Research Article

A Variational Method for Multivalued Boundary Value Problems

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In this paper, we investigate the existence of solution for differential systems involving a ϕ -Laplacian operator which incorporates as a special case the well-known p -Laplacian operator. In this purpose, we use a variational method which relies on Szulkin's critical point theory. We obtain the existence of solution when the corresponding Euler-Lagrange functional is coercive.

1. Introduction

This paper is devoted to the study of the following second-order differential systems:

$$\begin{cases} -(\phi(u'(t)))' + \varepsilon\phi(u(t)) = \nabla F(t, u(t)) \text{ a.e. on } \Omega =]0, T[\\ (\phi(u'(0)), -\phi(u'(T))) \in \partial j(u(0), u(T)), \end{cases} \quad (1)$$

where $\varepsilon \geq 0$ is fixed, $\phi: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a monotone homeomorphism, $j: \mathbb{R}^N \times \mathbb{R}^N \rightarrow (-\infty, +\infty]$ is a proper, convex, and lower semicontinuous function, $F: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Caratheodory mapping, continuously differentiable with respect to the second variable and satisfies some usual growth conditions, $\nabla F(t, x)$ is the gradient of $F(t, \cdot)$ at $x \in \mathbb{R}^N$ and ∂j denotes the sub-differential of j in the sense of convex analysis.

Second-order differential problems with multivalued boundary value conditions have been studied by many authors. In this direction, we can cite the works of Bader and Papageorgiou [1], Béhi et al. [2], Gasinski and Papageorgiou [4], Jebelean [5], Jebelean and Morosanu [6], and references therein. In [1, 4–6], the authors investigate differential systems driven by a homogeneous

p -Laplacian operator while in problem (1), we deal with a nonhomogeneous ϕ -Laplacian operator which incorporates as a special case the p -Laplacian operator. As a consequence, problem (1) is a generalization of the following problem studied, in 2005, by Jebelean and Morosanu [6]:

$$\begin{cases} -(\phi_p(u'(t)))' + \varepsilon\phi_p(u(t)) = \nabla F(t, u(t)) \text{ a.e. on } \Omega =]0, T[\\ (\phi_p(u'(0)), -\phi_p(u'(T))) \in \partial j(u(0), u(T)), \end{cases} \quad (2)$$

where ϕ_p denotes the homogeneous operator p -Laplacian. Indeed, in our work, the ϕ -Laplacian operators are non-homogeneous and are of the form $\phi(x) = a(x)\phi_p(x)$ with ϕ_p the p -Laplacian operator, $p > 1$, and $a: \mathbb{R}^N \rightarrow \mathbb{R}_+^*$ a continuous map.

In order to obtain existence result for problem (1), we will use a variational method which relies on Szulkin's critical point theory [7].

This paper is organized as follows: After introducing notations and preliminary results in Section 2, in Section 3 we give a variational approach to problem (1). In Section 4, using some results from Section 3, we prove our main result. In

Section 4, we give an example to illustrate the applicability of our result. Finally, Section 5 is reserved for the conclusion.

2. Preliminaries

Let us recall some notions and results in the framework of Szulkin's critical point theory [7] which are needed and also some notations which will be used in the sequel. At this end, $W^{1,p}((0, T), \mathbb{R}^N)$ is the Sobolev Banach space which will be endowed with the norm:

$$\|u\|_m = \left(m\|u\|_p^p + \|u'\|_p^p \right)^{1/p}, \quad (3)$$

where $m > 0$ and $\|\cdot\|_p$ is the norm on $L^p((0, T), \mathbb{R}^N)$ defined by

$$\|u\|_p = \left(\int_0^T |u|^p dt \right)^{1/p}. \quad (4)$$

We denote $\|\cdot\|_\infty$, the norm on the set $C([0, T], \mathbb{R}^N)$:

$$\|u\|_\infty = \max\{|u(t)| : t \in [0, T]\}. \quad (5)$$

Let $(X, \|\cdot\|)$ be a real Banach space and $B: X \rightarrow (-\infty, +\infty]$ be a functional of type

$$B = E + G, \quad (6)$$

where $E \in C^1(X, \mathbb{R})$ and G is proper, convex, and lower semicontinuous (lsc). A point $u \in X$ is said to be a critical point of B if it satisfies the inequality

$$\langle E'(u), v - u \rangle + G(v) - G(u) \geq 0, \quad \forall v \in X. \quad (7)$$

A number $c \in \mathbb{R}$ such that $B^{-1}(c)$ contains a *critical point* is called a critical value of B .

To establish existence results for (1), we will need the following proposition due to Szulkin (see [7] Proposition 1.1).

Proposition 1. *If B satisfies (7), each local minimum point of B is necessarily a critical point of B .*

The functional B is said to satisfy the *Palais-smale* (in short, (PS)) condition if every sequence $\{u_n\} \subset X$ for which $B(u_n) \rightarrow c \in \mathbb{R}$ and

$$\langle E'(u_n), v - u_n \rangle + G(v) - G(u_n) \geq \varepsilon_n \|v - u_n\|, \quad \forall v \in X, \quad (8)$$

where $\varepsilon_n \rightarrow 0$ possesses a convergent subsequence.

