that $\mathfrak{P}$ coincides with a closed sector of the complex plane with the vertex situated at the point zero. Let us denote by $\vartheta(R_{\tilde{W}})$ the angle of this sector. It is obvious that $\mathfrak{P} \subset \mathfrak{L}_0(\theta)$. Therefore, $0 \leq \vartheta(R_{\tilde{W}}) \leq 2\theta$. Let us prove that $0 < \vartheta(R_{\tilde{W}})$, i.e., the strict inequality holds. If we assume that $\vartheta(R_{\tilde{W}}) = 0$, then we get $e^{-i \arg z} = \varsigma, \forall z \in \mathfrak{P} \setminus 0$, where ς is a constant independent on z. In consequence to this fact, we have $\mathrm{Im}\ \Theta(\varsigma R_{\tilde{W}}) = 0$. Hence, the operator $\varsigma R_{\tilde{W}}$ is symmetric (see Problem 3.9 [34], p.269), and by virtue of the fact $\mathrm{D}(\varsigma R_{\tilde{W}}) = \mathfrak{H}$, one is selfadjoint. On the other hand, taking into account the equality $R_{\tilde{W}}^{*} = R_{\tilde{W}^{+}}$ (see the proof of Theorem 5), we have $(\varsigma R_{\tilde{W}}f, g)_{\mathfrak{H}} = (f, \bar{\varsigma}R_{\tilde{W}^{+}}g)_{\mathfrak{H}}, f, g \in \mathfrak{H}$. Hence, $\varsigma R_{\tilde{W}} = \bar{\varsigma}R_{\tilde{W}^{+}}$. In the particular case, we have $\forall f \in \mathfrak{H}, \mathrm{Im}\, f = 0 : \mathrm{Re}\ \varsigma R_{\tilde{W}}f = \mathrm{Re}\ \varsigma R_{\tilde{W}^{+}}f, \mathrm{Im}\ \varsigma R_{\tilde{W}}f = -\mathrm{Im}\ \varsigma R_{\tilde{W}^{+}}f$. It implies that $\mathrm{N}(R_{\tilde{W}}) \neq 0$. This contradiction concludes the proof of the fact $\vartheta(R_{\tilde{W}}) > 0$. Let us use Theorem 6.2 [33], p.305, according to which we have the following. If the following two conditions (a) and (b) are fulfilled, then the system of root vectors of the operator $R_{\tilde{W}}$ is complete.

(a) $\vartheta(R_{\tilde{W}}) = \pi/d$, where $d > 1$

(b) For some β, the operator $B := (e^{i\beta}R_{\tilde{W}})_{\mathfrak{J}} : s_i(B) = o(i^{-1/d}), i \longrightarrow \infty$.

Let us show that conditions (a) and (b) are fulfilled. Note that due to Lemma 1, we have $0 \leq \theta < \pi/2$. Hence, $0 < \vartheta(R_{\tilde{W}}) < \pi$. It implies that there exists $1 < d < \infty$ such that $\vartheta(R_{\tilde{W}}) = \pi/d$. Thus, condition (a) is fulfilled. Let us choose the certain value $\beta = \pi/2$ in condition (b) and notice that $(e^{i\pi/2}R_{\tilde{W}})_{\mathfrak{J}} = (R_{\tilde{W}})_{\mathfrak{R}}$. Since the operator $V := (R_{\tilde{W}})_{\mathfrak{R}}$ is selfadjoint, then we have $s_i(V) = \lambda_i(V), i \in \mathbb{N}$. In consequence to Theorem 5, we obtain

$$s_i(V)\, i^{1/d} = s_i(V)\, i^{\mu} \cdot i^{1/d-\mu} \leq C \cdot i^{1/d-\mu}, i \in \mathbb{N}. \tag{72}$$

Hence, to achieve condition (b), it is sufficient to show that $d > \mu^{-1}$. By virtue of the conditions $\vartheta(R_{\tilde{W}}) \leq 2\theta, \theta < \pi\mu/2$, we have $d = \pi/\vartheta(R_{\tilde{W}}) \geq \pi/2\theta > \mu^{-1}$. Hence, we obtain $s_i(V) =$

$o(i^{-1/d})$. Since both conditions (a) and (b) are fulfilled, then using Theorem 6.2 [33], p.305, we complete the proof.

