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Research Article

Initial Bounds for Certain Classes of Bi-Univalent Functions Defined by Horadam Polynomials

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The main purpose of this article is to make use of the Horadam polynomials $h_n(x)$ and the generating function $\Pi(x,z)$, in order to introduce three new subclasses of the bi-univalent function class σ. For functions belonging to the defined classes, we then derive coefficient inequalities and the Fekete–Szegö inequalities. Some interesting observations of the results presented here are also discussed. We also provide relevant connections of our results with those considered in earlier investigations.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} \tag{1}$$

be the set of positive integers. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in Δ.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \Delta) \tag{3}$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right), \tag{4}$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right)w^4 + \cdots. \tag{5}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in Δ if both the function f and its inverse f^{-1} are univalent in Δ. Let σ denote the class of bi-univalent functions in Δ given by (2).

In 2010, Srivastava et al. [1] revived the study of bi-univalent functions by their pioneering work on the study of coefficient problems. Various subclasses of the bi-univalent function class σ were introduced and nonsharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor–Maclaurin series expansion (2) were found in the recent investigations (see, for example, [2–23]) and including the references therein. The afore-cited all these papers on the subject were actually motivated by the work of Srivastava et al. [1]. However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \ldots)$ for functions $f \in \sigma$ is still open problem.

For analytic functions f and g in Δ, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0) = 0, \quad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in \Delta). \tag{6}$$

This subordination will be denoted here by

$$f \prec g \quad (z \in \Delta) \tag{7}$$

or, conventionally, by

$$f(z) \prec g(z) \quad (z \in \Delta). \tag{8}$$

In particular, when g is univalent in Δ,

$$f \prec g \quad (z \in \Delta) \iff f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta). \tag{9}$$

The Horadam polynomials $h_n(x, a, b; p, q)$, or briefly $h_n(x)$ are given by the following recurrence relation (see [22, 23]):

$$\begin{aligned} &h_1(x) = a, \quad h_2(x) = bx \quad \text{and} \\ &h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x) \quad (n \geq 3) \end{aligned} \tag{10}$$

for some real constants a, b, p, and q.

The generating function of the Horadam polynomials $h_n(x)$ (see [23]) is given by

$$\Pi(x, z) := \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a + (b - ap)xz}{1 - pxz - qz^2}. \tag{11}$$

Here, and in what follows, the argument $x \in \mathbb{R}$ is independent of the argument $z \in \mathbb{C}$; that is, $x \neq \mathscr{R}(z)$.

Note that for particular values of a, b, p, and q, the Horadam polynomial $h_n(x)$ leads to various polynomials, among those, we list a few cases here (see, [22, 23] for more details):

(1) For $a = b = p = q = 1$, we have the Fibonacci polynomials $F_n(x)$.
(2) For $a = 2$ and $b = p = q = 1$, we obtain the Lucas polynomials $L_n(x)$.
(3) For $a = q = 1$ and $b = p = 2$, we get the Pell polynomials $P_n(x)$.
(4) For $a = b = p = 2$ and $q = 1$, we attain the Pell-Lucas polynomials $Q_n(x)$.
(5) For $a = b = 1, p = 2$ and $q = -1$, we have the Chebyshev polynomials $T_n(x)$ of the first kind.
(6) For $a = 1, b = p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $U_n(x)$ of the second kind.

Recently, in literature, the coefficient estimates are found for functions in the class of univalent and bi-univalent functions associated with certain polynomials such as the Faber polynomial [8], the Chebyshev polynomials [6], and the Horadam polynomial [15]. Motivated in these lines, estimates on initial coefficients of the Taylor–Maclaurin series expansion (2) and Fekete–Szegö inequalities for certain classes of bi-univalent functions defined by means of Horadam polynomials are obtained. The classes introduced in this paper are motivated by the corresponding classes investigated in [2, 10, 14, 15].

2. Coefficient Estimates and Fekete–Szegö Inequalities

A function $f \in \mathcal{A}$ of the form (2) belongs to the class $\mathcal{S}_\sigma^*(\lambda, x)$ for $\lambda \geq 0$ and $z, w \in \Delta$, if the following conditions are satisfied:

$$\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} \prec \Pi(x, z) + 1 - a \tag{12}$$

and for $g(w) = f^{-1}(w)$

$$\frac{wg'(w)}{g(w)} + \lambda \frac{w^2 g''(w)}{g(w)} \prec \Pi(x, w) + 1 - a, \tag{13}$$

where the real constant a is as in (10).

Note that $S_\sigma^*(x) \equiv \mathcal{S}_\sigma^*(0, x)$ was introduced and studied by Srivastava et al. [15].

Remark 1. When $a = 1$, $b = p = 2$, $q = -1$ and $x = t$, the generating function in (11) reduces to that of the Chebyshev polynomial $U_n(t)$ of the second kind, which is given explicitly by

$$\begin{aligned} U_n(t) &= (n+1)\,{}_2F_1\left(-n, n+2; \frac{3}{2}; \frac{1-t}{2}\right) \\ &= \frac{\sin(n+1)\varphi}{\sin\varphi}, \quad (t = \cos\varphi) \end{aligned} \tag{14}$$

in terms of the hypergeometric function ${}_2F_1$.

