Research Article

Frequency Equations for the In-Plane Vibration of Circular Annular Disks

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This paper deals with the in-plane vibration of circular annular disks under combinations of different boundary conditions at the inner and outer edges. The in-plane free vibration of an elastic and isotropic disk is studied on the basis of the two-dimensional linear plane stress theory of elasticity. The exact solution of the in-plane equation of equilibrium of annular disk is attainable, in terms of Bessel functions, for uniform boundary conditions. The frequency equations for different modes can be obtained from the general solutions by applying the appropriate boundary conditions at the inner and outer edges. The presented frequency equations provide the frequency parameters for the required number of modes for a wide range of radius ratios and Poisson's ratios of annular disks under clamped, free, or flexible boundary conditions. Simplified forms of frequency equations are presented for solid disks and axisymmetric modes of annular disks. Frequency parameters are computed and compared with those available in literature. The frequency equations can be used as a reference to assess the accuracy of approximate methods.

1. Introduction

The out-of-plane vibration properties of circular disks subjected to a variety of boundary conditions have been extensively investigated (e.g., [1–4]). The in-plane vibration analyses of circular disk, however, have been gaining increasing attention only in the recent years. Much of the interest could be attributed to important significance of the in-plane vibration in various practical problems such as the vibration of railway wheels, disk brakes, and hard disk drives contributing to noise and structural vibration [5–7].

The in-plane vibration of circular disks was first attempted by Love [8] who formulated the equations of motion for a thin solid circular disk with free outer edge together with the general solution. The equations of motion were subsequently solved by Onoe [9] to obtain the exact frequency equations corresponding to different modes of a solid disk with free outer edge. Holland [10] evaluated the frequency parameters and the corresponding mode shapes for a wide range of Poisson's ratios and the vibration response to an in-plane force. The in-plane vibration characteristics of

solid disks clamped at the outer edge have been investigated in a few recent studies. Farag and Pan [11] evaluated the frequency parameters and the mode shapes of in-plane vibration of solid disks clamped at the outer edge using assumed deflection modes in terms of trigonometric and Bessel functions. Park [12] studied the exact frequency equation for the solid disk clamped at the outer edge.

The in-plane vibration analyses in the above reported studies were limited to solid disks with either free or clamped outer edge. The in-plane free vibration of annular disks with different boundary conditions has also been addressed in a few studies. The variations in the in-plane vibration frequency parameters of annular disks with free edges were investigated as function of the size of the opening by Ambati et al. [13]. The variation ranged from a solid disk to a thin ring, while the validity of the analytical results was demonstrated using the experimental data. Another study investigated the free vibration and dynamic response characteristics of an annular disk with clamped inner boundary and a concentrated radial force applied at the outer boundary [14]. Irie et al. [15] investigated the modal characteristics

of in-plane vibration of annular disks using transfer matrix formulation while considering free and clamped inner and outer edges.

The above reported studies on in-plane vibration of solid and annular disks have employed different methods of analyses. The finite-element technique has also been used to examine the validity of analytical methods (e.g., [11, 14]). The exact frequency equations of in-plane vibration, however, have been limited only to solid disks. Such analyses for the annular disks pose more complexities due to presence of different combinations of boundary conditions at the inner and outer edges. This study aims at generalized formulation for in-plane vibration analyses of circular annular disks under different combinations of clamped, free, or flexible boundary conditions at the inner and outer edges. The equations of motion are solved for the general case of annular disks. The exact frequency equations are presented for different combinations of boundary conditions, including the flexible boundaries, for various radius ratios, while the solid disk is considered as special cases of the generalized formulation.

2. Theory

The equations of the in-plane vibration of a circular disk are formulated for an annular disk shown in Figure 1. The disk is considered to be elastic with thickness h , outer radius b and inner radius a . The material is assumed to be isotropic with mass density ρ , Young's modulus E , and Poisson ratio ν . The equations of dynamic equilibrium in terms of in-plane displacements along the radial and circumferential directions can be found in many reported studies (e.g., [11, 16]). These equations of motion in the polar coordinate system (r, θ) can be written as