Theorem 7 is devoted to the description of s-number behavior, but questions related with asymptotic of the eigenvalues $\lambda_i(R_{\tilde{W}}), i \in \mathbb{N}$, are still relevant in our work. It is a well-known fact that for any bounded operator with the compact imaginary component, there is a relationship between the s-numbers of the imaginary component and the eigenvalues (see [33]). Similarly, using the information on s-numbers of the real component, we can obtain an asymptotic formula for the eigenvalues $\lambda_i(R_{\tilde{W}}), i \in \mathbb{N}$. This idea is realized in the following theorem.

Theorem 9. *The following inequality holds:*

$$\sum_{i=1}^{n} |\lambda_i(R_{\tilde{W}})|^p \le \sec^p\theta \left\|S^{-1}\right\| \sum_{i=1}^{n} \lambda_i^p(R_H), \tag{73}$$

$$(n = 1, 2, \cdots, \nu(R_{\tilde{W}})), 1 \le p < \infty. \tag{74}$$

Moreover, if $\nu(R_{\tilde{W}}) = \infty$ *and the order* $\mu(H) \ne 0$*, then the following asymptotic formula holds:*

$$|\lambda_i(R_{\tilde{W}})| = o(i^{-\mu+\varepsilon}), i \longrightarrow \infty, \forall\varepsilon > 0. \tag{75}$$

Proof. Let L be a bounded operator with a compact imaginary component. Note that according to Theorem 6.1 [33], p.81, we have

$$\sum_{m=1}^{k} |\mathrm{Im}\ \lambda_m(L)|^p \le \sum_{m=1}^{k} |s_m(L_{\mathfrak{J}})|^p,\ (k = 1, 2, \cdots, \nu_{\mathfrak{J}}(L)), 1 \le p < \infty, \tag{76}$$

where $\nu_{\mathfrak{J}}(L) \le \infty$ is the sum of all algebraic multiplicities corresponding to the not real eigenvalues of the operator L (see [33], p.79). It can easily be checked that

$$(iL)_{\mathfrak{J}} = L_{\mathfrak{R}},\ \mathrm{Im}\ \lambda_m(iL) = \mathrm{Re}\ \lambda_m(L),\ m \in \mathbb{N}. \tag{77}$$

By virtue of (70), we have $\mathrm{Re}\ \lambda_m(R_{\tilde{W}}) > 0, m = 1, 2, \cdots, \nu(R_{\tilde{W}})$. Combining this fact with (77), we get $\nu_{\mathfrak{J}}(iR_{\tilde{W}}) = \nu(R_{\tilde{W}})$. Taking into account the previous equality and combining (76) and (77), we obtain

$$\sum_{m=1}^{k} |\mathrm{Re}\ \lambda_m(R_{\tilde{W}})|^p \le \sum_{m=1}^{k} |s_m(V)|^p,\ (k = 1, 2, \cdots, ,\nu(R_{\tilde{W}})), V \coloneqq (R_{\tilde{W}})_{\mathfrak{R}}. \tag{78}$$

Note that by virtue of (70), we have

$$|\mathrm{Im}\ \lambda_m(R_{\tilde{W}})| \le \tan\theta\ \mathrm{Re}\ \lambda_m\,(R_{\tilde{W}}), m \in \mathbb{N}. \tag{79}$$

Hence

$$|\lambda_m(R_{\tilde{W}})| = \sqrt{|\mathrm{Im}\ \lambda_m(R_{\tilde{W}})|^2 + |\mathrm{Re}\ \lambda_m(R_{\tilde{W}})|^2} \le \le\sqrt{\tan^2\theta + 1}|\mathrm{Re}\,\lambda_m(R_{\tilde{W}})| = \sec\theta|\mathrm{Re}\ \lambda_m(R_{\tilde{W}})|, m \in \mathbb{N}. \tag{80}$$

