

Research Article
Uniform Hybrid Difference Scheme for Singularly Perturbed Differential-Difference Turning Point Problems Exhibiting Boundary Layers

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In this paper, a class of linear second-order singularly perturbed differential-difference turning point problems with mixed shifts exhibiting two exponential boundary layers is considered. For the numerical treatment of these problems, first we employ a second-order Taylor’s series approximation on the terms containing shift parameters and obtain a modified singularly perturbed problem which approximates the original problem. Then a hybrid finite difference scheme on an appropriate piecewise-uniform Shishkin mesh is constructed to discretize the modified problem. Further, we proved that the method is almost second-order \( \varepsilon \)-uniformly convergent in the maximum norm. Numerical experiments are considered to illustrate the theoretical results. In addition, the effect of the shift parameters on the layer behavior of the solution is also examined.

1. Introduction

Many real-life phenomena in different fields of science are modeled mathematically by delay differential or differential-difference equations (DDEs). Equations of this type arise widely in scientific fields such as physics, biosciences, ecology, control theory, economics, material science, medicine, and robotics, in which time evolution depends not only on present states but also on the states at or near a given time in the past. DDEs are also prominent in describing several aspects of infectious disease dynamics such as primary infection, drug therapy, and immune response. In addition, statistical analysis of ecological data has shown that there is evidence of delay effects in the population dynamics of many species; for the detail theory of DDEs, one can refer to the books [1, 2].

If we restrict the class of DDEs in which the highest derivative is multiplied by a small parameter, then it is said to be a singularly perturbed differential-difference equations (SPDDEs) [3]. In the past, less attention had been given for the solutions of SPDDEs. However, in recent years, there has been a growing interest in the treatment of such problems. This is due to their importance in the modeling of processes in various fields such as optical bistable devices [4], variational problems in control theory [5, 6], the hydrodynamics of liquid helium [7], the first exit-time problem [8], describing the human pupil-light reflex [9], microscale heat transfer [10], and a variety of models for physiological processes or diseases [11, 12].

The study of different classes of SPDDEs was initiated by Lange and Miura [8, 13, 14], where they used extension of the method of matched asymptotic expansions for approximating the solution. But in all the cases, they excluded the occurrence of turning points and left it for future study. On the other hand, Kadalbajoo and Sharma [3, 15, 16] initiated the numerical study of SPDDEs with mixed shifts by constructing a variety of numerical schemes. In recent years, different scholars further developed numerical
schemes for SPDEs with mixed shifts, to mention few [17–21]. Most of the works developed so far focuses only on SPDEs without turning points. In contrast, there are few works on singularly perturbed delay turning point problems. The papers by Ria and Sharma [22–25] are the first and also the only notable works in the treatment of such problems when the solutions exhibit both interior and boundary layers, where the authors used a fitted mesh and fitted operator methods and obtained an almost first-order uniform convergence. Therefore, it is natural to develop a robust numerical method for turning point problems having a better accuracy and efficiency.

In this paper, we consider the following second-order linear singularly perturbed differential-difference problem containing mixed shifts and with a turning point at \( x = 0 \):

\[
\begin{align*}
- \varepsilon y''(x) - a(x) y'(x) + b(x) y(x) - c(x) y(x - \delta) &= f(x), & \forall x \in \Omega = (-1, 1), \\
y(x) &= \phi(x), & -1 - \delta \leq x \leq -1, \\
y(x) &= \psi(x), & 1 \leq x \leq 1 + \eta,
\end{align*}
\]

where \( a(x), b(x), c(x), d(x), f(x), \phi(x), \) and \( \psi(x) \) are sufficiently smooth functions on \( \Omega = (-1, 1) \) is the singular perturbation parameter, and \( 0 < \varepsilon \ll 1 \) and \( 0 < \delta \ll 1 \) are the delay and advance parameters, respectively, together with the following assumptions:

\[
\begin{align*}
a(0) &= 0, & a'(0) &= 0, \\
b(x) &= b_0 > 0, & \forall x \in \overline{\Omega} = [-1, 1], \\
b(x) - c(x) - d(x) &\geq \beta > 0, & c(x) &\geq K_1 > 0, \\
d(x) &\geq K_2 > 0, & \forall x \in \overline{\Omega}, \\
|a'(x)| &\geq \frac{|a'(0)|}{2}, & \forall x \in \overline{\Omega},
\end{align*}
\]

under these assumptions, problems (1) and (2) possesses a unique solution having boundary layers of exponential type at \( x = \pm 1 \), i.e., at both end points [25].

Here for the numerical treatment of (1) and (2), we propose an hybrid finite difference scheme on an appropriate piecewise-uniform Shishikin mesh and analyze the stability and uniform convergence the proposed method. Further, we investigate the effect of the shift parameters on the behavior of the solution.

Throughout this paper, \( M \) (sometimes subscripted) denotes a generic positive constant independent of the singular perturbation parameter \( \varepsilon \) and in the case of discrete problems, also independent of the mesh parameter \( N \). The maximum norm (i.e., \( \|f\| = \max_{-1 \leq x \leq 1} |f(x)| \)) is used for studying the convergence of the approximate solution to the exact solution of the problem.