$$\begin{aligned} \frac{\partial^2 u_r}{\partial t^2} - C_L^2 \left[\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} \right] - \frac{C_T^2}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} \\ - C_T^2 \frac{1}{r} \frac{1+\nu}{1-\nu} \frac{\partial^2 u_\theta}{\partial r \partial \theta} + C_T^2 \frac{1}{r^2} \frac{3-\nu}{1-\nu} \frac{\partial u_\theta}{\partial \theta} = 0, \\ \frac{\partial^2 u_\theta}{\partial t^2} - C_T^2 \left[\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} \right] - \frac{C_L^2}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} \\ - C_T^2 \frac{1}{r} \frac{1+\nu}{1-\nu} \frac{\partial^2 u_r}{\partial r \partial \theta} + C_T^2 \frac{1}{r^2} \frac{3-\nu}{1-\nu} \frac{\partial u_r}{\partial \theta} = 0, \end{aligned} \quad (1)$$

where u_r and u_θ are the radial and circumferential displacements, respectively, along the r and θ directions, $C_L^2 = E/\rho(1-\nu^2)$ and $C_T^2 = E/2\rho(1+\nu)$.

Following Love's theory [8], the radial and circumferential displacements can be expressed in terms of the Lamé Potentials ϕ and ψ [17], as

$$u_r = \frac{1}{b} \left(\frac{\partial \phi}{\partial \xi} + \frac{1}{\xi} \frac{\partial \psi}{\partial \theta} \right), \quad (2)$$

$$u_\theta = \frac{1}{b} \left(\frac{1}{\xi} \frac{\partial \phi}{\partial \theta} - \frac{\partial \psi}{\partial \xi} \right), \quad (3)$$

where $\xi = r/b$.

FIGURE 1: Geometry and coordinate system used for in-plane vibration analysis of an annular disk.

Assuming harmonic oscillations corresponding to a natural frequency ω , the potential functions ϕ and ψ can be represented by

$$\phi(\xi, \theta, t) = \Phi(\xi) \cos n\theta \sin \omega t, \quad (4)$$

$$\psi(\xi, \theta, t) = \Psi(\xi) \sin n\theta \sin \omega t, \quad (5)$$

where n is the circumferential wave number or nodal diameter number. Upon substituting for u_r and u_θ in terms of ξ , Φ , and Ψ from (2) to (5), in (1), the equations of motion reduce to the following uncoupled form:

$$\nabla^2 \Phi = -\lambda_1^2 \Phi, \quad (6)$$

$$\nabla^2 \Psi = -\lambda_2^2 \Psi, \quad (7)$$

where λ_1 and λ_2 are nondimensional frequency parameters defined as

$$\begin{aligned} \lambda_1^2 &= \frac{\omega^2(1-\nu^2)\rho b^2}{E}, & \lambda_2^2 &= \frac{2\omega^2(1+\nu)\rho b^2}{E}, \\ \nabla^2 &= \frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{n^2}{\xi^2}. \end{aligned} \quad (8)$$

Equations (6) and (7) are the parametric Bessel equations and their general solutions are attainable in terms of the Bessel functions as [18]

$$\Phi = \{A_n J_n(\lambda_1 \xi) + B_n Y_n(\lambda_1 \xi)\}, \quad (9)$$

$$\Psi = \{C_n J_n(\lambda_2 \xi) + D_n Y_n(\lambda_2 \xi)\}, \quad (10)$$

where J_n and Y_n are the Bessel functions of the first and second kind of order n , respectively, and A_n, B_n, C_n , and D_n are the deflection coefficients.

The radial and circumferential displacements can then be expressed in terms of the Bessel functions by substituting for

Φ and Ψ in (2) and (3). The resulting expressions for the radial and circumferential displacements can be expressed as:

$$u_r = \frac{1}{b} \left\{ A_n X_n(\lambda_1 \xi) + B_n Z_n(\lambda_1 \xi) + \frac{n}{\xi} [C_n J_n(\lambda_2 \xi) + D_n Y_n(\lambda_2 \xi)] \right\} \cos n\theta \sin \omega t, \quad (11)$$

$$u_\theta = -\frac{1}{b} \left\{ C_n X_n(\lambda_2 \xi) + D_n Z_n(\lambda_2 \xi) + \frac{n}{\xi} [A_n J_n(\lambda_1 \xi) + B_n Y_n(\lambda_1 \xi)] \right\} \sin n\theta \sin \omega t,$$

where

$$X_n(\lambda_i \xi) = \frac{\partial}{\partial \xi} J_n(\lambda_i \xi) = -\frac{n}{\xi} J_n(\lambda_i \xi) + \lambda_i J_{n-1}(\lambda_i \xi),$$