In view of Remark 1, the bi-univalent function class $\mathcal{S}_\sigma^*(x)$ reduces to $\mathcal{S}_\sigma^*(t)$ and this class was studied earlier in [3, 12]. For functions in the class $\mathcal{S}_\sigma^*(\lambda, x)$, the following coefficient estimates and Fekete–Szegö inequality are obtained.

Theorem 1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_\sigma^*(\lambda, x)$. Then*

$$\begin{aligned} &|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|\left[(1+4\lambda)b - p(1+2\lambda)^2\right]bx^2 - qa(1+2\lambda)^2\right|}}, \text{ and} \\ &|a_3| \leq \frac{|bx|}{2+6\lambda} + \frac{b^2x^2}{(1+2\lambda)^2} \end{aligned} \tag{15}$$

and for $\nu \in \mathbb{R}$

$$\begin{aligned} &\text{if } |\nu - 1| \leq \frac{\left|\left[(1+4\lambda)b - p(1+2\lambda)^2\right]bx^2 - qa(1+2\lambda)^2\right|}{2b^2x^2(1+3\lambda)}, \\ &\text{if } |\nu - 1| \geq \frac{\left|\left[(1+4\lambda)b - p(1+2\lambda)^2\right]bx^2 - qa(1+2\lambda)^2\right|}{2b^2x^2(1+3\lambda)}. \end{aligned} \tag{16}$$

$$|a_3 - \nu a_2^2| \leq \begin{cases} \dfrac{|bx|}{2+6\lambda} \\ \dfrac{|bx|^3|\nu - 1|}{\left|\left[(1+4\lambda)b - p(1+2\lambda)^2\right]bx^2 - qa(1+2\lambda)^2\right|} \end{cases}$$

Proof. Let $f \in \mathcal{S}_\sigma^*(\lambda, x)$ be given by the Taylor–Maclaurin expansion (2). Then, there are analytic functions u and v such that

$$u(0) = 0; \quad v(0) = 0, \quad |u(z)| < 1 \text{ and } |v(z)| < 1 \quad (\forall z, w \in \Delta), \tag{17}$$

and we can write

$$\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} = \Pi(x, u(z)) + 1 - a \tag{18}$$

and

$$\frac{wg'(w)}{g(w)} + \lambda\frac{w^2g''(w)}{g(w)} = \Pi(x, v(w)) + 1 - a. \tag{19}$$

Equivalently,

$$\frac{zf'(z)}{f(z)} + \lambda\frac{z^2f''(z)}{f(z)} = 1 + h_1(x) - a + h_2(x)u(z) + h_3(x)[u(z)]^2 + \cdots \tag{20}$$

and

$$\frac{wg'(w)}{g(w)} + \lambda\frac{w^2g''(w)}{g(w)} = 1 + h_1(x) - a + h_2(x)v(w) + h_3(x)[v(w)]^2 + \cdots. \tag{21}$$

From (20) and (21) and in view of (11), we obtain

$$\frac{zf'(z)}{f(z)} + \lambda\frac{z^2f''(z)}{f(z)} = 1 + h_2(x)u_1z + \left[h_2(x)u_2 + h_3(x)u_1^2\right]z^2 + \cdots \tag{22}$$

and

$$\frac{wg'(w)}{g(w)} + \lambda\frac{w^2g''(w)}{g(w)} = 1 + h_2(x)v_1w + \left[h_2(x)v_2 + h_3(x)v_1^2\right]w^2 + \cdots. \tag{23}$$

If

$$u(z) = \sum_{n=1}^{\infty} u_nz^n \quad \text{and} \quad v(z) = \sum_{n=1}^{\infty} v_nw^n, \tag{24}$$

then it is well known that

$$|u_n| \le 1 \quad \text{and} \quad |v_n| \le 1 \quad (n \in \mathbb{N}). \tag{25}$$

Thus upon comparing the corresponding coefficients in (22) and (23), we have

$$(1 + 2\lambda)a_2 = h_2(x)u_1, \tag{26}$$

$$2(1 + 3\lambda)a_3 - (1 + 2\lambda)a_2^2 = h_2(x)u_2 + h_3(x)u_1^2, \tag{27}$$

$$-(1 + 2\lambda)a_2 = h_2(x)v_1 \tag{28}$$

and

$$(3 + 10\lambda)a_2^2 - 2(1 + 3\lambda)a_3 = h_2(x)v_2 + h_3(x)v_1^2. \tag{29}$$

From (26) and (28), we can easily see that

$$u_1 = -v_1, \quad \text{provided} \quad h_2(x) = bx \neq 0 \tag{30}$$

and

$$\begin{aligned} 2(1 + 2\lambda)^2a_2^2 &= \left[h_2(x)\right]^2\left(u_1^2 + v_1^2\right) \\ a_2^2 &= \frac{\left[h_2(x)\right]^2\left(u_1^2 + v_1^2\right)}{2(1 + 2\lambda)^2}. \end{aligned} \tag{31}$$

If we add (27) to (29), we get

$$2(1 + 4\lambda)a_2^2 = h_2(x)\left(u_2 + v_2\right) + h_3(x)\left(u_1^2 + v_1^2\right). \tag{32}$$

By substituting (31) in (32), we obtain

$$a_2^2 = \frac{\left[h_2(x)\right]^3\left(u_2 + v_2\right)}{2(1 + 4\lambda)\left[h_2(x)\right]^2 - 2h_3(x)(1 + 2\lambda)^2} \tag{33}$$

and by taking $h_2(x) = bx$ and $h_3(x) = bpx^2 + qa$ in (33), it further yields

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|\left[(1 + 4\lambda)b - p(1 + 2\lambda)^2\right]bx^2 - qa(1 + 2\lambda)^2\right|}}. \tag{34}$$