\section{The Continuous Problem}

Using Taylor’s series expansion to approximate the terms containing shift arguments gives us

\[
\begin{align*}
y(x - \delta) &= y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x), \\
y(x + \eta) &= y(x) + \eta y'(x) + \frac{\eta^2}{2} y''(x).
\end{align*}
\]

Substituting (7) into (1) and (2) and simplifying gives the following asymptotically equivalent two-point boundary value problem

\[
\begin{align*}
- \left( \varepsilon + \frac{\delta^2}{2} c(x) + \frac{\eta^2}{2} d(x) \right) y'' - (a(x) - \delta c(x)) y' + \eta d(x) y' + (b(x) - c(x) - d(x)) y &= f(x), \\
y(-1) &= \phi(-1), & y(1) &= \psi(1).
\end{align*}
\]

Since (8) is an approximation version of (1) and (2), it is better to use different notation (say \( u(x) \)) for the solution of this approximate equation. Thus, (8) can be written as

\[
\begin{align*}
Lu &\equiv - C_1 u''(x) - A(x) u'(x) + B(x) u(x) = f(x), \\
u(-1) &= \phi(-1) = \phi, \\
u(1) &= \psi(1) = \psi,
\end{align*}
\]

where \( C_1(x) = (\varepsilon + (\delta^2/2) c(x) + (\eta^2/2) d(x)), A(x) = a(x) - \delta c(x) + \eta d(x), \) and \( B(x) = b(x) - c(x) - d(x). \) Moreover, the terms \( a(x), c(x), d(x), \delta, \) and \( \eta \) are such that \( |A(x)| \geq 2\alpha > 0 \), for \( \tau < |x| \leq 1 \), for some \( \tau > 0 \), and later on, we will use the term \( C_1 \) to denote the constant part of \( C_1(x) \) (since \( c(x), d(x) \) are bounded and \( \delta, \eta \) are small parameters, we have \( C_1 = O(\varepsilon) \).

The solution of problem (9) is an approximation to the solution of the original problem (1) and (2).

We establish some a priori results about the solutions and their derivatives for the modified problem (9). Hereinafter, we divide the interval \( \overline{\Omega} \) in to three subintervals as \( \Omega_1 = [-1, -\tau], \Omega_2 = [-\tau, \tau], \) and \( \Omega_3 = [\tau, 1] \) such that \( \overline{\Omega} = \Omega_1 \cup \Omega_2 \cup \Omega_3 \), where \( 0 < \tau \leq 1/2 \).

First, we consider the following property of the operator \( L \) of (9).

\begin{lemma}
\text{(Maximum principle).} Let \( \pi(x) \) be any sufficiently smooth function satisfying \( \pi(-1) \geq 0 \) and \( \pi(1) \geq 0 \), such that \( L \pi(x) \geq 0 \) for all \( x \in \Omega \). Then, \( \pi(x) \geq 0 \) for all \( x \in \overline{\Omega} \).
\end{lemma}

\begin{proof}
Let \( x^* \) be an arbitrary point in \( \Omega = (-1, 1) \) such that \( \pi(x^*) = \min_{x \in \overline{\Omega}} |\pi(x)| \) and assume that \( \pi(x^*) < 0 \). Clearly \( x^* \notin [-1, 1] \), and from the definition of \( x^* \), we have \( \pi'(x^*) = 0 \) and \( \pi''(x^*) \geq 0 \). But then,
\end{proof}
\[ L \pi (x^*) = -C(x) \pi ''(x^*) - A(x^*) \pi '(x^*) + B(x^*) \pi (x^*) \leq 0, \]

which is a contradiction. It follows that our assumption \( \pi (x^*) < 0 \) is wrong, so \( \pi (x^*) \geq 0 \). Since \( x^* \) is an arbitrary point, \( \pi (x) \geq 0, \forall x \in \Omega = [-1, 1]. \)

Using the maximum principle, it is easy to prove that:

**Lemma 2 (Stability Result).** Let \( u(x) \) be the solution of the TPP (9). Then \( \forall C_\varepsilon > 0 \) we have
\[
\| u \| \leq \beta^{-1} \| f \| + \max (|\phi|, |\psi|), \quad \forall x \in \Omega = [-1, 1]. \tag{11}
\]

**Proof.** First we consider the barrier functions \( \varphi^+ \) defined by
\[
\varphi^+(x) = \beta^{-1} \| f \| + \max (|\phi|, |\psi|) + u(x). \tag{12}
\]

Then it is easy to show that \( \varphi^-(1) \geq 0 \) and \( \varphi^+(1) > 0 \), and
\[
L \varphi^+(x) = -C(x) (\varphi^+ (x))^\alpha - A(x)(\varphi^+ (x))' + B(x)\varphi^+(x) = B(x) (\beta^{-1} \| f \| + \max (|\phi|, |\psi|)) + \mu u(x)
\]

\[
\geq (\| f \| + \mu f(x)) + \beta \max (|\phi|, |\psi|) \geq 0. \tag{13}
\]

Therefore, from Lemma 1, we obtain \( \varphi^+(x) \geq 0 \) for all \( x \in [-1, 1] \), which gives the desired estimate. \( \square \)

The following theorem gives estimates for \( u \) and its derivatives in the interval \( \Omega_1 \) and \( \Omega_3 \), which excludes the turning point \( x = 0 \).

**Theorem 1.** Let \( A(x), B(x), \) and \( f(x) \in C^m(\Omega) \), \( m > 0 \), \( |A(x)| \geq 2\alpha > 0, \forall x \in \Omega_1 \cup \Omega_2 \). Then there exists a positive constant \( M \), such that for \( A(x) > 0 \) on \( \Omega_1 \), the solution \( u(x) \) of problem (9) satisfies
\[
\| u^{(i)}(x) \| \leq M \left[ 1 + C_\varepsilon \exp \left( \frac{-\alpha (1 - x)}{C_\varepsilon} \right) \right], \tag{14}
\]

for \( i = 1, \ldots, m + 1 \),

and for \( A(x) < 0 \) on \( \Omega_3 \), we have
\[
\| u^{(i)}(x) \| \leq M \left[ 1 + C_\varepsilon \exp \left( \frac{-\alpha (1 - x)}{C_\varepsilon} \right) \right], \tag{15}
\]

for \( i = 1, \ldots, m + 1 \).