3. A Variational Approach for Problem (1)

Before beginning the variational approach, we make the following hypotheses on the data of problem (1):

(H_a) : $a: \mathbb{R}^N \rightarrow \mathbb{R}_+^*$ is a continuous map such that there exists $\eta, M > 0$ such that

$$\eta \leq a(x) \leq M, \quad \forall x \in \mathbb{R}^N. \quad (9)$$

(H_ϕ) : $\phi: \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a monotone homeomorphism such that

- (a) $\phi(x) = a(x)\phi_p(x)$, $\forall x \in \mathbb{R}^N$, with ϕ_p the well-know p -Laplacian operator and $\nabla \Phi = \phi$ with $\Phi: \mathbb{R}^N \rightarrow \mathbb{R}$ of class C^1 on $\mathbb{R}^N$, strictly convex (i.e., Φ is a potential function corresponding to ϕ) and such that $\Phi(0) = 0$;
- (b) there exist $c_1, c_2, c_3 > 0$ such that

$$c_1|x|^p \leq \Phi(x) \leq c_2 + c_3|x|^p, \quad \forall x \in \mathbb{R}^N, \quad (10)$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$.

Remark 1. Suppose that $\forall x \in \mathbb{R}^N$, $a(x) = 1$, we have $\phi(x) = \phi_p(x) = |x|^{p-2}x$, $p > 1$. Then this function satisfies hypotheses (H_a) and (H_ϕ) . Other cases that satisfy Hypotheses (H_a) and (H_ϕ) are when $\eta < a(x) < M$ with $\eta, M > 0$ and the function $\phi: x \mapsto a(x)|x|^{p-2}x$ is a monotone homeomorphism on $\mathbb{R}^N$. This is the case, for example, when function a is equal to one of the following functions:

$$x \mapsto 1 + \frac{1}{(1 + |x|^p)^2} \quad (11)$$

$$\text{or } x \mapsto p + \frac{p - |x|}{e^{|x|}}, \quad \forall x \in \mathbb{R}^N,$$

with $p > 1$. Indeed, for these examples, the potentials functions Φ corresponding to ϕ are respectively the functions $x \mapsto (1/p)(1 + (1/(1 + |x|^p)))|x|^p$ and $x \mapsto (1 + (1/e^{|x|}))|x|^p$, $\forall x \in \mathbb{R}^N$.

(H_F) : $F: (0, T) \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Caratheodory mapping such that

$(H_F)_1$: $F(t, \cdot)$ is continuously differentiable, for a.e. $t \in (0, T)$

$(H_F)_2$: $F(\cdot, 0) \in L^1(0, T)$

$(H_F)_3$: for each $R > 0$ there is some $\alpha_R \in L^1(0, T)$ such that

$$|\nabla F(t, x)| \leq \alpha_R(t), \quad \text{a.e. } t \in (0, T), \forall x \in \mathbb{R}^N \text{ with } |x| \leq R. \quad (12)$$

Our notion of solution of problem (1) is defined as follows:

Definition 1. By a solution of the differential system (1), we will understand a function $u: [0, T] \rightarrow \mathbb{R}^N$ of class

$C^1(\Omega, \mathbb{R}^N)$ with $\phi(u')$ absolutely continuous, which satisfies the equality in (1) a.e. on $(0, T)$.

Now let us start the variational approach. In this purpose, for $\varepsilon > 0$, let $K_\varepsilon: W^{1,p}((0, T), \mathbb{R}^N) \longrightarrow \mathbb{R}$ be defined by

$$K_\varepsilon(u) = \int_0^T (\Phi(u'(s)) + \varepsilon \Phi(u(s))) ds, \quad \forall u \in W^{1,p}((0, T), \mathbb{R}^N). \quad (13)$$

Lemma 1. K_ε is proper, convex, and lower semicontinuous.

Proof. By hypothesis (H_ϕ) , it follows

$$c_1 \left(\|u'\|_p^p + \varepsilon \|u\|_p^p \right) \leq K_\varepsilon(u) \leq (1 + \varepsilon)c_2 + c_3 \left(\|u'\|_p^p + \varepsilon \|u\|_p^p \right), \quad \forall u \in W^{1,p}((0, T), \mathbb{R}^N). \quad (14)$$

Whence K_ε is proper. Also since Φ is convex, it follows that K_ε is convex. Finally, because of the lower semicontinuity of the functional norm on Banach space, K_ε is lower semicontinuous (in short, lsc). $\square$

Lemma 2. $K_\varepsilon \in C^1(W^{1,p}((0, T), \mathbb{R}^N), \mathbb{R})$ and

$$\langle K'_\varepsilon(u), w \rangle = \int_0^T \langle \phi(u'(t)), w'(t) \rangle_{\mathbb{R}^N} + \varepsilon \langle \phi(u(t)), w(t) \rangle_{\mathbb{R}^N} dt, \quad \forall u, w \in W^{1,p}((0, T), \mathbb{R}^N), \quad (15)$$

where $\langle \cdot \rangle_{\mathbb{R}^N}$ is the inner product in $\mathbb{R}^N$.