$$Z_n(\lambda_i \xi) = \frac{\partial}{\partial \xi} Y_n(\lambda_i \xi) = \left(-\frac{n}{\xi} Y_n(\lambda_i \xi) + \lambda_i Y_{n-1}(\lambda_i \xi) \right),$$

$$i = 1, 2. \quad (12)$$

2.1. Free and Clamped Boundary Conditions. Equations (11) represent the solutions for distributions of the radial and circumferential displacements for the general case of an annular disk. The evaluations of the natural frequencies and arbitrary deflection coefficients (A_n, B_n, C_n and D_n), however, necessitate the consideration of the in-plane free vibration response under different combinations of boundary conditions at the inner and the outer edges. For the annular disk clamped at the outer edge ($\xi = 1$), the application of boundary conditions ($u_r = 0$ and $u_\theta = 0$) must satisfy the following for the general solutions (11):

$$A_n X_n(\lambda_1) + B_n Z_n(\lambda_1) + n[C_n J_n(\lambda_2) + D_n Y_n(\lambda_2)] = 0, \quad (13)$$

$$C_n X_n(\lambda_2) + D_n Z_n(\lambda_2) + n[A_n J_n(\lambda_1) + B_n Y_n(\lambda_1)] = 0. \quad (14)$$

In a similar manner, the solution must satisfy the following for the clamped inner edge ($\xi = \beta$), where $\beta = a/b$ is the radius ratio between inner and outer radii of the disk:

$$A_n X_n(\lambda_1 \beta) + B_n Z_n(\lambda_1 \beta) + \frac{n}{\beta} [C_n J_n(\lambda_2 \beta) + D_n Y_n(\lambda_2 \beta)] = 0, \quad (15)$$

$$C_n X_n(\lambda_2 \beta) + D_n Z_n(\lambda_2 \beta) + \frac{n}{\beta} [A_n J_n(\lambda_1 \beta) + B_n Y_n(\lambda_1 \beta)] = 0. \quad (16)$$

The conditions involving at the free edges are satisfied when the radial (N_r) and circumferential ($N_{r\theta}$) in-plane forces at the edge are zero [11], such that

$$N_r = \frac{-Eh}{(1-\nu^2)b} \left[\frac{\partial u_r}{\partial \xi} + \frac{\nu}{\xi} \frac{\partial u_\theta}{\partial \theta} + \frac{\nu}{\xi} u_\xi \right] = 0, \quad (17)$$

$$N_{r\theta} = \frac{-Eh}{2(1+\nu)b} \left[\frac{1}{\xi} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \xi} - \frac{u_\theta}{\xi} \right] = 0. \quad (18)$$

A direct substitution of u_r and u_θ from (11) in the above equations would result in second derivatives of the Bessel functions. Alternatively, the above equation for the boundary conditions may be expressed in terms of Φ and Ψ through direct substitution of u_r and u_θ from (2) and (3), respectively. The boundary conditions in terms of N_r can thus be obtained as

$$\left(\frac{\partial^2 \Phi}{\partial \xi^2} + \frac{n}{\xi} \frac{\partial \Psi}{\partial \xi} - \frac{n}{\xi^2} \Psi \right) - \frac{n^2 \nu}{\xi^2} \Phi - \frac{n \nu}{\xi} \frac{\partial \Psi}{\partial \xi} + \frac{\nu}{\xi} \left(\frac{\partial \Phi}{\partial \xi} + \frac{n}{\xi} \Psi \right) = 0. \quad (19)$$

Rearranging (19) results in

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\nu}{\xi} \frac{\partial}{\partial \xi} - \frac{n^2 \nu}{\xi^2} \right) \Phi + (1-\nu) \left(\frac{n}{\xi} \frac{\partial}{\partial \xi} - \frac{n}{\xi^2} \right) \Psi = 0. \quad (20)$$

The second order derivative term ($\partial^2 \Phi / \partial \xi^2$) in (20) can be eliminated by adding and subtracting the term $((1/\xi)(\partial/\partial \xi) - (n^2/\xi^2) + \lambda_1^2)\Phi$, which yields

$$\left[\left(\frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{n^2}{\xi^2} + \lambda_1^2 \right) - \left(\frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{n^2}{\xi^2} + \lambda_1^2 \right) + \frac{\nu}{\xi} \frac{\partial}{\partial \xi} - \frac{n^2 \nu}{\xi^2} \right] \Phi + (1-\nu) \left(\frac{n}{\xi} \frac{\partial}{\partial \xi} - \frac{n}{\xi^2} \right) \Psi = 0. \quad (21)$$