**Proof.** For the proof of this theorem, the reader can refer [25]. \( \square \)

If \( \lambda = B(0)/\alpha'(0) < 0 \), then the solution \( u(x) \) is smooth near the turning point \( x = 0 \) [26]. Using this, the following theorem gives the bound for the derivatives of the solution in the interval \( \Omega_2 \) which contains the turning point \( x = 0 \).

**Theorem 2.** Let \( \lambda < 0 \). If \( u(x) \) is the solution of (9) and satisfies all conditions from (3) to (6), let \( A, B, \) and \( f \in C^m(\Omega) \), for \( m > 0 \). Then there exists a positive constant \( M \), such that
\[
\| u^{(i)}(x) \| \leq M, \quad \text{for } i = 1, \ldots, m \text{ and } \forall x \in \Omega_2. \tag{16}
\]

**Proof.** For the proof one can refer [25, 26]. \( \square \)

Finally, to prove the uniform convergence of the proposed numerical method, we need to consider the following theorem which provides bounds for the smooth and singular components of the exact solution \( u \) of problem (9).

**Theorem 3.** Let \( A, B \) and \( f \in C^4(\Omega) \), and assume that the solution \( u(x) \) of problem (9) is decomposed into the smooth and singular components as
\[
u^{(i)}(x) \leq M \left[ 1 + C_\varepsilon \exp \left( \frac{-\alpha (1 - x)}{C_\varepsilon} \right) \right], \tag{18}
\]

and the singular component satisfies
\[
\| w^{(i)}(x) \| \leq MC_\varepsilon \exp \left( \frac{-\alpha (1 - x)}{C_\varepsilon} \right), \quad \forall x \in \Omega, \tag{19}
\]

where \( \varepsilon(x, a) = \exp(-\alpha(1 + x)/C_\varepsilon) + \exp(-\alpha(1 - x)/C_\varepsilon) \).

**Proof.** For the proof of this theorem, the reader can refer [25, 27]. \( \square \)

### 3. Discrete Problem

In this section, we describe the piecewise-uniform Shishkin mesh for the discretization of the domain and study the behavior of the hybrid difference scheme used for the modified problem (9).

#### 3.1. Piecewise-Uniform Shishkin Mesh

Consider the domain \( \Omega = [-1, 1] \) and let \( N = 8k \) and \( k > 0 \) be a positive integer. Since the TPP (9) has two boundary layers at \( x = \pm 1 \), we construct a piecewise-uniform Shishkin mesh by subdividing the domain \( \Omega \) into three subintervals \( \Omega_\varepsilon = [-1, -1 + \tau], \Omega_C = [-1 + \tau, 1 - \tau], \) and \( \Omega_R = [1 - \tau, 1] \) such that \( \Omega = \Omega_\varepsilon \cup \Omega_C \cup \Omega_R \), where the transition parameter \( \tau \) satisfies \( 0 < \tau \leq 1/2 \) and defined by
\[
\tau = \min \left\{ \frac{1}{2}, \frac{2C_\varepsilon \ln N}{\alpha} \right\}, \quad \text{where } C_\varepsilon = \varepsilon + K\frac{\beta^2}{2} + K\frac{\eta^2}{2}. \tag{20}
\]

Then the discrete mesh \( \Omega^N \) is obtained by putting a uniform mesh with \( N/4 \) mesh elements in both \( \Omega_\varepsilon \) and \( \Omega_R \), and a uniform mesh with \( N/2 \) mesh elements in \( \Omega_C \), such that \( \Omega^N = \{ x_i \in [-1, 1]: i = 0, 1, \ldots, N \} \).

Let \( h_i = x_i - x_{i-1}, \) for \( i = 1, \ldots, N \), denotes the variable step size. Since the mesh is piecewise-uniform, then the mesh elements are given by
where \( h = 4\tau/N \) and \( H = 4(1 - \tau)/N \) are uniform mesh lengths for the piecewise-uniform meshes.

If \( C_\varepsilon > MN^{-1} \), then the mesh becomes equally spaced and the transition parameter becomes \( \tau = 1/2 \) such that

\[
C_\varepsilon^{-1} < 4\ln N \alpha.
\]

On the other hand, for \( C_\varepsilon < MN^{-1} \), the mesh is piecewise-uniform and \( \tau = 2C_\varepsilon/\alpha \ln N \). Here, we have \( 2N^{-1} \leq H \leq 4N^{-1} \) and

\[
h = \frac{8\alpha}{C_\varepsilon} N^{-1} \ln N,
\]

\[
e^{-\alpha(1+x_N/4)C_\varepsilon} e^{-\alpha(1-x_N/4)C_\varepsilon} = N^{-2}.
\]

### 3.2. Hybrid Difference Scheme

Before describing the scheme, for a given mesh function \( y(x_i) = y_i \), we define the forward, backward, and central difference operators \( D^+, D^- \), and \( D^0 \) by

\[
D^+ y_i = \frac{y_{i+1} - y_i}{h_{i+1}},
\]

\[
D^- y_i = \frac{y_i - y_{i-1}}{h_i},
\]

\[
D^0 y_i = \frac{y_{i+1} - y_{i-1}}{2h_i},
\]

respectively, and the second-order central difference operator \( \delta^2 \) by

\[
\delta^2 y_i = \frac{2(D^+ y_i - D^- y_i)}{h_i} = \frac{2}{h_i h_{i+1} h_i} (h_{i+1} y_{i+1} - h_i y_i + h_i y_{i+1}).
\]

where \( h_i = h_{i+1} \) for \( i = 1, \ldots, N - 1 \).