Proof. Let us consider the product space $H = \Pi_{i=1}^N L^q(\Omega, \mathbb{R})$, $(1/p) + (1/q) = 1$, equipped with the norm

$$\|x\|_H = \left(\sum_{i=1}^N \|x_i\|_q^q \right)^{1/q}, \quad \forall x = (x_1, x_2, \dots, x_{n-1}, x_n) \in H. \quad (16)$$

We define $g(g_1, g_2, \dots, g_{n-1}, g_N): W^{1,p}((0, T), \mathbb{R}^N) \longrightarrow H$ such that

$$g(u) = \phi(u') + \varepsilon \phi(u), \quad \forall u \in W^{1,p}((0, T), \mathbb{R}^N). \quad (17)$$

Let us show that g is bounded.

We have

$$\begin{aligned} \|g_i(u)\|_q^q &= \int_0^T |\phi_i(u'(t)) + \varepsilon \phi_i(u(t))|^q dt \\ &= \int_0^T |a(u'(t))| |u'(t)|^{p-2} u'_i(t) \\ &\quad + \varepsilon a(u(t)) |u(t)|^{p-2} u_i(t)|^q dt \\ &\leq M^q \int_0^T |u'(t)|^{(p-1)q} + \varepsilon |u(t)|^{(p-1)q} dt \\ &= M^q \left(\|u'\|_p^p + \varepsilon \|u\|_p^p \right) = M^q \|u\|_\varepsilon^p \\ &\leq M^q \eta_1 \|u\|_m^p, \quad \text{for some } \eta_1 > 0. \end{aligned} \quad (18)$$

It follows

$$\|g(u)\|_H \leq M(N\eta_1)^{1/q} \|u\|_m^{p-1}, \quad \forall u \in W^{1,p}(\Omega). \quad (19)$$

So g is bounded.

Let us show that g is continuous.

Let $\{u_n\}_{n \geq 1}$ be a sequence such that $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$. Then $u_n \longrightarrow u$ in $L^p(\Omega)$ and $u'_n \rightharpoonup u'$ in $L^p(\Omega)$. Whence $\phi_i(u'_n) \longrightarrow \phi_i(u')$ in $L^q(\Omega)$ and $\phi_i(u_n) \longrightarrow \phi_i(u)$ in $L^q(\Omega)$. We set

$$g_i(u_n) = \phi_i(u'_n) + \varepsilon \phi_i(u_n). \quad (20)$$

By the previous arguments, the sequence $g_i(u_n) \longrightarrow g_i(u)$ in $L^q(\Omega)$. We infer that the sequence $g(u_n) \longrightarrow g(u)$ in H . So g is continuous.

We consider the functional:

$$Q(v) = \int_0^T \langle \phi(u'(t)), v'(t) \rangle + \varepsilon \langle \phi(u(t)), v(t) \rangle dt, \quad (21)$$

for all $v \in W^{1,p}(\Omega)$.

Let us show that the linear operator Q is continuous on $W^{1,p}(\Omega)$.

Using Hölder's inequality, we obtain

$$\begin{aligned} |Q(v)| &= \left| \int_0^T \langle \phi(u'(t)), v'(t) \rangle + \varepsilon \langle \phi(u(t)), v(t) \rangle dt \right| \\ &\leq \left(\int_0^T |\phi(u'(t))|^q dt \right)^{1/q} \left(\int_0^T |v'(t)|^p dt \right)^{1/p} \\ &\quad + \varepsilon \left(\int_0^T |\phi(u(t))|^q dt \right)^{1/q} \left(\int_0^T |v(t)|^p dt \right)^{1/p} \\ &\leq M(1 + \varepsilon) \|u\|_m^{p/q} \|v\|_m, \quad \forall u \in W^{1,p}(\Omega). \end{aligned} \quad (22)$$

So Q is continuous.

Let us show that K_ε is Frechet differentiable in $u \in W^{1,p}(\Omega)$ and $K'_\varepsilon(u) = \phi(u') + \varepsilon\phi(u)$ in the sense of (15).