By subtracting (29) from (27) and in view of (30), we obtain

$$\begin{aligned} 4(1 + 3\lambda)a_3 - 4(1 + 3\lambda)a_2^2 &= h_2(x)\left(u_2 - v_2\right) + h_3(x)\left(u_1^2 - v_1^2\right) \\ a_3 &= \frac{h_2(x)\left(u_2 - v_2\right)}{4(1 + 3\lambda)} + a_2^2. \end{aligned} \tag{35}$$

Then in view of (31), (35) becomes

$$a_3 = \frac{h_2(x)\left(u_2 - v_2\right)}{4(1 + 3\lambda)} + \frac{\left[h_2(x)\right]^2\left(u_1^2 + v_1^2\right)}{2(1 + 2\lambda)^2}. \tag{36}$$

Applying (10), we deduce that

$$|a_3| \le \frac{|bx|}{2 + 6\lambda} + \frac{b^2x^2}{(1 + 2\lambda)^2}. \tag{37}$$

From (35), for $v \in \mathbb{R}$, we write

$$a_3 - va_2^2 = \frac{h_2(x)\left(u_2 - v_2\right)}{4(1 + 3\lambda)} + (1 - v)a_2^2. \tag{38}$$

By substituting (33) in (38), we have

$$\begin{aligned} a_3 - va_2^2 &= \frac{h_2(x)\left(u_2 - v_2\right)}{4(1 + 3\lambda)} + \left(\frac{(1 - v)\left[h_2(x)\right]^3\left(u_2 + v_2\right)}{2\left[(1 + 4\lambda)\left[h_2(x)\right]^2 - h_3(x)(1 + 2\lambda)^2\right]}\right) \\ &= h_2(x)\left\{\left(\Omega(v, x) + \frac{1}{4(1 + 3\lambda)}\right)u_2 + \left(\Omega(v, x) - \frac{1}{4(1 + 3\lambda)}\right)v_2\right\}, \end{aligned} \tag{39}$$

where

$$\Omega(v, x) = \frac{(1 - v)\left[h_2(x)\right]^2}{2(1 + 4\lambda)\left[h_2(x)\right]^2 - 2h_3(x)(1 + 2\lambda)^2}. \tag{40}$$

Hence, we conclude that

$$\left|a_3 - va_2^2\right| \le \begin{cases} \frac{|h_2(x)|}{2+6\lambda} & 0 \le |\Omega(v, x)| \le \frac{1}{4(1+3\lambda)} \\ 2\left|h_2(x)\right||\Omega(v, x)| & |\Omega(v, x)| \ge \frac{1}{4(1+3\lambda)} \end{cases} \tag{41}$$

and in view of (10), it evidently completes the proof of Theorem 1. □

For $\lambda = 0$, Theorem 1 readily yields the following coefficient estimates for $\mathcal{S}_\sigma^*(x)$.

Corollary 1. *Let $f(z) = z + \sum_{n=2}^{\infty} a_nz^n$ be in the class $\mathcal{S}_\sigma^*(x)$. Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|[b - p]bx^2 - qa\right|}}, \quad \text{and} \quad |a_3| \le \frac{|bx|}{2} + b^2x^2 \tag{42}$$

and for $v \in \mathbb{R}$

$$\left|a_3 - va_2^2\right| \le \begin{cases} \frac{|bx|}{2} & \text{if} \quad |v - 1| \le \frac{\left|[b - p]bx^2 - qa\right|}{2b^2x^2}, \\ \frac{|bx|^3|v - 1|}{\left|[b - p]bx^2 - qa\right|} & \text{if} \quad |v - 1| \ge \frac{\left|[b - p]bx^2 - qa\right|}{2b^2x^2}. \end{cases} \tag{43}$$

In view of Remark 1, Theorem 1 can be shown to yield the following result.

Corollary 2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{S}_\sigma^*(\lambda, t)$. Then*

$$|a_2| \le \frac{|2t|\sqrt{|2t|}}{\sqrt{\left|(1+2\lambda)^2 - 16\lambda^2 t^2\right|}}, \quad \text{and} \quad |a_3| \le \frac{|t|}{1+3\lambda} + \frac{4t^2}{(1+2\lambda)^2} \tag{44}$$

and for $\nu \in \mathbb{R}$

$$|a_3 - \nu a_2^2| \le \begin{cases} \dfrac{|t|}{1+3\lambda} & \text{if } |\nu - 1| \le \dfrac{\left|(1+2\lambda)^2 - 16\lambda^2 t^2\right|}{8t^2(1+3\lambda)}, \\ \dfrac{8|t|^3|\nu - 1|}{\left|(1+2\lambda)^2 - 16\lambda^2 t^2\right|} & \text{if } |\nu - 1| \ge \dfrac{\left|(1+2\lambda)^2 - 16\lambda^2 t^2\right|}{8t^2(1+3\lambda)}. \end{cases} \tag{45}$$

Remark 2. Results obtained in Corollary 1 coincide with results obtained in [15]. For $\lambda = 0$. Corollary 2 reduces to the results discussed in [3, 12].