Further, we define the midpoint upwind schemes \( L^N_M \) and the classical central difference scheme \( L^N_C \) used to approximate the continuous operator \( L \) as

\[
L^N_M y_i = -C_{\varepsilon i} \delta^2 y_i - A_{i-1/2} D^- y_i + B_{i-1/2} y_{i-1/2} = f_{i-1/2},
\]

\[
L^N_C y_i = -C_{\varepsilon i} \delta^2 y_i - A_i D^0 y_i + B_i y_i = f_i,
\]

where \( A_{i+1/2} = (A_i + A_{i+1})/2 \) and similarly for \( C_{\varepsilon i+1/2}, B_{i+1/2} \), and \( f_{i+1/2} \).

Now, we propose the hybrid difference scheme to discretize (9), which consists of the classical central difference scheme when \( C_\varepsilon > MN^{-1} \), and a proper combination of the midpoint upwind schemes in the outer region \( \Omega_c \) and the central difference scheme in the layer regions \( \Omega_l \) and \( \Omega_R \), whenever \( C_\varepsilon < MN^{-1} \). Hence, the proposed hybrid scheme on \( \Omega^+ \) takes the following form:

\[
\begin{align*}
L^N_M U_i &= f_i, & i &= 1, \ldots, N - 1, \\
U_0 &= u_0, & U_N &= u_N,
\end{align*}
\]

where

\[
L^N_M U_i, \quad i = 1, \ldots, N - 1, \text{ and } C_\varepsilon > MN^{-1},
\]

\[
L^N_C U_i, \quad i = 1, \ldots, N - 1, \text{ and } C_\varepsilon \leq MN^{-1},
\]

\[
L^N_M U_i, \quad i = N/4 + 1, \ldots, 3N/4, \text{ and } C_\varepsilon \leq MN^{-1},
\]

\[
L^N_C U_i, \quad i = 1, \ldots, N - 1, i = 3N/4 + 1, \ldots, N - 1, \text{ and } C_\varepsilon \leq MN^{-1},
\]

\[
L^N_M U_i, \quad i = N/4 + 1, \ldots, 3N/4, \quad C_\varepsilon \leq MN^{-1},
\]

\[
L^N_C U_i, \quad i = 1, \ldots, N - 1, \text{ and } C_\varepsilon \leq MN^{-1},
\]

\[
L^N_M U_i, \quad i = N/4 + 1, \ldots, 3N/4, \quad C_\varepsilon \leq MN^{-1},
\]

\[
L^N_C U_i, \quad i = 1, \ldots, N - 1, \text{ and } C_\varepsilon \leq MN^{-1},
\]

\[
L^N_M U_i, \quad i = N/4 + 1, \ldots, 3N/4, \quad C_\varepsilon \leq MN^{-1},
\]

\[
L^N_C U_i, \quad i = 1, \ldots, N - 1, \text{ and } C_\varepsilon \leq MN^{-1},
\]

\[
L^N_M U_i, \quad i = N/4 + 1, \ldots, 3N/4, \quad C_\varepsilon \leq MN^{-1},
\]

\[
L^N_C U_i, \quad i = 1, \ldots, N - 1, \text{ and } C_\varepsilon \leq MN^{-1},
\]
and the right hand side vector \( \overline{f}_i \) as

\[
\overline{f}_i = \begin{cases} 
  f_i, & i = 1, \ldots, N - 1, \text{ and } C_c > MN^{-1}, \\
  f_{i+1/2}, & i = \frac{N}{4}, \ldots, \frac{N}{2}, \text{ and } C_c \leq MN^{-1}, \\
  f_{i-1/2}, & i = \frac{N}{2} + 1, \ldots, \frac{3N}{4}, \text{ and } C_c \leq MN^{-1}, \\
  f_i, & i = 1, \ldots, \frac{3N}{4} - 1, i = \frac{3N}{4} + 1, \ldots, N - 1, \text{ and } C_c \leq MN^{-1}.
\end{cases}
\]

After rearranging the terms in (27), we obtain the following system of equations:

\[
\begin{align*}
  r^i_+ &= \frac{2C_{c,i}}{h_i h_{i+1}} + A_{i+1/2} \frac{A_i}{h_i} + B_{i+1/2}, \\
  r^i_- &= \frac{2C_{c,i+1/2}}{h_i h_{i+1}} + A_{i+1/2} \frac{A_i}{h_i} + B_{i+1/2}, \\
  r^i_+ &= \frac{2C_{c,i+1/2}}{h_i h_{i+1}} + A_{i+1/2} \frac{A_i}{h_i} + B_{i+1/2}, \\
  r^i_- &= \frac{2C_{c,i+1/2}}{h_i h_{i+1}} + A_{i+1/2} \frac{A_i}{h_i} + B_{i+1/2}, \\
  r^i_+ &= \frac{2C_{c,i+1/2}}{h_i h_{i+1}} + A_{i+1/2} \frac{A_i}{h_i} + B_{i+1/2}, \\
  r^i_- &= \frac{2C_{c,i+1/2}}{h_i h_{i+1}} + A_{i+1/2} \frac{A_i}{h_i} + B_{i+1/2},
\end{align*}
\]

where the coefficients are given by

\[
\begin{align*}
  &r^i_+ = \frac{2C_{c,i}}{h_i h_{i+1}} + A_{i+1/2} \frac{A_i}{h_i} + B_{i+1/2}, \\
  &r^i_- = \frac{2C_{c,i+1/2}}{h_i h_{i+1}} + A_{i+1/2} \frac{A_i}{h_i} + B_{i+1/2},
\end{align*}
\]

for \( i = 1, \ldots, N - 1 \), then \( U_i \leq D_o \) for \( i = 0, \ldots, N \). Hence, (27) has a unique solution.