Using Fubini's inequality, for arbitrary $v \in W^{1,p}(\Omega)$, we obtain

$$\begin{aligned}
 & K_\varepsilon(u+v) - K_\varepsilon(u) - \int_0^T \langle \phi(u'(t)), v'(t) \rangle + \varepsilon \langle \phi(u(t)), v(t) \rangle dt \\
 &= \int_0^T \int_0^1 \frac{d}{ds} (\Phi(u'(t) + sv'(t))) + \varepsilon \frac{d}{ds} (\Phi(u(t) + sv(t))) ds dt \\
 &\quad - \int_0^T \int_0^1 \langle \phi(u'(t)), v'(t) \rangle + \varepsilon \langle \phi(u(t)), v(t) \rangle ds dt \\
 &= \int_0^T \int_0^1 \langle \phi(u'(t) + sv'(t)), v'(t) \rangle + \varepsilon \langle \phi(u(t) + sv(t)), v(t) \rangle ds dt \\
 &\quad - \int_0^T \int_0^1 \langle \phi(u'(t)), v'(t) \rangle + \varepsilon \langle \phi(u(t)), v(t) \rangle ds dt \\
 &= \int_0^T \int_0^1 \langle \phi(u'(t) + sv'(t)) - \phi(u'(t)), v'(t) \rangle + \varepsilon \langle \phi(u(t) + sv(t)) - \phi(u(t)), v(t) \rangle ds dt \\
 &\leq \int_0^1 \left[\left(\int_0^T |\phi(u'(t) + sv'(t)) - \phi(u'(t))|^q dt \right)^{1/q} \left(\int_0^T |v'(t)|^p dt \right)^{1/p} \right] ds \\
 &\quad + \varepsilon \int_0^1 \left[\left(\int_0^T |\phi(u(t) + sv(t)) - \phi(u(t))|^q dt \right)^{1/q} \left(\int_0^T |v(t)|^p dt \right)^{1/p} \right] ds \\
 &\leq \int_0^1 (\|\phi(u' + sv') - \phi(u')\|_H + \varepsilon \|\phi(u + sv) - \phi(u)\|_H) ds \|v\|_m \\
 &\leq \int_0^1 (\|L(u + sv) - L(u)\|_H + \varepsilon \|\Lambda(u + sv) - \Lambda(u)\|_H) ds \|v\|_m,
 \end{aligned} \tag{23}$$

with $L: W^{1,p}(\Omega) \longrightarrow H$ and $L(u) = \phi(u')$;
 $\Lambda: W^{1,p}(\Omega) \longrightarrow H$ and $\Lambda(u) = \phi(u)$.

Arguing as in the proof of the continuity of g and the fact it is bounded on $W^{1,p}(\Omega)$, we show that L and Λ are

continuous and bounded on $W^{1,p}(\Omega)$. Moreover, using Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
 & \frac{K_\varepsilon(u+v) - K_\varepsilon(u) - \int_0^T \langle \phi(u'(t)), v'(t) \rangle + \varepsilon \langle \phi(u(t)), v(t) \rangle dt}{\|v\|_m} \\
 &\leq \int_0^1 (\|L(u + sv) - L(u)\|_H + \varepsilon \|\Lambda(u + sv) - \Lambda(u)\|_H) ds \longrightarrow 0, \quad \text{as } \|v\|_m \longrightarrow 0.
 \end{aligned} \tag{24}$$

Therefore, (15) is proved.

We know that $K'_\varepsilon: W^{1,p}(\Omega) \longrightarrow (W^{1,p}(\Omega))^*$ where $(W^{1,p}(\Omega))^*$ denotes the dual of $W^{1,p}(\Omega)$.

Let us show that K'_ε is continuous.

We have

$$\begin{aligned}
 \langle K'_\varepsilon(u) - K'_\varepsilon(v), w \rangle &= \int_0^T \langle \phi(u'(t)) + \varepsilon\phi(u(t)) \\
 &\quad - \phi(v'(t)) - \varepsilon\phi(v(t)), w \rangle dt \\
 &= \int_0^T \langle g(u)(t) - g(v)(t), w \rangle dt.
 \end{aligned} \tag{25}$$

Using Hölder's inequality, we have

$$\langle K'_\varepsilon(u) - K'_\varepsilon(v), w \rangle \leq k \|g(u) - g(v)\|_H \|w\|_m, \quad (26)$$

$$\forall u, v, w \in W^{1,p}(\Omega).$$

Whence

$$\|K'_\varepsilon(u) - K'_\varepsilon(v)\|_{(W^{1,p}(\Omega))^*} \leq k \|g(u) - g(v)\|_H. \quad (27)$$

Thus, since g is continuous, we obtain the continuity of K'_ε .

We introduce also the functional: $J: W^{1,p}((0, T), \mathbb{R}^N) \longrightarrow (-\infty, +\infty]$ defined by

$$J(u) = j(u(0), u(T)), \quad \forall u \in W^{1,p}((0, T), \mathbb{R}^N). \quad (28)$$