Next, a function $f \in \mathcal{A}$ of the form (2) belongs to the class $\mathcal{M}_\sigma(\alpha, x)$ for $0 \leqq \alpha \leqq 1$ and $z, w \in \Delta$, if the following conditions are satisfied:

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \Pi(x, z) + 1 - a \tag{46}$$

and for $g(w) = f^{-1}(w)$

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \Pi(x, w) + 1 - a, \tag{47}$$

where the real constant a is as in (10).

Note that the class $\mathcal{M}_\sigma(\alpha, x)$ reduces to the classes $S_\sigma^*(x)$ and $K_\sigma(x)$ as $\mathcal{M}_\sigma(0, x) \equiv S_\sigma^*(x)$ and $\mathcal{M}_\sigma(1, x) \equiv K_\sigma(x)$. In view of Remark 1, the bi-univalent function classes $\mathcal{M}_\sigma^*(\alpha, x)$ would become the class $\mathcal{M}_\sigma^*(\alpha, t)$ introduced and studied by Altınkaya and Yalçin [4]. For functions in the class $\mathcal{M}_\sigma(\alpha, x)$, the following coefficient estimates and Fekete–Szegö inequality are obtained.

Theorem 2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{M}_\sigma(\alpha, x)$. Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{\left|\left[(1+\alpha)b - p(1+\alpha)^2\right]bx^2 - qa(1+\alpha)^2\right|}}, \text{ and}$$
$$|a_3| \le \frac{|bx|}{2+4\alpha} + \frac{b^2x^2}{(1+\alpha)^2} \tag{48}$$

and for $\nu \in \mathbb{R}$

$$|a_3 - \nu a_2^2| \le \begin{cases} \dfrac{|bx|}{2+4\alpha} \\ \dfrac{|bx|^3|\nu - 1|}{\left|\left[(1+\alpha)b - p(1+\alpha)^2\right]bx^2 - qa(1+\alpha)^2\right|} \end{cases}$$

$$\text{if } |\nu - 1| \le \frac{\left|\left[(1+\alpha)b - p(1+\alpha)^2\right]bx^2 - qa(1+\alpha)^2\right|}{b^2x^2(2+4\alpha)},$$
$$\text{if } |\nu - 1| \ge \frac{\left|\left[(1+\alpha)b - p(1+\alpha)^2\right]bx^2 - qa(1+\alpha)^2\right|}{b^2x^2(2+4\alpha)}. \tag{49}$$

Proof. Let $f \in \mathcal{M}_\sigma(\alpha, x)$ be given by the Taylor–Maclaurin expansion (2). Then, there are analytic functions u and v such that

$$u(0) = 0; \quad v(0) = 0, \quad |u(z)| < 1 \quad \text{and} \quad |v(z)| < 1 \quad (\forall z, w \in \Delta) \tag{50}$$

and

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \Pi(x, u(z)) + 1 - a \tag{51}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) = \Pi(x, v(w)) + 1 - a. \tag{52}$$

Equivalently,

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + h_1(x) - a + h_2(x)u(z) + h_3(x)[u(z)]^2 + \cdots \tag{53}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 + h_1(x) - a + h_2(x)v(w) + h_3(x)[v(w)]^2 + \cdots. \tag{54}$$

From (53), (54) and in view of (11), we obtain

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + h_2(x)u_1 z + \left[h_2(x)u_2 + h_3(x)u_1^2\right]z^2 + \cdots \tag{55}$$

and

$$(1-\alpha)\frac{wg'(w)}{g(w)} + \alpha\left(1 + \frac{wg''(w)}{g'(w)}\right) = 1 + h_2(x)v_1 w + \left[h_2(x)v_2 + h_3(x)v_1^2\right]w^2 + \cdots. \tag{56}$$

If

$$u(z) = \sum_{n=1}^{\infty} u_n z^n \quad \text{and} \quad v(z) = \sum_{n=1}^{\infty} v_n w^n, \tag{57}$$

then it is well known that

$$|u_n| \le 1 \quad \text{and} \quad |v_n| \le 1 \quad (n \in \mathbb{N}). \tag{58}$$

Thus upon comparing the corresponding coefficients in (55) and (56), we have

$$(1+\alpha)\, a_2 = h_2(x)u_1 \tag{59}$$

$$2(1+2\alpha)\,a_3-(1+3\alpha)a_2^2=h_2(x)u_2+h_3(x)u_1^2 \tag{60}$$

$$-(1+\alpha)a_2=h_2(x)v_1 \tag{61}$$

and

$$(3+5\alpha)a_2^2-2(1+2\alpha)a_3=h_2(x)v_2+h_3(x)v_1^2. \tag{62}$$

From (59) and (61), we can easily see that

$$u_1=-v_1,\qquad \text{provided}\qquad h_2(x)=bx\neq 0 \tag{63}$$

and

$$\begin{aligned}2(1+\alpha)^2a_2^2&=\left[h_2(x)\right]^2\left(u_1^2+v_1^2\right)\\ a_2^2&=\frac{\left[h_2(x)\right]^2\left(u_1^2+v_1^2\right)}{2(1+\alpha)^2}.\end{aligned} \tag{64}$$

If we add (60) to (62), we get

$$2(1+\alpha)a_2^2=h_2(x)\left(u_2+v_2\right)+h_3(x)\left(u_1^2+v_1^2\right). \tag{65}$$