Combining (78), (80), we get

$$\sum_{m=1}^{k} |\lambda_m(R_{\tilde{W}})|^p \le \sec^p\theta \sum_{m=1}^{k} |s_m(V)|^p,\ (k = 1, 2, \cdots, ,\nu(R_{\tilde{W}})). \tag{81}$$

Using (34), we complete the proof of inequality (73).

Suppose $\nu(R_{\tilde{W}}) = \infty, \mu(H) \ne 0$, and let us prove (75). Note that for $\mu > 0$ and for any $\varepsilon > 0$, we can choose p so that $\mu p > 1, \mu - \varepsilon < 1/p$. Using the condition $\mu p > 1$, we obtain convergence of the series on the left side of (73). It implies that

$$|\lambda_i(R_{\tilde{W}})| i^{1/p} \longrightarrow 0,\ i \longrightarrow \infty. \tag{82}$$

It is obvious that

$$|\lambda_i(R_{\tilde{W}})| i^{\mu-\varepsilon} < |\lambda_i(R_{\tilde{W}})| i^{1/p}, i \in \mathbb{N}. \tag{83}$$

Taking into account (82), we obtain (75).

4. Applications

We begin with Definitions. Suppose Ω is a convex domain of the n-dimensional Euclidian space with the sufficient smooth boundary, $L_2(\Omega)$ is a complex Lebesgue space of summable with square functions, $H^2(\Omega)$,$H^1(\Omega)$ are complex Sobolev spaces, $D_i f \coloneqq \partial f/\partial x_i, 1 \le i \le n$ is the weak partial derivatives of the function f. Consider a sum of a uniformly elliptic operator and the extension of the Kipriyanov fractional differential operator of order $0 < \alpha < 1$ (see Lemma 2.5 [37]):

$$\begin{aligned} Lu &\coloneqq -D_j\left(a^{ij} D_i f\right) + \mathfrak{D}_{0+}^{\alpha} f, \\ \mathrm{D}(L) &= H^2(\Omega) \cap H_0^1(\Omega), \end{aligned} \tag{84}$$

with the following assumptions relative to the real-valued coefficients

$$a^{ij}(Q) \in C^1\left(\bar{\Omega}\right), a^{ij}\xi_i\xi_j \ge a|\xi|^2, a > 0. \tag{85}$$

It was proved in the paper [37] that the operator $L^+ f \coloneqq -D_i(a^{ij} D_j f) + \mathfrak{D}_{d-}^{\alpha} f, \mathrm{D}(L^+) = \mathrm{D}(L)$ is formal adjoint with respect to L. Note that in accordance with Theorem 2 [43], we have $\mathrm{R}(L) = \mathrm{R}(L^+) = L_2(\Omega)$, due to the reasonings of Theorem 3.1 [44], the operators L, L^+ are strictly accretive. Taking into account these facts, we can conclude that the operators L, L^+ are closed (see Problem 5.15 [34], p.165). Consider the operator $L_{\mathfrak{R}}$. Having made the absolutely analogous reasonings as in the previous case, we conclude that the operator $L_{\mathfrak{R}}$ is closed. Applying the reasonings of Theorem 4.3 [37], we obtain that the operator $L_{\mathfrak{R}}$ is selfadjoint and strictly accretive. Recall that to apply the methods described in the paper [19], we must have some decomposition of the initial operator L on a sum $L = \mathcal{T} + \mathcal{A}$, where $\mathcal{T}$ must be an operator of a special type either a selfadjoint or a normal operator. Note that the uniformly elliptic operator of second-order is neither selfadjoint no normal in the general case. To demonstrate the significance of the method obtained in this paper, we would like to note that a search for a convenient decomposition of L on a sum of a selfadjoint operator and some operator does not seem to be a reasonable way. Now, to justify this claim, we consider one of possible decompositions of L on a sum. Consider a selfadjoint strictly accretive operator $\mathcal{T} : \mathfrak{H} \longrightarrow \mathfrak{H}$.