Recall that j is proper, convex, and lsc. Then, J is also proper, convex, and lsc. Let us set

$$\Delta_\varepsilon = K_\varepsilon + J. \quad (29)$$

Since K_ε and J are proper, convex and lsc, it follows that Δ_ε is proper, convex and lsc on $W^{1,p}((0, T), \mathbb{R}^N)$. Assuming that hypotheses $(H_F)_1$, $(H_F)_2$ and $(H_F)_3$ on the Caratheodory function F hold, for each $R > 0$, we obtain:

$$|F(t, x)| \leq R \alpha_R(t) + |F(t, 0)|, \quad \text{for a.e } t \in (0, T) \text{ with } |x| \leq R, \quad (30)$$

with $\alpha_R \in L^1(0, T)$. Equation (30) comes from inequality (12) and the estimation:

$$\begin{aligned} |F(t, x)| &= \left| \int_0^1 \frac{d}{ds} F(t, sx) ds \right| + |F(t, 0)| \\ &\leq \left| \int_0^1 \langle \nabla F(t, sx), x \rangle_{\mathbb{R}^N} ds \right| + |F(t, 0)| \\ &\leq |x| \left| \int_0^1 \nabla F(t, sx) ds \right| + |F(t, 0)|, \end{aligned} \quad (31)$$

equation (30) and the embedding $W^{1,p}((0, T), \mathbb{R}^N) \subset C((0, T), \mathbb{R}^N)$ allow us to introduce the functional: $\Psi_F: W^{1,p}((0, T), \mathbb{R}^N) \longrightarrow \mathbb{R}$ defined by

$$\Psi_F(u) = - \int_0^T F(t, u(t)) dt + \int_0^T F(t, 0) dt, \quad (32)$$

$$\forall u \in W^{1,p}((0, T), \mathbb{R}^N). \quad \square$$

Lemma 3. *If the hypotheses $(H_F)_1$, $(H_F)_2$, and $(H_F)_3$ hold, then*

$$\Psi_F \in C^1(W^{1,p}(\Omega, \mathbb{R}^N), \mathbb{R}), \quad (33)$$

$$\begin{aligned} \langle \Psi'_F(u), v \rangle &= - \int_0^T \langle \nabla F(t, u), v \rangle dt, \\ \forall u, v &\in W^{1,p}((0, T), \mathbb{R}^N). \end{aligned} \quad (34)$$

Proof. The proof is similar to the one of Theorem 2.6 of [2].

Considering the functional framework in Section 2, we set $X = W^{1,p}((0, T), \mathbb{R}^N)$, $E = \Psi_F$ in (32), $G = \Delta_\varepsilon$ in (29) and $B_{F,\varepsilon} = B$:

$$B_{F,\varepsilon} = \Psi_F + \Delta_\varepsilon. \quad (35) \quad \square$$

Proposition 2. *Let the Caratheodory function $F: (0, T) \times \mathbb{R}^N \longrightarrow \mathbb{R}$ satisfies $(H_F)_1$, $(H_F)_2$, and $(H_F)_3$ and let $u \in W^{1,p}((0, T), \mathbb{R}^N)$. If u is a critical point of the functional $B_{F,\varepsilon}$ defined by (35), in the sense (7), i.e.,*

$$\begin{aligned} \langle \Psi'_F(u), w - u \rangle + \Delta_\varepsilon(w) - \Delta_\varepsilon(u) &\geq 0, \\ \forall w &\in W^{1,p}((0, T), \mathbb{R}^N), \end{aligned} \quad (36)$$

then u is a solution of problem (1). The converse implication is also true.

Proof. We suppose that u is a critical point of $B_{F,\varepsilon}$. In the inequality (36), we take $w = u + sv$. Then dividing by s and letting $s \longrightarrow 0^+$, we obtain

$$\begin{aligned} \langle \Psi'_F(u), v \rangle + \langle K'_\varepsilon(u), v \rangle + J'(u; v) &\geq 0, \\ \forall v &\in W^{1,p}((0, T), \mathbb{R}^N), \end{aligned} \quad (37)$$

where $J'(u; v)$ is the directional derivative of the convex function J at u in the direction v . From (28) and (37), it follows

$$\begin{aligned} \langle \Psi'_F(u), v \rangle + \langle K'_\varepsilon(u), v \rangle + j'((u(0), u(T)); \\ (v(0), v(T))) &\geq 0, \quad \forall v \in W^{1,p}((0, T), \mathbb{R}^N). \end{aligned} \quad (38)$$

Since $C_0^\infty((0, T), \mathbb{R}^N) \subset W^{1,p}((0, T), \mathbb{R}^N)$, using Hahn-Banach's theorem (see Theorem I.1 of Brezis [3]), (38) implies

$$\langle \Psi'_F(u), v \rangle + \langle K'_\varepsilon(u), v \rangle = 0, \quad \forall v \in C_0^\infty((0, T), \mathbb{R}^N). \quad (39)$$

Using (15), (34), and (39), we obtain

$$\begin{aligned} \int_0^T \langle \phi(u'(t)), v'(t) \rangle_{\mathbb{R}^N} dt \\ = \int_0^T \langle -\varepsilon \phi(u(t)) + \nabla F(t, u(t)), v(t) \rangle_{\mathbb{R}^N} dt, \\ \forall v \in C_0^\infty((0, T), \mathbb{R}^N). \end{aligned} \quad (40)$$

From hypothesis $(H_\phi)(b)$ and $u \in W^{1,p}((0, T), \mathbb{R}^N)$, it follows that

$$\phi(u), \phi(u') \in L^q((0, T), \mathbb{R}^N), \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1. \quad (41)$$