From (6), it can be seen that the terms within the first parenthesis are identically equal to zero. Equation (21) describing the boundary condition associated with N_r can be further simplified upon substitutions for $\lambda_1^2 = \lambda_2^2(1-\nu)/2$, which yields

$$\left(\frac{n^2}{\xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{1}{2} \lambda_2^2 \right) \Phi + n \left(\frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{1}{\xi^2} \right) \Psi = 0. \quad (22)$$

Similarly, the boundary condition equation associated with $N_{r\theta}$ (18) can be simplified as

$$\frac{2n}{\xi} \left(-\frac{\partial}{\partial \xi} + \frac{1}{\xi} \right) \Phi + \left(\frac{2}{\xi} \frac{\partial}{\partial \xi} - \frac{2n^2}{\xi^2} + \lambda_2^2 \right) \Psi = 0. \quad (23)$$

Upon substitutions for Φ and Ψ from (16) in (26) and (27), the boundary condition equations for the free edges are obtained, which involve only first derivatives of the Bessel functions. For an annular disk with free inner and outer edges, (22) and (23) represent the conditions at both the inner and the outer boundaries ($\xi = 1$ and $\xi = \beta$). The equations for the free edge boundary conditions can be expressed in the matrix form in the four deflection coefficients, as

TABLE 1: Frequency equations for the solid disks corresponding to free and clamped edge conditions.

Boundary conditions at $\xi = 1$		Clamped	Free
$n = 0$	Radial	$J_1(\lambda_1) = 0$	$\lambda_1 J_0(\lambda_1) = (1 - \nu) J_1(\lambda_1)$
	Circumferential	$J_1(\lambda_2) = 0$	$\lambda_2 J_0(\lambda_2) = 2 J_1(\lambda_2)$
	$n = 1$	$X_n(\lambda_1) X_n(\lambda_2) = n^2 J_n(\lambda_1) J_n(\lambda_2)$	$[X_1(\lambda_2) J_1(\lambda_1)] + [X_1(\lambda_1) J_1(\lambda_2)] = (2 - (\lambda_2^2/2)) J_1(\lambda_2) J_1(\lambda_1)$
	$n > 1$		$[X_n(\lambda_2) - Q J_n(\lambda_2)][X_n(\lambda_1) - Q J_n(\lambda_1)] = n^2(Q - 1)^2$ where $Q = \lambda_2^2/2(n^2 - 1)$

$$\begin{bmatrix}
\left(n^2 - \frac{\lambda_2^2}{2}\right) J_n(\lambda_1) - X_n(\lambda_1) & \left(n^2 - \frac{\lambda_2^2}{2}\right) Y_n(\lambda_1) - Z_n(\lambda_1) & n X_n(\lambda_2) - n J_n(\lambda_2) & n Z_n(\lambda_2) - n Y_n(\lambda_2) \\
2n J_n(\lambda_1) - 2n X_n(\lambda_1) & 2n Y_n(\lambda_1) - 2n Z_n(\lambda_1) & 2X_n(\lambda_2) - (2n^2 - \lambda_2^2) J_n(\lambda_2) & 2Z_n(\lambda_2) - (2n^2 - \lambda_2^2) Y_n(\lambda_2) \\
\left(\frac{n^2}{\beta^2} - \frac{\lambda_2^2}{2}\right) J_n(\lambda_1 \beta) - \frac{1}{\beta} X_n(\lambda_1 \beta) & \left(\frac{n^2}{\beta^2} - \frac{\lambda_2^2}{2}\right) Y_n(\lambda_1 \beta) - \frac{1}{\beta} Z_n(\lambda_1 \beta) & \frac{n}{\beta} X_n(\lambda_2 \beta) - \frac{n}{\beta^2} J_n(\lambda_2 \beta) & \frac{n}{\beta} Z_n(\lambda_2 \beta) - \frac{n}{\beta^2} Y_n(\lambda_2 \beta) \\
\frac{2n}{\beta^2} J_n(\lambda_1 \beta) - \frac{2n}{\beta} X_n(\lambda_1 \beta) & \frac{2n}{\beta^2} Y_n(\lambda_1 \beta) - \frac{2n}{\beta} Z_n(\lambda_1 \beta) & \frac{2}{\beta} X_n(\lambda_2 \beta) - \left(\frac{2n^2}{\beta^2} - \lambda_2^2\right) J_n(\lambda_2 \beta) & \frac{2}{\beta} Z_n(\lambda_2 \beta) - \left(\frac{2n^2}{\beta^2} - \lambda_2^2\right) Y_n(\lambda_2 \beta)
\end{bmatrix} \quad (24)$$

$$\times \begin{Bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{Bmatrix} = \{0\}.$$

The determinant of the above matrix yields the frequency equation for the annular disk with free inner and outer edge conditions.