Definition 10. In accordance with the definition of the paper [19], a quadratic form $\mathfrak{a} := \mathfrak{a}[f]$ is called a $\mathcal{T}$-subordinated form if the following condition holds:

$$|\mathfrak{a}[f]| \le b\, \mathfrak{t}[f] + M\|f\|^2_{\mathfrak{H}},\ \mathrm{D}(\mathfrak{a}) \supset \mathrm{D}(\mathfrak{t}),\ b < 1,\ M > 0, \quad (86)$$

where $\mathfrak{t}[f] = \|\mathcal{T}^{1/2}\|^2_{\mathfrak{H}}, f \in \mathrm{D}(\mathcal{T}^{1/2})$. The form $\mathfrak{a}$ is called a completely $\mathcal{T}$-subordinated form if, besides of (86), we have the following additional condition $\forall \varepsilon > 0 \exists b, M > 0 : b < \varepsilon$.

Let us consider the trivial decomposition of the operator L on the sum $L = 2L_{\mathfrak{R}} - L^+$ and let us use the notation $\mathcal{T} := 2L_{\mathfrak{R}}, \mathcal{A} := -L^+$. Then, we have $L = \mathcal{T} + \mathcal{A}$. Due to the sectorial property proven in Theorem 4.2 [37], we have

$$\begin{aligned} |(\mathcal{A}f, f)_{L_2}| &= \sec\theta_f |\operatorname{Re}(\mathcal{A}f, f)_{L_2}| \\ &= \sec\theta_f \frac{1}{2}(\mathcal{T}f, f)_{L_2}, f \in \mathrm{D}(\mathcal{T}), \end{aligned} \quad (87)$$

where $0 \le \theta_f \le \theta, \theta_f := |\arg(L^+ f, f)_{L_2}|, L_2 := L_2(\Omega)$ and θ is the semiangle corresponding to the sector $\mathfrak{L}_0(\theta)$. Due to Theorem 4.3 [37], the operator $\mathcal{T}$ is m-accretive. Hence, in consequence to Theorem 3.35 [34], p.281, we conclude that $\mathrm{D}(\mathcal{T})$ is a core of the operator $\mathcal{T}^{1/2}$. It implies that we can extend relation (87) to

$$\frac{1}{2}\, \mathfrak{t}[f] \le |\mathfrak{a}[f]| \le \sec\theta \frac{1}{2}\, \mathfrak{t}[f], f \in \mathrm{D}(\mathfrak{t}), \quad (88)$$

where $\mathfrak{a}$ is a quadratic form generated by $\mathcal{A}$ and $\mathfrak{t}[f] = \|\mathcal{T}^{1/2} f\|^2_{\mathfrak{H}}$. If we consider the case $0 < \theta < \pi/3$, then it is obvious that there exist constants $b < 1$ and $M > 0$ such that the following inequality holds:

$$|\mathfrak{a}[f]| \le b\, \mathfrak{t}[f] + M\|f\|^2_{L_2},\ f \in \mathrm{D}(\mathfrak{t}). \quad (89)$$

Hence, the form $\mathfrak{a}$ is a $\mathcal{T}$-subordinated form. In accordance with the definition given in the paper [19], it means $\mathcal{T}$-subordination of the operator $\mathcal{A}$ in the sense of form. Assume that $\forall \varepsilon > 0 \exists b, M > 0 : b < \varepsilon$. Using inequality (88), we get

$$\frac{1}{2}\mathfrak{t}[f] \le \varepsilon\, \mathfrak{t}[f] + M(\varepsilon)\|f\|^2_{L_2};\ \mathfrak{t}[f] \le \frac{2M(\varepsilon)}{(1-2\varepsilon)}\|f\|^2_{L_2}, f \in \mathrm{D}(\mathfrak{t}), \varepsilon < 1/2. \quad (90)$$