Also, $(H_F)_3$ implies

$$\nabla F(., u) \in L^1((0, T), \mathbb{R}^N). \quad (42)$$

Equations (40) and (42) imply

$$\phi(u') \in W^{1,1}((0, T), \mathbb{R}^N), \quad (43)$$

$$-(\phi(u'(t)))' = -\varepsilon \phi(u(t)) + \nabla F(t, u(t)). \quad (44)$$

With ϕ being a homeomorphism, (43) ensures that $u \in C^1((0, T), \mathbb{R}^N)$. This together with (44) shows that u is the solution of the differential system (1). Furthermore, (38) and (44) yield

$$\begin{aligned} j'((u(0), u(T)); (v(0), v(T))) &\geq \langle \phi(u'(0)), v(0) \rangle \\ &\quad - \langle \phi(u'(T)), v(T) \rangle, \\ &\quad \forall v \in W^{1,p}((0, T), \mathbb{R}^N). \end{aligned} \quad (45)$$

Thus,

$$\begin{aligned} j'((u(0), u(T)); (x, y)) &\geq \langle \phi(u'(0)), x \rangle - \langle \phi(u'(T)), y \rangle, \\ &\quad \forall x, y \in \mathbb{R}^N, \end{aligned} \quad (46)$$

which, by a standard result from convex analysis (Theorem 23.2 of Rockafellar [8]), means that the boundary conditions in (1) are true.

Now let us show the converse implication.

By multiplying (1) with $v \in W^{1,p}((0, T), \mathbb{R}^N)$ and integrating by parts on $(0, T)$, we obtain

$$\begin{aligned} &\int_0^T (\langle \phi(u'(t)), v'(t) \rangle_{\mathbb{R}^N} + \varepsilon \langle \phi(u(t)), v(t) \rangle_{\mathbb{R}^N}) dt \\ &\quad + \langle \phi(u'(0)), v'(0) \rangle_{\mathbb{R}^N} - \langle \phi(u'(T)), v'(T) \rangle_{\mathbb{R}^N} \\ &= \int_0^T \langle \nabla F(t, u(t)), v(t) \rangle_{\mathbb{R}^N} dt. \end{aligned} \quad (47)$$

Using inequalities (15) and (34) in (47), it follows

$$\begin{aligned} &\langle \Psi'_F(u), v \rangle + \langle K'_\varepsilon(u), v \rangle + \langle \phi(u'(0)), v'(0) \rangle_{\mathbb{R}^N} \\ &\quad - \langle \phi(u'(T)), v'(T) \rangle_{\mathbb{R}^N} = 0. \end{aligned} \quad (48)$$

By using Theorem 23.2 of Rockafellar [8], we obtain

$$\langle \Psi'_F(u), v \rangle + \langle K'_\varepsilon(u), v \rangle + j'((u(0), u(T)); (v(0), v(T))) \geq 0. \quad (49)$$

Using some previous arguments, we infer that

$$\langle \Psi'_F(u), w - u \rangle + \Delta_\varepsilon(w) - \Delta_\varepsilon(u) \geq 0, \quad \forall u \in W^{1,p}((0, T), \mathbb{R}^N). \quad (50)$$

□

4. Existence Results for Problem (1)

At first, let us introduce the constant

$$\begin{aligned} \lambda_1 &= \lambda_1(p, j, \varepsilon) = \varepsilon \\ &\quad + \inf \left\{ \frac{\|u'\|_p^p}{\|u\|_p^p} : u \in W^{1,p}((0, T), \mathbb{R}^N) \setminus \{0\}, \right. \\ &\quad \left. (u(0), u(T)) \in D(j) \right\}, \end{aligned} \quad (51)$$

for $\varepsilon \geq 0$. If $D(j) = \mathbb{R}^N \times \mathbb{R}^N$, then $\lambda_1(p, j, 0) = 0$ and if $D(j) = \{(0; 0)\}$, then $\lambda_1(p, j, 0) > 0$. To obtain the existence result, we make the following hypothesis:

$$(H_{\lambda_1}) \quad \lambda_1(p, j, \varepsilon) > 0.$$

Proposition 3. *If (H_{λ_1}) is true, then*

$$2^{1/p} \|u\|_{\lambda_1} \leq \left(\|u'\|_p^p + \varepsilon \|u\|_p^p \right)^{1/p} \leq \|u\|_{\lambda_1}, \quad \forall u \in D(J), \quad (52)$$

where J is defined by (28).