For the clamped inner and outer edges, the equations for the boundary conditions can be obtained directly from (13) to (16), such that

$$\begin{bmatrix}
X_n(\lambda_1) & Z_n(\lambda_1) & n J_n(\lambda_2) & n Y_n(\lambda_2) \\
n J_n(\lambda_1) & n Y_n(\lambda_1) & X_n(\lambda_2) & Z_n(\lambda_2) \\
X_n(\lambda_1 \beta) & Z_n(\lambda_1 \beta) & \frac{n}{\beta} J_n(\lambda_2 \beta) & \frac{n}{\beta} Y_n(\lambda_2 \beta) \\
\frac{n}{\beta} J_n(\lambda_1 \beta) & \frac{n}{\beta} Y_n(\lambda_1 \beta) & X_n(\lambda_2 \beta) & Z_n(\lambda_2 \beta)
\end{bmatrix} \begin{Bmatrix} A_n \\ B_n \\ C_n \\ D_n \end{Bmatrix} = \{0\}. \quad (25)$$

In the above equations, (24) and (25), the top two rows describe the boundary condition at the outer edge, while the bottom two rows are associated with those at the inner edge. The equations for the boundary conditions involving combinations of free and clamped edges can thus be directly obtained from the above two equations. For the free inner edge and clamped outer edge, denoted as “free-clamped” condition, the matrix equation comprises the top two rows of the matrix in (25) and the lower two rows from (24). For the clamped inner edge and free outer edge, denoted as “Clamped-Free” condition, the matrix equation is formulated in the similar manner using the lower and upper two rows from (25) and (24), respectively.

The in-plane vibration analysis of a solid disk can be shown as a special case of the above generalized formulations. Upon eliminating the coefficients associated with Bessel function of the second kind, Equations (24) and (25) reduce to those reported by Onoe [9] for free solid disk and by Park [12] for the clamped solid disk. The frequency equation corresponding to different values of n for the solid disks

involving the two boundary conditions are summarized in Table 1, where $X_n(\lambda_1)$ is the derivative of the Bessel function J_n evaluated at the outer edge ($\xi = 1$). For annular disks, simplified frequency equations can be obtained for the axisymmetric modes. These equations were expressed in Table 2 for the four combinations of boundary conditions.

2.2. Flexible Boundary Conditions. In the above analysis, the boundary conditions considered are either clamped or free. However, Flexible boundary conditions may be considered more representative of many practical situations. The proposed formulations can be further employed to study the in-plane vibration of solid as well as annular disks with flexible boundary conditions. Artificial springs may be applied to describe the flexible boundary conditions at the inner or the outer edge of an annular disk. A number of studies on the analysis of out-of-plane vibration characteristics of circular plates and cylindrical shells have employed uniformly distributed artificial springs around the edge to represent a flexible boundary conditions or a flexible joint [19–22].

The effects of flexible boundary conditions on the in-plane free vibration of circular disks have been considered in a recent study by the authors [23] using the Rayleigh-Ritz approach. Artificial springs, distributed along the radial and circumferential directions at the free outer and/or inner edges, were considered to simulate for flexible boundary conditions. The exact solution of the frequency equations for the disk with flexible supports can be attained from (11) together with the consideration of the flexible boundary conditions. The conditions involving flexible edge supports at the inner and outer edges are satisfied when the radial (N_r) and circumferential ($N_{r\theta}$) in-plane forces at the edges

TABLE 2: Frequency equations of axisymmetric modes for annular disks.