Using the strictly accretive property of the operator L (see inequality (4.9) [37]), we obtain

$$\|f\|^2_{H_0^1} C \le \mathfrak{t}[f], f \in \mathrm{D}(\mathfrak{t}). \quad (91)$$

On the other hand, using the results of the paper [37], it is easy to prove that $H_0^1(\Omega) \subset \mathrm{D}(\mathfrak{t})$. Taking into account the facts considered above, we get

$$\|f\|_{H_0^1} \le C\|f\|_{L_2}, f \in H_0^1(\Omega), \quad (92)$$

but as it is well known, this inequality is not true. This contradiction shows us that the form $\mathfrak{a}$ is not a completely $\mathcal{T}$-subordinated form. It implies that we cannot use Theorem 8.4 [19] which could give us an opportunity to describe the spectral properties of the operator L. Note that the reasonings corresponding to another trivial decomposition of L on a sum are analogous.

This rather particular example does not aim at showing the inability of using remarkable methods considered in the paper [19] but only creates prerequisite for some value of another method based on using spectral properties of the real component of the initial operator L. Now, we would like to demonstrate the effectiveness of this method. Suppose $\mathfrak{H} := L_2(\Omega)$, $\mathfrak{H}^+ := H_0^1(\Omega)$, $Tf := -D_j(a^{ij}D_i f)$, $Af := \mathfrak{D}^{\alpha}_{0+} f$, $\mathrm{D}(T)$, $\mathrm{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$; then, due to the Rellich-Kondrachov theorem, we conclude that condition (2) is fulfilled. Due to the results obtained in the paper [37], we conclude that condition (4) is fulfilled. Applying the results obtained in the paper [37], we conclude that the operator $L_{\mathfrak{R}}$ has nonzero order. Hence, we can apply the abstract results of this paper to the operator L. In fact, Theorems 7–9 describe the spectral properties of the operator L.

We deal with the differential operator acting in the complex Sobolev space and defined by the following expression

$$\begin{aligned} &\mathcal{L}f := \left(c_k f^{(k)}\right)^{(k)} + \left(c_{k-1} f^{(k-1)}\right)^{(k-1)} + \cdots + c_0 f, \\ &\mathrm{D}(\mathcal{L}) = H^{2k}(I) \cap H_0^k(I), k \in \mathbb{N}, \end{aligned} \quad (93)$$

where $I := (a, b) \subset \mathbb{R}$, and the complex-valued coefficients $c_j(x) \in C^{(j)}(\bar{I})$ satisfy the condition $\operatorname{sign}(\operatorname{Re} c_j) = (-1)^j, j = 1, 2, \cdots, k$. It is easy to see that

$$\begin{aligned} \operatorname{Re}(\mathcal{L}f, f)_{L_2(I)} &\ge \sum_{j=0}^{k} |\operatorname{Re} c_j| \left\|f^{(j)}\right\|^2_{L_2(I)} \\ &\ge C\left\|f^{(j)}\right\|^2_{H_0^k(I)}, f \in \mathrm{D}(\mathcal{L}). \end{aligned} \quad (94)$$

On the other hand

$$\left|(\mathscr{L}f,f)_{L_2(I)}\right| = \left|\sum_{j=0}^{k}(-1)^j\left(c_jf^{(j)},g^{(j)}\right)_{L_2(I)}\right| \le \sum_{j=0}^{k}\left|\left(c_jf^{(j)},g^{(j)}\right)_{L_2(I)}\right| \le$$
$$\le C\sum_{j=0}^{k}\left\|f^{(j)}\right\|_{L_2(I)}\left\|g^{(j)}\right\|_{L_2(I)} \le \|f\|_{H_0^k(I)}\|g\|_{H_0^k(I)}, f \in \mathrm{D}(\mathscr{L}). \quad (95)$$