Proof. Since

$$D(J) = \{u \in W^{1,p}((0, T), \mathbb{R}^N) : (u(0), u(T)) \in D(j)\}, \quad \text{by (51) we have}$$

$$\begin{aligned} \lambda_1 &= \lambda_1(p, j, \varepsilon) = \varepsilon + \inf \left\{ \frac{\|u'\|_p^p}{\|u\|_p^p} : u \in W^{1,p}((0, T), \mathbb{R}^N) \setminus \{0\}, \right. \\ &\quad \left. (u(0), u(T)) \in D(j) \right\}. \end{aligned} \quad (53)$$

Whence,

$$\lambda_1 \leq \varepsilon + \frac{\|u'\|_p^p}{\|u\|_p^p} \quad \forall u \in W^{1,p}((0, T), \mathbb{R}^N) \setminus \{0\}, \quad (54)$$

$$(u(0), u(T)) \in D(j).$$

Then

$$\lambda_1 \|u\|_p^p \leq \varepsilon \|u\|_p^p + \|u'\|_p^p. \quad (55)$$

It follows

$$\|u'\|_p^p + \lambda_1 \|u\|_p^p \leq 2 \|u'\|_p^p + \varepsilon \|u\|_p^p. \quad (56)$$

Whence

$$2^{-(1/p)} \|u\|_{\lambda_1} \leq \left(\|u'\|_p^p + \varepsilon \|u\|_p^p \right)^{1/p}. \quad (57)$$

Furthermore, we have

$$\varepsilon + \frac{\|u'\|_p^p}{\|u\|_p^p} \leq \lambda_1 + \frac{\|u'\|_p^p}{\|u\|_p^p}. \quad (58)$$

Then

$$\|u'\|_p^p + \varepsilon \|u\|_p^p \leq \|u'\|_p^p + \lambda_1 \|u\|_p^p. \quad (59)$$

So

$$\left(\|u'\|_p^p + \varepsilon \|u\|_p^p \right)^{1/p} \leq \|u\|_{\lambda_1}. \quad (60)$$

Inequalities (57) and (60) yield the result.

If the nonlinearity F lies asymptotically on the left of λ_1 , then problem (1) is solvable. $\square$

Theorem 1. Assume $(H_F)_1$, $(H_F)_2$, $(H_F)_3$, and H_{λ_1} . If

$$\lim_{|x| \rightarrow \infty} \sup \frac{F(t, x)}{c_1 |x|^p} < \lambda_1, \quad \text{uniformly for a.e. } t \in (0, T), \quad (61)$$

the problem (1) has at least a solution.

Proof. Let us show that the functional $B_{F,\varepsilon}$ in (35) is sequentially weakly lsc and coercive on the space $(W^{1,p}((0, T), \mathbb{R}^N), \|\cdot\|_{\lambda_1})$.

Let us show that Ψ_F in (32) is sequentially weakly continuous.

Let $u, v \in W^{1,p}((0, T), \mathbb{R}^N)$ be such that $\|u\|_{\infty}, \|v\|_{\infty} \leq M$, with some $M > 0$. By $(H_F)_3$, there is an $\alpha_{2M} \in L^1(0, T)$ such that

$$\begin{aligned} |\nabla F(t, x)| &\leq \alpha_{2M}, \\ \text{for a.e. } t \in (0, T), \forall x \in \mathbb{R}^N \text{ with } |x| &\leq 2M. \end{aligned} \quad (62)$$

We estimate

$$\begin{aligned} |\Psi_F(u) - \Psi_F(v)| &= \left| \int_0^T (F(t, u) - F(t, v)) dt \right| \\ &= \left| \int_0^T \int_0^1 \frac{d}{ds} F(t, u + s(v - u)) ds dt \right| \\ &= \left| \int_0^T \int_0^1 \langle \nabla F(t, u + s(v - u)), v - u \rangle_{\mathbb{R}^N} ds dt \right| \\ &\leq \int_0^T \int_0^1 |\nabla F(t, u + s(v - u))| ds \|u - v\|_{\infty} dt, \end{aligned} \quad (63)$$

and by (62), it follows

$$|\Psi_F(u) - \Psi_F(v)| \leq \left(\int_0^T \alpha_{2M}(t) dt \right) \|u - v\|_{\infty}. \quad (64)$$

By the compactness of the embedding $W^{1,p}((0, T), \mathbb{R}^N) \subset C((0, T), \mathbb{R}^N)$ and (64), it follows that Ψ_F is sequentially weakly continuous on $W^{1,p}((0, T), \mathbb{R}^N)$. Then, by the weak lower semicontinuity of Δ_ε in (29), $B_{F,\varepsilon}$ is sequentially weakly lower semicontinuous.

Furthermore, from (61), there are constant $\sigma \in (0, \lambda_1)$ and $\rho > 0$ such that

$$\begin{aligned} F(t, x) &\leq (\lambda_1 - \sigma)c_1 |x|^p, \\ \text{for a.e. } t \in (0, T), \forall x \in \mathbb{R}^N \text{ with } |x| &> \rho. \end{aligned} \quad (65)$$