Boundary conditions		radial	Circumferential
inner	outer		
Clamped	Clamped	$J_1(\lambda_1)Y_1(\lambda_1\beta) - J_1(\lambda_1\beta)Y_1(\lambda_1) = 0$	$J_1(\lambda_2)Y_1(\lambda_2\beta) - J_1(\lambda_2\beta)Y_1(\lambda_2) = 0$
Free	Free	$[(-\lambda_1/(1-\nu))J_0(\lambda_1) + J_1(\lambda_1)][(-\lambda_1/(1-\nu))Y_0(\lambda_1\beta) + (1/\beta)Y_1(\lambda_1\beta)] - \lambda_1/(1-\nu)Y_0(\lambda_1) + Y_1(\lambda_1)[(-\lambda_1/(1-\nu))J_0(\lambda_1\beta) + (1/\beta)J_1(\lambda_1\beta)] = 0$	$[-2J_1(\lambda_2) + \lambda_2 J_0(\lambda_2)][-(2/\beta)Y_1(\lambda_2\beta) + \lambda_2 Y_0(\lambda_2\beta)] - [-(2/\beta)J_1(\lambda_2\beta) + \lambda_2 J_0(\lambda_2\beta)][-2Y_1(\lambda_2) + \lambda_2 Y_0(\lambda_2)] = 0$
Clamped	Free	$[(\lambda_1/(1-\nu))J_0(\lambda_1) - J_1(\lambda_1)]Y_1(\lambda_1\beta) - J_1(\lambda_1\beta)[(\lambda_1/(1-\nu))Y_0(\lambda_1) - Y_1(\lambda_1)] = 0$	$[2J_1(\lambda_2) - \lambda_2 J_0(\lambda_2)]Y_1(\lambda_2\beta) - J_1(\lambda_2\beta)[2Y_1(\lambda_2) - \lambda_2 Y_0(\lambda_2)] = 0$
Free	Clamped	$J_1(\lambda_1)[(-\lambda_1/(1-\nu))Y_0(\lambda_1\beta) + (1/\beta)Y_1(\lambda_1\beta)] + [(-\lambda_1/(1-\nu))J_0(\lambda_1\beta) + (1/\beta)J_1(\lambda_1\beta)]Y_1(\lambda_1) = 0$	$J_1(\lambda_2)[-(2/\beta)Y_1(\lambda_2\beta) + \lambda_2 Y_0(\lambda_2\beta)] - [-(2/\beta)J_1(\lambda_2\beta) + \lambda_2 J_0(\lambda_2\beta)]Y_1(\lambda_2) = 0$

TABLE 3: Exact frequency parameters of in-plane vibration of a solid disk with free edge ($\nu = 0.3$).

Mode	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	1.6176	1.3877	2.1304	2.7740
2	3.5291	2.5112	3.4517	4.4008
3	4.0474	4.5208	5.3492	6.1396
4	5.8861	5.2029	6.3695	7.4633
5	6.9113	6.7549	7.6186	8.5007
6	7.7980	8.2639	9.3470	10.2350
7	9.6594	8.7342	9.8366	11.0551

TABLE 4: Exact frequency parameters of in-plane vibration of a solid disk with clamped edge ($\nu = 0.33$).

Mode	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	1.9441	3.0185	3.0185	4.7021
2	3.1126	4.0127	4.0127	5.8985
3	4.9104	5.7398	5.7398	7.3648
4	5.3570	6.7079	6.7079	8.9816
5	6.7763	7.6442	7.6442	9.5296
6	8.4938	9.4356	9.4356	11.1087
7	8.6458	9.9894	9.9894	12.5940

are equal to the respective radial and circumferential spring forces, such that

$$N_r = \frac{-Eh}{(1-\nu^2)b} \left[\frac{\partial u_r}{\partial \xi} + \frac{\nu}{\xi} \frac{\partial u_\theta}{\partial \theta} + \frac{\nu}{\xi} u_r \right] = K_r u_r(\xi, \theta),$$

$$N_{r\theta} = \frac{-Eh}{2(1+\nu)b} \left[\frac{1}{\xi} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial \xi} - \frac{u_\theta}{\xi} \right] = K_\theta u_\theta(\xi, \theta), \quad (26)$$

where K_r and K_θ are the radial and circumferential stiffness coefficients, respectively. Introducing the nondimensional stiffness parameters, $\bar{K}_r = K_r b(1-\nu^2)/Eh$ and $\bar{K}_\theta = K_\theta b(1+\nu)/Eh$, (26) can be written as

$$\left(\frac{n^2}{\xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{1}{2} \lambda_2^2 + \bar{K}_r \frac{\partial}{\partial \xi} \right) \Phi$$

$$+ n \left(\frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{1}{\xi^2} + \frac{1}{\xi} \bar{K}_r \right) \Psi = 0,$$

$$n \left(\frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{1}{\xi^2} - \frac{1}{\xi} \bar{K}_\theta \right) \Phi$$

$$+ \left(\frac{n^2}{\xi^2} - \frac{1}{\xi} \frac{\partial}{\partial \xi} - \frac{1}{2} \lambda_2^2 - \bar{K}_\theta \frac{\partial}{\partial \xi} \right) \Psi = 0. \quad (27)$$