By substituting (64) in (65), we obtain

$$a_2^2=\frac{\left[h_2(x)\right]^3\left(u_2+v_2\right)}{2(1+\alpha)\left[h_2(x)\right]^2-2h_3(x)(1+\alpha)^2} \tag{66}$$

and by taking $h_2(x)=bx$ and $h_3(x)=bpx^2+qa$ in (66), it further yields

$$\left|a_2\right|\leq\frac{|bx|\sqrt{|bx|}}{\sqrt{\left|\left[(1+\alpha)b-p(1+\alpha)^2\right]bx^2-qa(1+\alpha)^2\right|}}. \tag{67}$$

By subtracting (62) from (60) and in view of (63), we obtain

$$\begin{aligned}4(1+2\alpha)a_3-4(1+2\alpha)a_2^2&=h_2(x)\left(u_2-v_2\right)+h_3(x)\left(u_1^2-v_1^2\right)\\ a_3&=\frac{h_2(x)\left(u_2-v_2\right)}{4(1+2\alpha)}+a_2^2.\end{aligned} \tag{68}$$

Then in view of (64), (68) becomes

$$a_3=\frac{h_2(x)\left(u_2-v_2\right)}{4(1+2\alpha)}+\frac{\left[h_2(x)\right]^2\left(u_1^2+v_1^2\right)}{2(1+\alpha)^2}. \tag{69}$$

Applying (10), we deduce that

$$\left|a_3\right|\leq\frac{|bx|}{2+4\alpha}+\frac{b^2x^2}{(1+\alpha)^2}. \tag{70}$$

From (68), for $\nu\in\mathbb{R}$, we write

$$a_3-\nu a_2^2=\frac{h_2(x)\left(u_2-v_2\right)}{4(1+2\alpha)}+(1-\nu)a_2^2. \tag{71}$$

By substituting (66) in (71), we have

$$\begin{aligned}a_3-\nu a_2^2&=\frac{h_2(x)\left(u_2-v_2\right)}{4(1+2\alpha)}+\frac{(1-\nu)\left[h_2(x)\right]^3\left(u_2+v_2\right)}{2(1+\alpha)\left[h_2(x)\right]^2-2h_3(x)(1+\alpha)^2}\\ &=h_2(x)\left\{\left(\Omega(\nu,x)+\frac{1}{4(1+2\alpha)}\right)u_2+\left(\Omega(\nu,x)-\frac{1}{4(1+2\alpha)}\right)v_2\right\},\end{aligned} \tag{72}$$

where

$$\Omega(\nu,x)=\frac{(1-\nu)\left[h_2(x)\right]^2}{2(1+\alpha)\left[h_2(x)\right]^2-2h_3(x)(1+\alpha)^2}. \tag{73}$$

Hence, we conclude that

$$\left|a_3-\nu a_2^2\right|\leq\begin{cases}\dfrac{\left|h_2(x)\right|}{2+4\alpha} & 0\leq|\Omega(\nu,x)|\leq\dfrac{1}{4(1+2\alpha)}\\ 2\left|h_2(x)\right||\Omega(\nu,x)| & |\Omega(\nu,x)|\geq\dfrac{1}{4(1+2\alpha)},\end{cases} \tag{74}$$

which in view of (10), evidently completes the proof of Theorem 2. □

For $\alpha=1$, Theorem 2 readily yields the following coefficient estimates for $\mathcal{K}_\sigma(x)$.

Corollary 3. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be in the class $\mathcal{K}_\sigma(x)$. Then*

$$\left|a_2\right|\leq\frac{|bx|\sqrt{|bx|}}{\sqrt{\left|[2b-4p]bx^2-4qa\right|}},\qquad \textit{and}\qquad \left|a_3\right|\leq\frac{|bx|}{6}+\frac{b^2x^2}{4} \tag{75}$$

and for $\nu\in\mathbb{R}$

$$\left|a_3-\nu a_2^2\right|\leq\begin{cases}\frac{|bx|}{6} & \textit{if } |\nu-1|\leq\frac{\left|[b-2p]bx^2-4qa\right|}{3b^2x^2}\\ \frac{|bx|^3|\nu-1|}{\left|[2b-4p]bx^2-4qa\right|} & \textit{if } |\nu-1|\geq\frac{\left|[b-2p]bx^2-4qa\right|}{3b^2x^2}.\end{cases} \tag{76}$$

In view of Remark 1, Theorem 2 yields the following result.

Corollary 4. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be in the class $\mathcal{M}_\sigma(\alpha,t)$. Then*

$$\left|a_2\right|\leq\frac{2|t|\sqrt{2|t|}}{\sqrt{\left|(1+\alpha)^2-4\alpha(1+\alpha)t^2\right|}},\qquad \textit{and}\qquad \left|a_3\right|\leq\frac{|t|}{1+2\alpha}+\frac{4t^2}{(1+\alpha)^2} \tag{77}$$

and for $\nu\in\mathbb{R}$

$$\left|a_3-\nu a_2^2\right|\leq\begin{cases}\frac{|t|}{1+2\alpha} & \textit{if } |\nu-1|\leq\frac{\left|(1+\alpha)^2-4\alpha(1+\alpha)t^2\right|}{8t^2(1+2\alpha)}\\ \frac{8|t|^3|\nu-1|}{\left|(1+\alpha)^2-4\alpha(1+\alpha)t^2\right|} & \textit{if } |\nu-1|\geq\frac{\left|(1+\alpha)^2-4\alpha(1+\alpha)t^2\right|}{8t^2(1+2\alpha)}.\end{cases} \tag{78}$$

In view of Remark 1, Corollary 3 yields the following result.