Consider the Riemann-Liouville operators of fractional differentiation of arbitrary nonnegative order α (see [32], p.44) defined by the expressions

$$D_{a+}^{\alpha}f = \left(\frac{d}{dx}\right)^{[\alpha]+1} I_{a+}^{1-\{\alpha\}}f\,;$$
$$D_{b-}^{\alpha}f = \left(-\frac{d}{dx}\right)^{[\alpha]+1} I_{b-}^{1-\{\alpha\}}f, \quad (96)$$

where the fractional integrals of arbitrary positive order α are defined by

$$(I_{a+}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt,$$
$$(I_{b-}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}dt, f \in L_1(I). \quad (97)$$

Suppose $0<\alpha<1, f \in AC^{l+1}(\bar{I}), f^{(j)}(a)=f^{(j)}(b)=0, j=0, 1,\cdots,l$; then, the next formulas follow from Theorem 2.2 [32], p.46:

$$D_{a+}^{\alpha+l}f = I_{a+}^{1-\alpha}f^{(l+1)},\ D_{b-}^{\alpha+l}f = (-1)^{l+1}I_{b-}^{1-\alpha}f^{(l+1)}. \quad (98)$$

Further, we need the following inequalities (see [45]):

$$\mathrm{Re}\,(D_{a+}^{\alpha}f,f)_{L_2(I)} \ge C\|f\|_{L_2(I)}^2, f \in I_{a+}^{\alpha}(L_2),\ \ \mathrm{Re}\,(D_{b-}^{\alpha}f,f)_{L_2(I)}$$
$$\ge C\|f\|_{L_2(I)}^2, f \in I_{b-}^{\alpha}(L_2), \quad (99)$$

where $I_{a+}^{\alpha}(L_2), I_{b-}^{\alpha}(L_2)$ are the classes of the functions representable by the fractional integrals (see [32]). Consider the following operator with the constant real-valued coefficients:

$$\mathscr{D}f := p_nD_{a+}^{\alpha_n} + q_nD_{b-}^{\beta_n} + p_{n-1}D_{a+}^{\alpha_{n-1}} + q_{n-1}D_{b-}^{\beta_{n-1}} + \cdots + p_0D_{a+}^{\alpha_0} + q_0D_{b-}^{\beta_0},$$
$$\mathrm{D}(\mathscr{D}) = H^{2k}(I)\cap H_0^k(I), n \in \mathbb{N}, \quad (100)$$

where $\alpha_j, \beta_j \ge 0, 0 \le [\alpha_j], [\beta_j] < k,\ j=0,1,\cdots,n,$

$$q_j \ge 0,\ \ \mathrm{sign}\ p_j = \begin{cases}(-1)^{[\alpha_j]+1/2}, & [\alpha_j] = 2m-1, m \in \mathbb{N},\\ (-1)^{[\alpha_j]/2}, & [\alpha_j]=2m,\ m\in\mathbb{N}_0.\end{cases} \quad (101)$$

Using (98) and (99), we get

$$\begin{aligned}\left(p_jD_{a+}^{\alpha_j}f,f\right)_{L_2(I)} &= p_j\left(\left(\frac{d}{dx}\right)^m D_{a+}^{m-1+\{\alpha_j\}}f,f\right)_{L_2(I)}\\ &= (-1)^m p_j\left(I_{a+}^{1-\{\alpha_j\}}f^{(m)},f^{(m)}\right)_{L_2(I)}\\ &\ge C\left\|I_{a+}^{1-\{\alpha_j\}}f^{(m)}\right\|_{L_2(I)}^2 = C\left\|D_{a+}^{\{\alpha_j\}}f^{(m-1)}\right\|_{L_2(I)}^2\\ &\ge C\left\|f^{(m-1)}\right\|_{L_2(I)}^2,\end{aligned} \quad (102)$$