**Proof.** To prove the results, it is enough to show that the operator given by (30) is an M-matrix. For this, we need to show that (30) satisfies

\[
\begin{align*}
  r^i_- &< 0, \\
  r^i_+ &< 0, \\
  r^i_- + r^i_+ + r^i_+ &> 0,
\end{align*}
\]

for all the operators defined in (31)–(33). Here, we separately consider the following two cases based on the relation between \( C_c \) and \( N \).

**Case 1:** When \( C_c > MN^{-1} \), the mesh is uniform, and we used central difference scheme on the entire domain. Thus, for \( M \geq \|A\| \) and using (22) into (31), we get
\[ r_i^- = \frac{C_{ij}}{h^2} + \frac{A_j}{2h} = \frac{1}{h^2} (C_{ij} + \frac{1}{2} A_j h) \leq \frac{1}{h^2} (-C_i + \|A\| N^{-1}) < 0, \]
\[ r_i^+ = \frac{C_{ij}}{h^2} - \frac{A_j}{2h} = \frac{1}{h^2} (C_{ij} - \frac{1}{2} A_j h) \leq \frac{1}{h^2} (-C_i + \|A\| N^{-1}) < 0, \]
(36)
and some simple calculations gives \( r_i^- + r_i^+ + r_i^+ = B_i > 0 \), for all \( 1 \leq i \leq N - 1 \).

Case 2: when \( C_r \leq MN^{-1} \), different operators are used in the layer regions and the outer regions.

In the layer regions, it is apparent that \( r_i^+ < 0 \) and \( r_i^- < 0 \), for \( 1 \leq i < N/4 \) and \( 3N/4 < i \leq N - 1 \), respectively. Further, using the first assumption of (34) and (23) in (31) we get
\[ r_i = \frac{C_{ij}}{h^2} + \frac{A_j}{2h} \leq \frac{1}{h^2} \left( \frac{-\alpha N}{8 \ln N} + \|A\| \right) < 0, \]
\[ r_i^+ = \frac{C_{ij}}{h^2} - \frac{A_j}{2h} \leq \frac{1}{h^2} \left( \frac{-\alpha N}{8 \ln N} + \|A\| \right) < 0, \]
(37)
for \( 1 \leq i \leq N/4 - 1 \) and \( 3N/4 + 1 \leq i \leq N - 1 \), respectively.

In both the layer regions, we simply obtain \( r_i^- + r_i^+ + r_i^+ = B_i > 0 \).

Finally, in the outer regions, it is straightforward to verify that \( r_i^- < 0 \) for \( N/4 < i \leq 2N/3 \) and \( r_i^+ < 0 \) for \( 2N/3 + 1 < i \leq 3N/4 \). In addition, using \( H \leq 4N^{-1} \) and the second assumption of (34) in (32) and (33) gives us
\[ r_i^- = \frac{2C_{i+1/2}}{h_{i+1}} - \frac{A_{i+1/2}}{h_{i+1}} \leq \frac{2C_{i+1/2}}{h_{i+1}} - \frac{A_{i+1/2}}{h_{i+1}} = \frac{-\alpha N}{4} + \|B\| < 0, \]
\[ r_i^+ = \frac{2C_{i-1/2}}{h_i} + \frac{A_{i-1/2}}{h_i} \leq \frac{2C_{i-1/2}}{h_i} + \frac{A_{i-1/2}}{h_i} = \frac{-\alpha N}{4} + \|B\| < 0, \]
(38)
for \( N/4 \leq i \leq N/2 \) and \( 2N/3 + 1 \leq i \leq 3N/4 \), respectively.

Moreover, for all \( N/4 \leq i \leq 3N/4 \), it is easy to verify that \( r_i^- + r_i^+ + r_i^+ > 0 \).

For all the cases, it is verified that the operator (30) satisfies the conditions in (35). Hence, the matrix is an \( M \)-matrix. Therefore, the solution of (27) exists and the maximum principle easily follows. For more details the reader can refer to [28, 29].

Whenever, the conditions of the maximum principle are satisfied, we can take \( \{D_i\} \) as a barrier function for \( \{U_i\} \).

4. Convergence Analysis of the Proposed Method

In this section, we establish the \( \epsilon \)-uniform error estimate of the hybrid scheme (27). For this, we consider the two cases \( C_r > MN^{-1} \) and \( C_r \leq MN^{-1} \) separately.

For both the cases, analogous to the continuous solution \( u \), we decompose the discrete solution \( U \) into a smooth component \( V \) and a singular component \( W \), such that \( U = V + W \), where \( V \) is the solution of the nonhomogeneous problem given by
\[ L_N^i V_i = f, \quad \text{for } i = 1, \ldots, N - 1, \quad V_0 = v(0), \quad V_N = v(1), \]
(39)
and \( W \) the solution of the homogeneous problem
\[ L_N^i W_i = 0, \quad \text{for } i = 1, \ldots, N - 1, \quad W_0 = w(0), \quad W_N = w(1). \]
(40)

Then the error at each mesh point is
\[ U_i - u(x_i) = (V_i - v(x_i)) + (W_i - w(x_i)), \]
(41)
which implies
\[ |U_i - u(x_i)| \leq |V_i - v(x_i)| + |W_i - w(x_i)|, \]
(42)
and so the error in the smooth and singular components of the solution can be estimated separately.

First, to bound the errors we need to consider the truncation error of associated with the discrete operators in (27). For any smooth function \( y(x) \), the truncation errors \( L_N^i \) applied to \( y \) at \( x_{i+1/2} \) and \( L_N^i \) applied to \( y \) at \( x_0 \) becomes \( T_{1+} = L_{N+}^i (y(x)) - (L_N^i)(x_{i+1/2}) \) and \( T_{1-} = L_{N-}^i (y(x)) - (L_N^i)(x_i) \) respectively, where \( y := y(x_i) \). Thus, the bounds are given in the following Lemma.