Corollary 5. *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be in the class $\mathcal{K}_\sigma(t)$. Then*

$$\left|a_2\right|\leq\frac{|t|\sqrt{2|t|}}{\sqrt{\left|1-2t^2\right|}},\qquad \textit{and}\qquad \left|a_3\right|\leq\frac{|t|}{3}+t^2 \tag{79}$$

and for $\nu\in\mathbb{R}$

$$\left|a_3-\nu a_2^2\right|\leq\begin{cases}\frac{|t|}{3} & \textit{if } |\nu-1|\leq\frac{\left|1-2t^2\right|}{6t^2}\\ \frac{2|t|^3|\nu-1|}{\left|1-2t^2\right|} & \textit{if } |\nu-1|\geq\frac{\left|1-2t^2\right|}{6t^2}.\end{cases} \tag{80}$$

Remark 3. The results obtained in Corollary 4 and 5 coincide with results of Altınkaya and Yalçin [4].

Next, a function $f \in \sigma$ of the form (2) belongs to the class $\mathcal{L}_\sigma(\mu, x)$ for $0 \leq \mu \leq 1$, and $z,\ w \in \Delta$, if the following conditions are satisfied:

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\mu} \prec \Pi(x,z)+1-a \quad (81)$$

and for $g(w) = f^{-1}(w)$

$$\left(\frac{wg'(w)}{g(w)}\right)^{\mu}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\mu} \prec \Pi(x,w)+1-a, \quad (82)$$

where the real constants a is as in (10).

This class also reduces to $\mathcal{S}^*_\sigma(x)$ and $\mathcal{K}_\sigma(x)$. In view of Remark 1, the bi-univalent function class $\mathcal{L}^*_\sigma(\mu, x)$ would become the class $\mathcal{L}^*_\sigma(\mu, t)$. For functions in the class $\mathcal{L}_\sigma(\mu, x)$, the following coefficient estimates are obtained.

Theorem 3. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{L}_\sigma(\mu, x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{2|bx|}}{\sqrt{\left|\left[\left(\mu^2-3\mu+4\right)b-2p(2-\mu)^2\right]bx^2-2qa(2-\mu)^2\right|}} \text{ and}$$
$$|a_3| \leq \frac{|bx|}{6-4\mu}+\frac{b^2x^2}{(2-\mu)^2} \quad (83)$$

and for $\nu \in \mathbb{R}$

$$\text{if } |\nu-1| \leq \frac{\left|\left[\left(\mu^2-3\mu+4\right)b-2p(2-\mu)^2\right]bx^2-2qa(2-\mu)^2\right|}{4b^2x^2(3-2\mu)}$$
$$\text{if } |\nu-1| \geq \frac{\left|\left[\left(\mu^2-3\mu+4\right)b-2p(2-\mu)^2\right]bx^2-2qa(2-\mu)^2\right|}{4b^2x^2(3-2\mu)}. \quad (84)$$

$$\left|a_3-\nu a_2^2\right| \leq \begin{cases} \dfrac{|bx|}{6-4\mu} \\ \dfrac{2|bx|^3|\nu-1|}{\left|\left[\left(\mu^2-3\mu+4\right)b-2p(2-\mu)^2\right]bx^2-2qa(2-\mu)^2\right|} \end{cases}$$

Proof. Let $f \in \mathcal{L}_\sigma(\mu, x)$ be given by the Taylor–Maclaurin expansion (2). Then, there are analytic functions u and v such that

$$u(0)=0;\quad v(0)=0,\quad |u(z)|<1 \text{ and } |v(z)|<1 \quad (\forall\ z,\ w \in \Delta), \quad (85)$$

and

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\mu} = \Pi(x,u(z))+1-a \quad (86)$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\mu}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\mu} = \Pi(x,v(w))+1-a. \quad (87)$$

Equivalently,

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\mu} = 1+h_1(x)-a+h_2(x)u(z)+h_3(x)[u(z)]^2+\cdots \quad (88)$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\mu}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\mu} = 1+h_1(x)-a+h_2(x)v(w)+h_3(x)[v(w)]^2+\cdots. \quad (89)$$

From (88) and (89) and in view of (11), we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1+\frac{zf''(z)}{f'(z)}\right)^{1-\mu} = 1+h_2(x)u_1z+\left[h_2(x)u_2+h_3(x)u_1^2\right]z^2+\cdots \quad (90)$$

and

$$\left(\frac{wg'(w)}{g(w)}\right)^{\mu}\left(1+\frac{wg''(w)}{g'(w)}\right)^{1-\mu} = 1+h_2(x)v_1w+\left[h_2(x)v_2+h_3(x)v_1^2\right]w^2+\cdots. \quad (91)$$

If

$$u(z)=\sum_{n=1}^{\infty}u_nz^n \quad \text{and} \quad v(z)=\sum_{n=1}^{\infty}v_nw^n, \quad (92)$$

then it is well known that

$$|u_n| \leq 1 \quad \text{and} \quad |v_n| \leq 1 \quad (n \in \mathbb{N}). \quad (93)$$

