Research Article

Free and Forced Vibrations of Elastically Connected Structures

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1. Introduction

The general theory for the free and forced response of strings, shafts, beams, and axially loaded beams is well documented [1–8]. Investigators have examined the free and forced response of elastically connected strings [9, 10], Euler-Bernoulli beams [11–14], and Timoshenko beams [15]. These analyses focused on a pair of elastically connected structures using a normal-mode solution for the free response and a modal analysis for the forced response. Each of these papers uses a normal-mode solution or a modal analysis specific to the problem to obtain a solution.

Ru [16–18] proposed that model for multiwalled carbon nanotubes to be modeled by elastically connected structures with the elastic layers representing interatomic vanDer Waals forces. Ru [16] proposed a model of concentric beams connected by elastic layers to model buckling of carbon nanotubes and elastic shell models [17, 18]. Yoon et al. [19] and Li and Chou [20] modeled free vibrations of multiwalled nanotubes by a series of concentric elastically connected Euler-Bernoulli beams, while Yoon et al. [21, 22] modeled nanotubes as concentric Timoshenko beams connected by an elastic layer. Xu et al. [23] modeled the nonlinearity of the vanDer Waals forces. Elishakoff and Pentaras [24] gave approximate formulas for the natural frequencies of double-walled nanotubes noting that if developed from the eigenvalue relation, the computations can be computationally intensive and difficult.

Kelly and Srinivas [25] developed a Rayleigh-Ritz method for elastically connected stretched structures. This paper develops a general theory within which a finite set of parallel structures connected by elastic layers of a Winkler type can be analyzed. The theory shows that the determination of the natural frequencies for uniform parallel structures such as shafts and Euler-Bernoulli beams can be reduced to matrix eigenvalue problems. The general theory is also used to develop a modal analysis for forced response of a set of parallel structures.

2. Problem Formulation

The problem considered is that of \( \text{n} \) structural elements in parallel but connected by elastic layers. Each elastic layer is modeled by a Winkler foundation, a layer of distributed stiffness across the span of the element. For generality, it is assumed that the outermost structures are connected to fixed foundations through elastic layers, as illustrated in Figure 1.
Each layer has a uniform stiffness per unit length \( k_i \); \( i = 0, 1, 2, \ldots, n \).

Let \( w_i(x) \) represent the displacement of the \( i \)th structure. If isolated from the system, the nondimensional differential equation governing the time-dependent motion of this structure is written as

\[
L_i w_i(x) + M_i \ddot{w}_i = G_i(x, t),
\]

(1)

where \( L_i \) is the stiffness operator for the element, \( M_i \) is an inertia operator for the element, and \( G_i(x, t) \) is the force per unit length acting on the structure which includes the forces from the elastic layer as well as any externally applied forces. The stiffness operator is a differential operator of order \( k \) \((k = 2 \) for strings and shafts, \( k = 4 \) for Euler-Bernoulli beams\), where the inertia operator is a function of the independent variable \( x \).

Each structure has the same end supports, and therefore their differential equations are subject to the same boundary conditions. Let \( S \) be the subspace of \( C^k[0,1] \) defined by the boundary conditions; all elements in \( S \) satisfy all boundary conditions.

The external forces acting on the \( i \)th structure are

\[
G_i(x, t) = \begin{cases} 
-\lambda_0 w_1 - \lambda_1 (w_1 - w_2) + F_1(x, t) & i = 1, \\
-\lambda_{i-1} (w_i - w_{i-1}) & i = 2, 3, \ldots, n-1, \\
-\lambda_{n-1} (w_n - w_{n-1}) - \lambda_n w_n + F_n(x, t) & i = n,
\end{cases}
\]

(2)

where \( \lambda_i, i = 0, 1, \ldots, n \) are the nondimensional stiffness coefficients connecting the \( i \)th and \( i+1 \)st structures. Substitution of (2) into (1) leads to a coupled set of differential equations which are written in a matrix form as

\[
(K + K_c) \mathbf{W} + M \mathbf{W} = \mathbf{F},
\]

(3)

where \( \mathbf{W} = \begin{bmatrix} w_1(x, t) & w_2(x, t) & \cdots & w_n(x, t) \end{bmatrix}^T \), \( \mathbf{F} = \begin{bmatrix} F_1(x, t) & F_2(x, t) & \cdots & F_n(x, t) \end{bmatrix}^T \), \( K \) is an \( n \times n \) diagonal operator matrix with \( k_{ii} = L_i \), \( M \) is an \( n \times n \) diagonal mass matrix with \( m_{ii} = M_i \), and \( K_c \) is a tridiagonal \( n \times n \) stiffness coupling matrix with

\[
(k_c)_{i,i-1} = -\lambda_{i-1} & i = 2, 3, \ldots, \\
(k_c)_{i,i} = \lambda_{i-1} + \lambda_i & i = 1, 2, \ldots, n, \\
(k_c)_{i,i+1} = -\lambda_i & i = 1, 2, \ldots, n-1.
\]

(4)

The vector \( \mathbf{W} \) is an element of the vector space \( U = S \times \mathbb{R}^n \); an element of \( U \) is an \( n \)-dimensional vector, whose elements all belong to \( S \).