**Lemma 4.** Let \( y(x) \) be a smooth function defined on \([-1, 1]\). Then there exists a positive constant \( M \) such that
\[ \left| T_{1+} \right| \leq MC_r \int_{x_{i+1/2}}^{x_i} |y'''(t)| dt + Mh_{i+1} \int_{x_{i+1/2}}^{x_i} |y''(t)| dt, \]
(43)
Proof. By repeated use of the fundamental theorem of calculus, one can obtain the proof as in Lemma 3.3 of [28].

To bound the truncation error of the scheme the comparison principle of Lemma 3 alone is not enough, so we will consider the following lemma which enables us to bound the error.

**Lemma 5.** Let \( Z_i = 2 + x_i \) for \( 0 \leq i \leq N \) be the mesh function for (27). Then there exists a positive constant \( M \) such that
\[ L_N^i Z_i \geq M, \quad \text{for } 1 \leq i \leq N - 1. \]
(44)
Proof. The proof is an easy computation.

Sometimes, the truncation error contains a term of magnitude greater than the desired order of convergence, when this happens we shall combine Lemma 3 with the following results.

Whenever \( C_r \leq MN^{-1} \), we define the auxiliary discrete function on the mesh \( \Omega^N \) by using the mesh elements given in (21) as
where \( \gamma \) is a positive constant. \( \square \)

**Lemma 6.** For any \( \gamma > 0 \) the discrete function \([S_i]\) from (45) is such that

\[
L_N^{N}S_i \geq \begin{cases} 
CN^{-1}, & \text{for } i \leq \frac{3}{4}N, \\
C, & \text{for } i > \frac{3}{4}N.
\end{cases}
\]

**Proof.** The proof is similarly to Lemma 3.3 of [30]. \( \square \)

**Lemma 7.** Let \( \bar{S}_i := (2\alpha/\beta)(h + H)S_{3N/4} + S_i \) be a discrete function, where \([S_i]\) is from (45) with \( \gamma = \alpha \), then

\[
\bar{S}_i \geq e^{-a(1-x)/\varepsilon} \quad \text{for } i = 0, N, \text{and } L_N^{N}\bar{S}_i \geq 0.
\]
Proof. For the proof, one can follow similarly like Lemma 3.4 of [30]. □

Remark 1. Because of the symmetry of the mesh and the adaptive nature of the hybrid scheme, it is easy to derive a similar result like Lemmas 7 and 8 using the mesh function \(S_{N-1}\) related to the layer function \(e^{-a(1+x)/C}\).

Now we have assembled the tools for the proof of the \(\varepsilon\)-uniform convergence.

**Theorem 4.** Assume that the conditions of (34) holds true. Then the hybrid scheme (27) satisfies the following error estimates.

1. **Case 1:** for \(C_\varepsilon > MN^{-1}\), we have

\[
|U_i - u(x_i)| \leq MN^{-2}ln^3 N, \quad \text{for } i = 0, \ldots, N. \tag{51}
\]

2. **Case 2:** for \(C_\varepsilon \leq MN^{-1}\), we have

\[
|L_H^N(V_i - v(x_i))| \leq Mh \int_{x_{i-1}}^{x_i} \left[ C_i \left| v^{(4)}(t) \right| + \left| v''(t) \right| \right] dt
\]

\[
\leq Mh^2 \left( C_\varepsilon + 1 \right) + Mh \int_{x_{i-1}}^{x_i} \left( e^{-a(1+t)/C_\varepsilon} + e^{-a(1-t)/C_\varepsilon} \right) dt
\]

\[
= Mh^2 \left( C_\varepsilon + 1 \right) + MhC_\varepsilon \left( e^{-a(1+x)/C_\varepsilon} + e^{-a(1-x)/C_\varepsilon} \right) \sinh \left( \frac{ah}{C_\varepsilon} \right)
\]

\[
\leq Mh^2 \left( C_\varepsilon + 1 \right) + Mh^2 \left( e^{-a(1+x)/C_\varepsilon} + e^{-a(1-x)/C_\varepsilon} \right), \quad \text{sinh} t \leq Mt \quad \text{for } 0 \leq t \leq 1
\]

\[
\leq Mh^2 \left( C_\varepsilon + 1 \right) + Mh^2 \leq Mh^2. \tag{53}
\]

Proof. Here we estimate the error bounds by considering the two cases separately as follows:

1. **Case 1:** when \(C_\varepsilon > MN^{-1}\), we employed the central difference scheme on the entire domain. First let us compute the nodal error for the smooth part \(V_i\). Using Lemma 4 and the bound of \(e\)-uniform truncation error of the scheme becomes

\[
|L_H^N(W_i - w(x_i))| = |L_C^N(W_i - w(x_i))| = \left| L - L_C^N \right| w(x_i)
\]

\[
\leq Mh \int_{x_{i-1}}^{x_i} \left[ C_i \left| w^{(4)}(t) \right| + \left| w''(t) \right| \right] dt. \tag{56}
\]

Application of Theorem 3 and using (22) on the above inequality and simplifying gives

\[
|L_H^N(W_i - w(x_i))| \leq Mh^2C_\varepsilon^3 \leq MN^{-2}ln^3 N, \quad \text{for } i = 1, \ldots, N-1. \tag{57}
\]

Now, arguing similarly like the smooth part, we obtain

\[
|W_i - w(x_i)| \leq MN^{-3}ln^3 N, \quad \text{for } i = 0, \ldots, N. \tag{58}
\]

Using (55) and (58) in to (42) gives the required result of the first case (51).
Case 2: for $C_e \leq MN^{-1}$, the mesh becomes piecewise-uniform and we employed combinations of midpoint upwind and central difference schemes. First, let us compute the nodal error for the smooth part $V_i$.