The application of the above conditions yields the matrix equations for the disk with flexible supports at the inner and outer edges, similar to (24). The frequency parameters are subsequently obtained through solution of the matrix

equations. The above boundary equations reduce to those in (22) and (23) for the free edge conditions by letting $\bar{K}_r = 0$ and $\bar{K}_\theta = 0$. Furthermore, the clamped edge condition can be represented by considering infinite values of $\bar{K}_r$ and $\bar{K}_\theta$. Equations (27) further show that the flexible edge support conditions involve combinations of the free and clamped edge conditions.

3. Results

The frequency parameters (λ_1) derived for different combinations of boundary conditions are compared with those reported in different studies to demonstrate the validity of the proposed formulation. For this purpose, the frequency parameters of a solid disk with free and clamped outer edge are initially evaluated and compared with those reported by Holland [10] and Park [12], respectively. The results presented in Tables 3 and 4 for the free and clamped outer edge conditions, respectively, were found to be identical to those reported in [10, 12] for solid disks with clamped edge.

The exact frequency parameters for the annular disks were subsequently obtained under different combinations of boundary conditions at the inner and outer edges. The edge conditions are presented for the inner followed by that of the outer edge. For instance, a “Free-Clamped” condition refers to free inner edge and clamped outer edge. The solutions obtained for conditions involving free and clamped edges (“Free-Free”, “Free-Clamped”, “Clamped-Clamped”, and “Clamped-Free”) are compared with those reported

TABLE 5: Frequency parameters of in-plane vibration of an annular disk with “Free-Free” conditions.

Radius ratio (β)	$n = 1$		$n = 2$		$n = 3$		$n = 4$	
	Reference [15]	present study	Reference [15]	Present study	Reference [15]	Present study	Reference [15]	Present study
0.2	1.652	1.651	1.110	1.110	2.071	2.071	2.767	2.766
	3.842	3.842	2.403	2.402	3.401	3.400	4.389	4.388
0.4	1.683	1.682	0.721	0.721	1.618	1.619	2.482	2.482
	4.044	4.044	2.451	2.450	3.346	3.346	4.227	4.226

TABLE 6: Frequency parameters of in-plane vibration of an annular disk with “Free-Clamped” conditions.

Radius ratio (β)	$n = 1$		$n = 2$		$n = 3$		$n = 4$	
	Reference [15]	Present study	Reference [15]	Present study	Reference [15]	Present study	Reference [15]	Present study
0.2	2.104	2.103	2.553	2.553	3.688	3.688	4.712	4.711
	3.303	3.302	3.948	3.948	4.859	4.858	5.894	5.893
0.4	2.517	2.517	2.721	2.721	3.214	3.214	3.955	3.956
	3.508	3.508	4.147	4.147	4.998	4.998	5.874	5.873

TABLE 7: Frequency parameters of in-plane vibration of an annular disk with “Clamped-Clamped” conditions.

Radius ratio (β)	$n = 1$		$n = 2$		$n = 3$		$n = 4$	
	Reference [15]	present study	Reference [15]	present study	Reference [15]	present study	Reference [15]	present study
0.2	2.783	2.783	3.378	3.378	4.066	4.065	4.802	4.800
	4.060	4.060	4.360	4.359	5.104	5.103	6.003	6.001
0.4	3.429	3.429	4.023	4.022	4.707	4.707	5.287	5.286
	5.306	5.306	5.311	5.311	5.619	5.619	6.289	6.288

TABLE 8: Frequency parameters of in-plane vibration of an annular disk with “Clamped-Free” conditions.

Radius ratio (β)	$n = 1$		$n = 2$		$n = 3$		$n = 4$	
	Reference [15]	present study	Reference [15]	present study	Reference [15]	present study	Reference [15]	present study
0.2	0.919	0.919	1.542	1.541	2.157	2.157	2.778	2.777
	2.121	2.121	2.605	2.604	3.473	3.472	4.408	4.406
0.4	1.281	1.281	1.965	1.964	2.445	2.445	2.911	2.911
	2.691	2.691	2.908	2.907	3.604	3.603	4.492	4.491

TABLE 9: Frequency parameters of in-plane vibration of an annular disk with flexible boundary conditions.