Thus upon comparing the corresponding coefficients in (90) and (91), we have

$$(2-\mu)a_2 = h_2(x)u_1 \quad (94)$$

$$2(3-2\mu)a_3+\left(\mu^2+5\mu-8\right)\frac{a_2^2}{2} = h_2(x)u_2+h_3(x)u_1^2 \quad (95)$$

$$-(2-\mu)a_2 = h_2(x)v_1 \quad (96)$$

and

$$\left(\mu^2-11\mu+16\right)\frac{a_2^2}{2}-2(3-2\mu)a_3 = h_2(x)v_2+h_3(x)v_1^2. \quad (97)$$

From (94) and (96), we can easily see that

$$u_1=-v_1, \quad \text{provided} \quad h_2(x)=bx \neq 0 \quad (98)$$

and

$$2(2-\mu)^2a_2^2 = [h_2(x)]^2\left(u_1^2+v_1^2\right)$$
$$a_2^2 = \frac{[h_2(x)]^2\left(u_1^2+v_1^2\right)}{2(2-\mu)^2}. \quad (99)$$

If we add (95) to (97), we get

$$\left(\mu^2 - 3\mu + 4\right)a_2^2 = h_2(x)(u_2 + v_2) + h_3(x)\left(u_1^2 + v_1^2\right). \quad (100)$$

By substituting (99) in (100), we obtain

$$a_2^2 = \frac{[h_2(x)]^3(u_2 + v_2)}{\left(\mu^2 - 3\mu + 4\right)[h_2(x)]^2 - 2h_3(x)(2-\mu)^2} \quad (101)$$

and by taking $h_2(x) = bx$ and $h_3(x) = bpx^2 + qa$ in (101), it further yields

$$|a_2| \le \frac{|bx|\sqrt{2|bx|}}{\sqrt{\left|\left[\left(\mu^2 - 3\mu + 4\right)b - 2p(2-\mu)^2\right]bx^2 - 2qa(2-\mu)^2\right|}}. \quad (102)$$

By subtracting (97) from (95) and in view of (98), we obtain

$$4(3-2\mu)a_3 - 4(3-2\mu)a_2^2 = h_2(x)(u_2 - v_2) + h_3(x)\left(u_1^2 - v_1^2\right)$$
$$a_3 = \frac{h_2(x)(u_2 - v_2)}{4(3-2\mu)} + a_2^2. \quad (103)$$

Then in view of (99), (103) becomes

$$a_3 = \frac{h_2(x)(u_2 - v_2)}{4(3-2\mu)} + \frac{[h_2(x)]^2\left(u_1^2 + v_1^2\right)}{2(2-\mu)^2}. \quad (104)$$

Applying (10), we deduce that

$$|a_3| \le \frac{|bx|}{6-4\mu} + \frac{b^2x^2}{(2-\mu)^2}. \quad (105)$$

From (103), for $v \in \mathbb{R}$, we write

$$a_3 - va_2^2 = \frac{h_2(x)(u_2 - v_2)}{4(3-2\mu)} + (1-v)a_2^2. \quad (106)$$

By substituting (101) in (106), we have

$$\begin{aligned} a_3 - va_2^2 &= \frac{h_2(x)(u_2 - v_2)}{4(3-2\mu)} + \frac{(1-v)[h_2(x)]^3(u_2 + v_2)}{\left(\mu^2 - 3\mu + 4\right)[h_2(x)]^2 - 2h_3(x)(2-\mu)^2} \\ &= h_2(x)\left\{\left(\Omega(v,x) + \frac{1}{4(3-2\mu)}\right)u_2 + \left(\Omega(v,x) - \frac{1}{4(3-2\mu)}\right)v_2\right\}, \end{aligned} \quad (107)$$

where

$$\Omega(v,x) = \frac{(1-v)[h_2(x)]^2}{\left(\mu^2 - 3\mu + 4\right)[h_2(x)]^2 - 2h_3(x)(2-\mu)^2}. \quad (108)$$

Hence, we conclude that

$$\left|a_3 - va_2^2\right| \le \begin{cases} \frac{|h_2(x)|}{6-4\mu} & 0 \le |\Omega(v,x)| \le \frac{1}{4(3-2\mu)} \\ 2|h_2(x)||\Omega(v,x)| & |\Omega(v,x)| \ge \frac{1}{4(3-2\mu)} \end{cases} \quad (109)$$

which in view of (10) evidently completes the proof of Theorem 2. □

In view of Remark 1, Theorem 3 yields.

Corollary 6. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathcal{L}_\sigma(\mu, t)$. Then*

$$|a_2| \le \frac{2|t|\sqrt{2|t|}}{\sqrt{\left|(2-\mu)^2 - 2\left(\mu^2 - 5\mu + 12\right)t^2\right|}} \quad \text{and} \quad |a_3| \le \frac{|t|}{3-2\mu} + \frac{4t^2}{(2-\mu)^2} \quad (110)$$

and for $v \in \mathbb{R}$

$$\left|a_3 - va_2^2\right| \le \begin{cases} \frac{|t|}{3-2\mu} & \text{if } |v-1| \le \frac{\left|(2-\mu)^2 - 2\left(\mu^2-5\mu+12\right)t^2\right|}{8t^2(3-2\mu)} \\ \frac{8|t|^3|v-1|}{\left|(2-\mu)^2 - 2\left(\mu^2-5\mu+12\right)t^2\right|} & \text{if } |v-1| \ge \frac{\left|(2-\mu)^2 - 2\left(\mu^2-5\mu+12\right)t^2\right|}{8t^2(3-2\mu)}. \end{cases} \quad (111)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there have no conflicts of interest.