\[
\left| L^N_H ((V_i - v(x_i)) \right| = \begin{cases} 
| (Lv)_{x_{i+1/2}} - L^N_M v(x_i) |, & \text{for } i = \frac{N}{4}, \ldots, \frac{N}{2}, \\
| (Lv)_{x_{i-1/2}} - L^N_M v(x_i) |, & \text{for } i = \frac{N}{2} + 1, \ldots, \frac{3N}{4}, \\
| (L - L^N_M) v(x_i) |, & \text{otherwise}, 
\end{cases}
\]

then using the bound for the truncation error of Lemma 4 and the bound for $v(x)$ from Theorem 3, we get

\[
\left| L^N_H ((V_i - v(x_i)) \right| \leq \begin{cases} 
M (C_e + h_i) (h_i + h_{i+1}), & \text{for } i = \frac{N}{4}, \ldots, \frac{N}{2}, \\
M (C_e + h_i) (h_i + h_{i+1}), & \text{for } i = \frac{N}{2} + 1, \ldots, \frac{3N}{4}, \\
M h^2, & \text{otherwise}.
\end{cases}
\]

Since, $C_e \leq MN^{-1}$ and $h_i \leq 4N^{-1}$, then using these in the above inequality gives us

\[
\left| L^N_H ((V_i - v(x_i)) \right| \leq MN^{-2}, \quad \text{for } i = 1, \ldots, N - 1. \quad (61)
\]

Now, arguing similarly like the smooth part of the previous case, we obtain

\[
\left| V_i - v(x_i) \right| \leq MN^{-2}, \quad \text{for } i = 0, \ldots, N. \quad (62)
\]

Next, we analyze the error bounds for the singular component $W$. A different argument is used to bound $|W - w|$ in the outer region and layer regions.

In the subinterval with no boundary layer $\Omega_L$, both $W$ and $w$ are small, and by the triangle inequality, we have

\[
| (W - w) (x_i) | \leq | W (x_i) | + | w (x_i) |,
\]

so it suffices to bound $W(x_i)$ and $w(x_i)$ separately. Theorem 3 for $i = N/4, \ldots, 3N/4$ gives

\[
| w (x_i) | \leq M \left( e^{-\alpha (1+x_i)/C_e} + e^{-\alpha (1-x_i)/C_e} \right) 
\leq M \left( e^{-\alpha (1+x_i)/C_e} + e^{-\alpha (1-x_i)/C_e} \right).
\]

Similarly like the smooth part of Case 1, the truncation error becomes

\[
\left| L^N_H ((V_i - v(x_i)) \right| \leq \begin{cases} 
M (C_e + h_{i+1}) (h_i + h_{i+1}), & \text{for } i = \frac{N}{4}, \ldots, \frac{N}{2}, \\
M (C_e + h_i) (h_i + h_{i+1}), & \text{for } i = \frac{N}{2} + 1, \ldots, \frac{3N}{4}, \\
M h^2, & \text{otherwise}.
\end{cases}
\]

Then using (23) in the above inequality, we get

\[
| w (x_i) | \leq MN^{-2}, \quad \text{for } i = \frac{N}{4}, \ldots, \frac{3N}{4}. \quad (65)
\]

To obtain a similar bound for $W(x_i)$, we set $D_i = M_i \left( N^{-1} S_i ^* + N^{-1} S_{N-i} ^* + S_i + S_{N-i} \right)$ for $i = 0, \ldots, N$, where $\{ S_i \}$ and $\{ S_i \}$ are from Lemmas 7 and 8, respectively. Now for sufficiently large $M_i$, using Theorem 3 in (40) and Lemma 8, we get

\[
| W_0 | = | w (-1) | \leq M \left( e^{-\alpha (1+x_0)/C_e} + e^{-\alpha (1-x_0)/C_e} \right) 
\leq M_i \left( S_0 + S_N \right) \leq D_0,
\]

\[
| W_N | = | w (-1) | \leq M \left( e^{-\alpha (1+x_N)/C_e} + e^{-\alpha (1-x_N)/C_e} \right) 
\leq M_i \left( S_N + S_0 \right) \leq D_N.
\]

Further, for $i = 1, \ldots, N - 1$, the property of the discrete operator from Lemmas 7 and 8 implies
since \( C_\varepsilon \leq MN^{-1} \) implies \( MC_\varepsilon^{-1}N^{-1} \geq 1 \), using this in the above inequality, we get
\[
L_H^N D_i \geq M_1 \left( \varepsilon^{-a(1+x_i)/C_i} + e^{-a(1-x_i)/C_i} \right),
\]
for \( i = 1, \ldots, \frac{N}{4} - 1 \),
\[
M_1 \left( \varepsilon^{-a(1+x_i)/C_i} + e^{-a(1-x_i)/C_i} \right),
\]
for \( i = \frac{N}{4}, \ldots, \frac{3N}{4} \),
\[
M_1 \left( \varepsilon^{-a(1+x_i)/C_i} + MC_\varepsilon^{-1}N^{-1} e^{-a(1+x_i)/C_i} \right),
\]
for \( i = \frac{3N}{4} + 1, \ldots, N - 1 \),

Further, taking \( i = 3N/4 \) in (72), we get
\[
|W_{3N/4} - w_{(3N/4)}| \leq MN^{-2},
\]
and for \( i = N \), the boundary condition in (40) gives
\[
|W_N - w(x_N)| = 0.
\]

From (66)–(69), we can easily observe that \( D_i \) can be a barrier function for \( W_i \) for \( M_1 \) sufficiently large. Therefore, by the discrete maximum principle of Lemma 3 we get
\[
|W_i| \leq D_i, \quad \text{for } i = 0, \ldots, N.
\]