Boundary conditions	$n = 1$		$n = 2$		$n = 3$		$n = 4$	
	Reference [19]	present study	Reference [19]	present study	Reference [19]	present study	Reference [19]	present study
Elastic-Free	0.771	0.771	1.408	1.408	2.121	2.121	2.772	2.772
	1.906	1.906	2.524	2.524	3.444	3.444	4.397	4.397
Elastic-Clamped	2.590	2.590	3.117	3.116	3.928	3.926	4.759	4.760
	3.625	3.625	4.134	4.136	5.000	5.001	5.957	5.957
Elastic-Elastic	1.494	1.492	1.815	1.813	2.474	2.472	3.123	3.120
	2.603	2.601	3.209	3.207	4.004	4.002	4.859	4.857
Free-Elastic	1.040	1.046	1.505	1.504	2.414	2.416	3.114	3.116
	2.432	2.432	3.018	3.018	3.895	3.896	4.833	4.834
Clamped-Elastic	1.686	1.686	1.960	1.959	2.514	2.512	3.129	3.127
	2.764	2.765	3.319	3.317	4.061	4.060	4.879	4.877

by Irie et al. [15] in Tables 5, 7, 8, 9, respectively. The simulations results were obtained for two different values of the radial ratios ($\beta = 0.2$ and 0.4), and $\nu = 0.3$. The results show excellent agreements of the values obtained in the present study with those reported in [15], irrespective of the boundary condition and radius ratio considered.

The exact frequency parameters of the annular disk are further investigated for boundary conditions involving different combinations of free, clamped, and elastic edges. The solutions corresponding to selected modes are obtained for ("Elastic-Free", "Elastic-Clamped", "Elastic-Elastic", "Free-Elastic", and "Clamped-Elastic") conditions are presented in Table 9. The results were attained for $\beta = 0.2$ and $\nu = 0.3$. The nondimensional radial and circumferential stiffness parameters were chosen as $\bar{K}_r = 1$ and $\bar{K}_\theta = 1$. The results are also compared with those obtained using the Raleigh-Ritz methods, as reported in [23]. The comparisons reveal very good agreements between the analytical and the reported results irrespective of the boundary condition considered. The results suggest that the proposed frequency equations could serve as the reference for approximate methods on in-plane vibration characteristics of the annular disks with different combinations of edge conditions.

4. Conclusions

The characteristics of in-plane vibration for circular disks are investigated under different combinations of boundary conditions. The governing equations are solved to obtain the exact frequency equation of solid and annular disks. Frequency equations are presented for different combinations of boundary conditions, including flexible boundaries, at the inner and outer edges. The nondimensional frequency parameters obtained by the present approach compare very well with those available in literature, irrespective of the boundary condition and radius ratio considered. The exact frequency parameters can serve as a reference to assess the accuracy of approximate methods. The presented frequency equations can be numerically evaluated to obtain the in-plane modal characteristics of circular disk for a wide range of constraints conditions and geometric parameters.

Nomenclature

A_n, B_n, C_n, D_n :	Deflection coefficients of the exact solution
a :	Inner radius of the annular disk
b :	Outer radius of the annular disk
C_L^2 :	$E/\rho(1 - \nu^2)$
C_T^2 :	$E/2\rho(1 + \nu)$
E :	Young's modulus of disk
h :	Thickness of the annular disk
$J_n(\xi)$:	Bessel function of the first kind of order n
K_r, K_θ :	Radial and circumferential stiffness coefficients
$\bar{K}_r, \bar{K}_\theta$:	Nondimensional radial and circumferential stiffness parameters
n :	Nodal diameter number
$N_r, N_{r\theta}$:	Radial and circumferential in-plane forces

r :	Radial coordinate
u_r, u_θ, u_z :	Radial, circumferential, and normal displacements of the disk
$Y_n(\xi)$:	Bessel function of the second kind of order n
z :	Normal coordinate
β :	Radius ratio
θ :	Circumferential coordinate
λ :	Nondimensional frequency parameters
ν :	Poisson's ratio
ξ :	Nondimensional radial coordinate of the disk
ρ :	Mass density of the disk
ϕ, ψ :	Lamé Potentials
Φ, Ψ :	Radial variations of Lamé Potentials
ω :	Radian natural frequency.

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