Authors' Contributions

All authors contributed equally towards writing, reading, and approval of this manuscript.

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References

[1] H. M. Srivastava, A. K. Mishra, and P. Gochhayat, "Certain subclasses of analytic and bi-univalent functions," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1188–1192, 2010.

[2] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramanian, "Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 344–351, 2012.

[3] S. Altınkaya and S. Yalçin, "On the Chebyshev polynomial coefficient problem of some subclasses of bi-univalent functions," *Gulf Journal of Mathematics*, vol. 5, no. 3, pp. 34–40, 2017.

[4] S. Altınkaya and S. Yalçin, "Chebyshev polynomial coefficient bounds for subclass of bi-univalent functions," pp. 1–7, 2017, https://arxiv.org/abs/1605.08224.

[5] M. Çağlar, E. Deniz, and H. M. Srivastava, "Second Hankel determinant for certain subclasses ofbi-univalent functions," *Turkish Journal of Mathematics*, vol. 41, pp. 694–706, 2017.

[6] V. B. Girgaonkar and S. B. Joshi, "Coefficient estimates for certain subclass of bi-univalent functions associated with Chebyshev polynomial," *Ganita*, vol. 68, no. 1, pp. 79–85, 2018.

[7] H. O. Güney, G. Murugusundaramoorthy, and J. Sokół, "Subclasses of bi-univalent functionsrelated to shell-like

curves connected with Fibonacci numbers," *Acta Universitatis Sapientiae, Mathematica*, vol. 10, no. 1, pp. 70–84, 2018.

[8] J. M. Jahangiri, S. G. Hamidi, and S. A. Halim, "Coefficients of bi-univalent functions with positive real part derivatives," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 37, no. 3, pp. 633–640, 2014.

[9] S. Kanas, E. Analouei Adegani, and A. Zireh, "An unified approach to second Hankel determinant of bi-subordinate functions," *Mediterranean Journal of Mathematics*, vol. 14, no. 6, Article ID 233, 2017.

[10] S. K. Lee, V. Ravichandran, and S. Supramaniam, "Initial coefficients of biunivalent functions," *Abstract and Applied Analysis*, vol. 2014, Article ID 640856, pp. 1–6, 2014.

[11] X.-F. Li and A.-P. Wang, "Two new subclasses of bi-univalent functions," *International Mathematical Forum*, vol. 7, no. 29-32, pp. 1495–1504, 2012.

[12] N. Magesh and S. Bulut, "Chebyshev polynomial coefficient estimates for a class of analytic bi-univalent functions related to pseudo-starlike functions," *Afrika Matematika*, vol. 29, no. 1-2, pp. 203–209, 2018.

[13] H. Orhan, N. Magesh, and V. K. Balaji, "Second Hankel determinant for certain class of bi-univalent functions defined by Chebyshev polynomials," *Asian-European Journal of Mathematics*, vol. 12, no. 1, pp. 1–16, 2019.

[14] Z. Peng and Q. Han, "On the coefficients of several classes of bi-univalent functions," *Acta Mathematica Sinica Series B (English Edition)*, vol. 34, no. 1, pp. 228–240, 2014.

[15] H. M. Srivastava, S. Altınkaya, and S. Yalçin, "Certain subclasses of bi-univalent functions associated with the Horadam polynomials," *Iranian Journal of Science and Technology, Transactions A: Science*, pp. 1–7, 2018.

[16] H. M. Srivastava, F. M. Sakar, and H. Özlem Güney, "Some general coefficient estimates for a new class of analytic and bi-univalent functions defined by a linear combination," *Filomat*, vol. 32, no. 4, pp. 1313–1322, 2018.

[17] H. M. Srivastava, S. S. Eker, S. G. Hamidi, and J. M. Jahangiri, "Faber polynomial coefficient estimates for bi-univalent functions defined by the Tremblay fractional derivative operator," *Bulletin of the Iranian Mathematical Society*, vol. 44, no. 1, pp. 149–157, 2018.

[18] H. M. Srivastava, N. Magesh, and J. Yamini, "Initial coefficient estimates for bi-λ-convex and bi-μ-starlike functions connected with arithmetic and geometric means," *Electronic Journal of Mathematical Analysis and Applications*, vol. 2, no. 2, pp. 152–162, 2014.

[19] Z. Tu and C. L. Xiong, "Coefficient problems for unified starlike and convex classes of m-fold symmetric bi-univalent functions," *Journal of Mathematical Inequalities*, vol. 12, no. 4, pp. 921–932, 2018.

[20] L. Xiong and X. Liu, "Some extensions of coefficient problems for bi-univalent Ma-Minda starlike and convex functions," *Filomat*, vol. 29, no. 7, pp. 1645–1650, 2015.

[21] P. Zaprawa, "On the Fekete–Szegö problem for classes of bi-univalent functions," *Bulletin of the Belgian Mathematical Society - Simon Stevin*, vol. 21, no. 1, pp. 169–178, 2014.

[22] A. F. Horadam and J. M. Mahon, "Pell and Pell-Lucas polynomials," *Fibonacci Quarterly*, vol. 23, pp. 7–20, 1985.

[23] T. Hörçum and E. Gökçen Koçer, "On some properties of Horadam polynomials," *International Mathematical Forum*, vol. 4, pp. 1243–1252, 2009.