In particular, in the coarser region, (70) and Lemmas 7 and 8 together imply that
\[
|W_i| \leq D_i \leq MN^{-2}, \quad \text{for } i = \frac{N}{4}, \ldots, \frac{3N}{4}.
\]

Therefore, combining (63), (65), and (71) we get
\[
|W_i - w(x_i)| \leq MN^{-2}, \quad \text{for } i = \frac{N}{4}, \ldots, \frac{3N}{4}.
\]

It remains to prove the bound for the singular component in the layer regions \( \Omega_L \) and \( \Omega_R \). First we estimate the bounds for the singular component in \( \Omega_L \). Since we employ central difference scheme in \( \Omega_R \), so as we did for the smooth component, we use the truncation error to bound the error. Thus, from Lemma 4 and Theorem 3 we get
\[
\left| L_H^N (W_i - w(x_i)) \right| \leq Mh \int_{x_{i-1}}^{x_i} \left| C_i |w^*(t)| + |w''(t)| \right| dt
\]
\[
\leq MhC_\varepsilon^{-2} \int_{x_{i-1}}^{x_i} e^{-a(1-t)/C_i} dt
\]
\[
= MhC_\varepsilon^{-2} \left( e^{-a(1-x_{i-1})/C_i} - e^{-a(1-x_i)/C_i} \right)
\]
\[
= MhC_\varepsilon^{-2} e^{-a(1-x_i)} C_i \sinh \left( \frac{\alpha h}{C_i} \right).
\]

Clearly, the first assumption of (34) implies \( \alpha h/C_i \leq 1 \) and since \( \sinh t \leq Mt \) for \( 0 \leq t \leq 1 \), so \( \sinh(\alpha h/C_i) \leq Mh/C_i \). Thus, for \( i = 3N/4 + 1, \ldots, N - 1 \) the above inequality is reduced to
\[
\left| L_H^N (W_i - w(x_i)) \right| \leq M \left( \frac{h}{C_i} \right)^2 C_i^{-1} e^{-a(1-x_i)/C_i}.
\]

Example 1. Consider the following homogeneous SPDDE with a turning point at \( x = 0 \):
Given the differential equation:

\[-\varepsilon y''(x) + xy'(x) + 3y(x) - y(x - \delta) - 0.5y(x + \eta) = 0, \quad x \in (-1, 1)\]

\[y(x) = 1, \quad -1 - \delta \leq x \leq -1, \quad y'(x) = 1, \quad 1 \leq x \leq 1 + \eta.\]

(80)

In Tables 1 and 2, the maximum pointwise errors and the rate of convergences for Example 1 are displayed for different values of \(N\) and \(\varepsilon\), respectively. The plots of the approximate solutions for \(N = 64\), \(\varepsilon = 0.1\), with different values of \(\delta\) and \(\eta\) are shown in Figures 1 and 2.
The values of the app. soln. $Y$

$\begin{align*}
\delta = 0.5 \ast \epsilon, & \eta = 0.0 \ast \epsilon \\
\delta = 0.5 \ast \epsilon, & \eta = 1.0 \ast \epsilon \\
\delta = 0.0 \ast \epsilon, & \eta = 1.5 \ast \epsilon
\end{align*}$

Consider the following nonhomogeneous example 2.

Table 3: Result of Example 2 (maximum pointwise error) for $\delta = \eta = 0.5 \ast \epsilon$.

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<th>$\varepsilon / N \rightarrow$</th>
<th>32</th>
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<th>128</th>
<th>256</th>
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Table 4: Result of Example 2 (rate of convergence) for $\delta = \eta = 0.5 \ast \epsilon$.

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</tr>
</tbody>
</table>

Example 2. Consider the following nonhomogeneous singularly perturbed differential-difference turning point problem with a turning point at $x = 1/2$:

\[-\varepsilon y''(x) - 2(1 - 2x)y'(x) + 6y(x) - y(x - \delta) \]
\[-y(x + \eta) = 4(1 - 4x), x \in (0, 1) \quad (81) \]

$y(x) = 1, -\delta \leq x \leq 0, y(x) = 1, 1 \leq x \leq 1 + \eta.$

In Tables 3 and 4 the maximum pointwise errors and the rate of convergences for Example 2 are displayed for different values of $N$ and $\varepsilon$, respectively. The plots of the approximate solutions for $N = 256, \varepsilon = 0.1$, with different values of $\delta$ and $\eta$ are shown in Figures 3 and 4.

6. Discussion

Singularly perturbed differential-difference turning point problems exhibiting twin boundary layers which contain small mixed shifts on the reaction coefficients are considered. For the numerical treatment of these problems, first we employ a second-order Taylor’s series approximation on the terms containing shift parameters and obtain a modified singularly perturbed problem which approximates the original problem. And then an efficient hybrid difference scheme on an appropriate piecewise-uniform Shishkin mesh is developed for the modified problem. The proposed method is analyzed for stability and convergence, and it has been shown that the method is $\varepsilon$-uniformly convergent with...
an almost second-order rate of convergence. Further, two numerical experiments are examined to support the theoretical results and to illustrate the effect of the small shifts on the layer behavior of the solutions. Tables 1–4 present the computed maximum pointwise error and the rate of convergence for the considered examples. These results demonstrate that the method is robust, i.e., converges for all $\varepsilon$. We also observed that the maximum pointwise error and the rate of convergence stabilizes as $\varepsilon \to 0$ for each appropriate $N$. Further, the numerical results clearly support the theoretical error bounds and order of convergence.

In addition, to demonstrate the effect of the small shifts on the behavior of the solution graphs of the considered problems are plotted in Figures 1–4 for different values of $\delta$ and $\eta$. It is observed that the boundary layers are maintained but layer gets shifted as delay/advance argument changes. Shifts in the layer depend upon the type of shift as well as on the value of the coefficients of the term containing delay/advance.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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**References**