where $f \in \mathrm{D}(\mathscr{D})$ is a real-valued function and $[\alpha_j]=2m-1$, $m\in\mathbb{N}$. Similarly, we obtain for orders $[\alpha_j]=2m$, $m\in\mathbb{N}_0$

$$\begin{aligned}\left(p_jD_{a+}^{\alpha_j}f,f\right)_{L_2(I)} &= p_j\left(D_{a+}^{2m+\{\alpha_j\}}f,f\right)_{L_2(I)}\\ &= (-1)^m p_j\left(D_{a+}^{m+\{\alpha_j\}}f,f^{(m)}\right)_{L_2(I)}\\ &= (-1)^m p_j\left(D_{a+}^{\{\alpha_j\}}f^{(m)},f^{(m)}\right)_{L_2(I)}\\ &\ge C\left\|f^{(m)}\right\|_{L_2(I)}^2.\end{aligned} \quad (103)$$

Thus, in both cases, we have

$$\left(p_jD_{a+}^{\alpha_j}f,f\right)_{L_2(I)} \ge C\left\|f^{(s)}\right\|_{L_2(I)}^2,\ s=\left[[\alpha_j]/2\right]. \quad (104)$$

In the same way, we obtain the inequality

$$\left(q_jD_{b-}^{\alpha_j}f,f\right)_{L_2(I)} \ge C\left\|f^{(s)}\right\|_{L_2(I)}^2,\ s=\left[[\alpha_j]/2\right]. \quad (105)$$

Hence, in the complex case, we have

$$\mathrm{Re}\,(\mathscr{D}f,f)_{L_2(I)} \ge C\|f\|_{L_2(I)}^2,\ f\in\mathrm{D}(\mathscr{D}). \quad (106)$$

Combining Theorem 2.6 [32], p.53, with (98), we get

$$\begin{aligned}\left\|p_j D_{a+}^{\alpha_j} f\right\|_{L_2(I)} &= \left\|I_{a+}^{1-\{\alpha_j\}} f^{([\alpha_j]+1)}\right\|_{L_2(I)} \\ &\le C\left\|f^{([\alpha_j]+1)}\right\|_{L_2(I)} \\ &\le C\|f\|_{H_0^k(I)}; \\ \left\|q_j D_{b-}^{\alpha_j} f\right\|_{L_2(I)} &\le C\|f\|_{H_0^k(I)},\ f \in \mathrm{D}(\mathscr{D}).\end{aligned} \tag{107}$$

Hence, we obtain

$$\|\mathscr{D}f\|_{L_2(I)} \le C\|f\|_{H_0^k(I)},\ f \in \mathrm{D}(\mathscr{D}). \tag{108}$$

Now, we can formulate the main result. Consider the operator

$$\begin{aligned} G &= \mathscr{L} + \mathscr{D}, \\ \mathrm{D}(G) &= H^{2k}(I) \cap H_0^k(I). \end{aligned} \tag{109}$$

Suppose $\mathfrak{H} := L_2(\mathrm{I}), \mathfrak{H}^+ := H_0^k(I), T := \mathscr{L}, A := \mathscr{D};$ then due to the well-known fact of the Sobolev spaces theory, condition (2) is fulfilled, due to the reasonings given above, condition (4) is fulfilled. Taking into account the equality

$$\begin{aligned}\mathscr{L}_{\mathfrak{R}} f = &\left(\mathrm{Re}\ c_k f^{(k)}\right)^{(k)} + \left(\mathrm{Re}\ c_{k-1} f^{(k-1)}\right)^{(k-1)} \\ &+ \cdots + \mathrm{Re}\ c_0 f,\ f \in \mathrm{D}(\mathscr{D})\end{aligned} \tag{110}$$

and using the method described in the paper [46], we can prove that the operator $\tilde{G}_{\mathfrak{R}}$ has nonzero order. Hence, we can successfully apply the abstract results of this paper to the operator G. Now, it is easily seen that Theorems 7–9 describe the spectral properties of the operator G.

Data Availability

The data (References) used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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