Then, (30) and (65) yield

$$\begin{aligned} F(t, x) &\leq \rho \alpha_p(t) + |F(t, 0)| + (\lambda_1 - \sigma)c_1 |x|^p, \\ \text{for a.e. } t \in (0, T), \forall x \in \mathbb{R}^N, \end{aligned} \quad (66)$$

which, by (32), give

$$\Psi_F(u) \geq -\gamma(\rho) - (\lambda_1 - \sigma)c_1 \|u\|_p^p, \quad \forall u \in W^{1,p}((0, T), \mathbb{R}^N), \quad (67)$$

with $\gamma(\rho) = \rho \int_0^T \alpha_p(t) dt + 2 \int_0^T |F(0, t)| dt$. Using (29), (67), and (53) and the Proposition 3, we estimate $B_{F,\varepsilon}$ on $D(J)$ as follows:

$$\begin{aligned} B_{F,\varepsilon}(u) &= \Psi_F(u) + \Delta_\varepsilon(u) \\ &\geq -\gamma(\rho) - (\lambda_1 - \sigma)c_1 \|u\|_p^p + c_1 \left(\|u'\|_p^p + \varepsilon \|u\|_p^p \right) + J(u) \\ &\geq -\gamma(\rho) - (\lambda_1 - \sigma)c_1 \frac{\left(\|u'\|_p^p + \varepsilon \|u\|_p^p \right)}{\lambda_1} \\ &\quad + c_1 \left(\|u'\|_p^p + \varepsilon \|u\|_p^p \right) + J(u) \\ &= -\gamma(\rho) + \frac{\sigma}{\lambda_1} c_1 \left(\|u'\|_p^p + \varepsilon \|u\|_p^p \right) + J(u) \\ &\geq -\gamma(\rho) + \frac{\sigma}{2\lambda_1} c_1 \|u\|_{\lambda_1}^p + J(u), \quad \forall u \in D(J). \end{aligned} \quad (68)$$

Since j is convex and lsc, it is bounded from below by an affine functional. Therefore, by (28) there are constants $k_1, k_2, k_3 \geq 0$ such that

$$\begin{aligned} B_{F,\varepsilon}(u) &\geq -\gamma(\rho) + \frac{\sigma}{2\lambda_1} c_1 \|u\|_{\lambda_1}^p - k_1 |u(0)| - k_2 |u(T)| - k_3, \\ &\quad \forall u \in D(J). \end{aligned} \quad (69)$$

From (69) and the continuity embedding of $W^{1,p}((0, T), \mathbb{R}^N) \subset C((0, T), \mathbb{R}^N)$, we obtain

$$\begin{aligned} B_{F,\varepsilon}(u) &\geq -\gamma(\rho) + \frac{\sigma}{2\lambda_1} c_1 \|u\|_{\lambda_1}^p - \bar{k}_1 \|u\|_{\lambda_1} - \bar{k}_2 \|u\|_{\lambda_1} - k_3, \\ &\quad \forall u \in D(J). \end{aligned} \quad (70)$$

with some $\bar{k}_1, \bar{k}_2 \geq 0$. Consequently,

$$B_{F,\varepsilon}(u) \longrightarrow +\infty, \quad \text{as } \|u\|_{\lambda_1} \longrightarrow \infty, \quad (71)$$

meaning that $B_{F,\varepsilon}$ is coercive on $(W^{1,p}, \|\cdot\|_{\lambda_1})$.

Since the functional $B_{F,\varepsilon}$ is sequentially weakly lsc and coercive, by a well-known result from calculus of variations, $B_{F,\varepsilon}$ is bounded from below and attains its infimum at some $u \in W^{1,p}((0, T), \mathbb{R}^N)$. Then, by Proposition 1 and 3, u is a solution of problem (1).

Remarque 2. The problem (1) incorporates classical problems of Dirichlet, Neumann, Periodic, antiperiodic, and Sturm–Liouville (see Remark 3.9 of Jebelean [5]). $\square$

5. Example

We consider the following problem:

$$\left\{ \begin{array}{l} -\left(\left(1 + \frac{1}{(1+|u'(t)|^p)^2} \right) \phi_p(u'(t)) \right)' + \varepsilon \left(1 + \frac{1}{(1+|u'(t)|^{p-1})^2} \right) \phi_p(u(t)) = \beta(t) \phi_p(u(t)), \\ \text{a.e on } \Omega =]0, T[\\ \left(\left(1 + \frac{1}{(1+|u'(0)|^p)^2} \right) \phi_p(u'(0)), -\left(1 + \frac{1}{(1+|u'(T)|^p)^2} \right) \phi_p(u(T)) \right) \in \partial j(u(0), u(T)), \end{array} \right. \quad (72)$$

where $\phi_p(x) = |x|^{p-2}x, \forall x \in \mathbb{R}^N$, is the well-known p -Laplacian operator. Let us set $\Phi(x) = (1/p)(1 + (1/(1+|x|^p)))|x|^p$. Whence $\nabla\Phi(x) = \phi(x) = (1 + (1/(1+|x|^p)))\phi_p(x)$. When, $p > 1$, we see that ϕ is a monotone homeomorphism such that $\phi(0) = 0$. Moreover, Φ satisfies hypothesis $(H_\phi)(b)$. Therefore, by Theorem 1, problem (72) has at least a solution.

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6. Conclusion

In this article, using variational method which relies on Szulkin theory, we have established existence for second-order problems with multivalued boundary conditions. We give an example but more examples and applications can be given.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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