### 3. General Theory

Let \( f(x) \) and \( g(x) \) be arbitrary elements of \( S \). A standard inner product on \( S \) is defined as

\[
(f, g)_S = \int_0^1 f(x) g(x) dx.
\]

(5)

If the stiffness operator is self-adjoint \([L_i f, g]_S = (f, L_i g)_S\) and positive definite \([L_i f, f]_S \geq 0 \) and \([L_i f, f]_S = 0 \) if and only if \( f = 0 \), then a potential energy inner product is defined as

\[
(f, g)_L = (L_i f, g)_S.
\]

(6)

Clearly each \( M_i \) is positive definite and self-adjoint with respect to the standard inner product, and thus a kinetic energy inner product can be defined as

\[
(f, g)_M = \int_0^1 (M_i f) g dx,
\]

(7)

for any \( f \) and \( g \) in \( S \).

The standard inner product on \( U \) is defined as

\[
(f, g)_U = \int_0^1 g^T f dx,
\]

(8)

for any \( f \) and \( g \) in \( U \). It is easy to show that \( M \) is self-adjoint with respect to the inner product of (8) and a kinetic energy inner product on \( U \) is defined as

\[
(f, g)_M = (M f, g)_U.
\]

(9)

Define \( \tilde{K} = K + K_c \). Since \( K \) is self-adjoint with respect to the standard inner product on \( S \) and \( K_c \) is a symmetric matrix, it can be shown that \( \tilde{K} \) is self-adjoint with respect to the standard inner product on \( U \). The positive definiteness of \( \tilde{K} \) with respect to the standard inner product on \( U \) is determined by considering \((\tilde{K}_f, f)_U = (K f, f)_U + (K_c f, f)_U \).

If \( K \) is a positive definite matrix with respect to the standard inner product on \( \mathbb{R}^n \), then \((K f, f)_U \geq 0 \) and \((K f, f)_U = 0 \) if and only if \( f = 0 \). If each of the operators \( L_i, i = 1, 2, \ldots, n \) is positive definite with respect to the standard inner product on \( C^k[0,1] \) then \((K f, f)_U \geq 0 \) and \((K f, f)_U = 0 \) if and only if \( f = 0 \). Thus, \( \tilde{K} \) is positive definite with respect to the standard inner product on \( U \) if either \( K_c \) is a positive definite
matrix with respect to the standard inner product on $R^n$ or each of the operators $L_i, i = 1, 2, \ldots, n$ is positive definite with respect to the standard inner product on $C^0[0, 1]$. Under either of these conditions, a potential energy inner product is defined on $U$ by

$$ (f, g)_K = (Kf, g)_U. $$

(10)

The operator $K$ is not positive definite only when the structures are unrestrained and $\lambda_0 = 0$ and $\lambda_n = 0$.

Define $D = M^{-1}K$. It is possible to show that $D$ is self-adjoint with respect to the kinetic energy inner product of (9) and the potential energy inner product of (10). If $K$ is positive definite with respect to the standard inner product, then $D$ is positive definite with respect to both inner products.

4. Free Response

First consider the free response of the structures, $F = 0$. A normal-mode solution is assumed as

$$ W = we^{i\omega t}, $$

(11)

where $\omega$ is a natural frequency and $w = \begin{bmatrix} w_1(x) & w_2(x) & \cdots & w_{n-1}(x) & w_n(x) \end{bmatrix}^T$ is a vector of mode shapes corresponding to that natural frequency. Substitution of (11) into (3) leads to

$$ M^{-1}(K + K_c)w = \omega^2 w, $$

(12)

where the partial derivatives have been replaced by ordinary derivatives in the definition of $K$. From (12), it is clear that the natural frequencies are the square roots of the eigenvalues of $D = M^{-1}(K + K_c)$, and the mode shape vectors are the corresponding eigenvectors.

It is well known [18] that eigenvalues of a self-adjoint operator are all real and that eigenvectors corresponding to distinct eigenvalues are orthogonal with respect to the inner product for which the operator is self-adjoint. Thus, if $\omega_i$ and $\omega_j$ are distinct natural frequencies with corresponding mode shape vectors $w_i$ and $w_j$, respectively, then

$$ (w_i, w_j)_M = 0, $$

(13)

$$ (w_i, w_j)_K = 0. $$

(14)

The mode shape vectors can be normalized by requiring

$$ (w_i, w_i)_M = 1, $$

(15)

which then leads to

$$ (w_i, w_j)_K = \omega_i^2 \delta_{ij}. $$

5. Forced Response

Since $D$ is a self-adjoint operator, its eigenvectors, the mode shape vectors, can be shown to be complete in $U$. An expansion theorem then implies that for any $f$ in $U$ there exists coefficients $\alpha_i, i = 1, 2, \ldots, n$, such that

$$ f = \sum_i \alpha_i w_i, $$

(16)

where $w_i$ are the normalized mode shape vectors, and the summation is carried out over all modes. For a given $f$ in $U$, the coefficients are calculated by

$$ \alpha_i = (f, w_i)_M. $$

(17)

Let $W(t)$ represent the response due to the force vector $F(t)$. Since $W$ must be in $U$, the expansion theorem may be applied at any $t$, leading to

$$ W(x, t) = \sum_i c_i(t)w_i(x). $$

(18)

Substitution of (18) into (3) results in

$$ \sum_i c_i(t)Mw_i + \sum_i c_i(t)Kw_i = F(x, t). $$

(19)

Taking the standard inner product on $U$ of both sides of (26) with $w_j(x)$ for an arbitrary $j = 1, 2, \ldots, n$ and using mode-shape orthogonality properties of (13)–(15) leads to an uncoupled set of differential equations of the form

$$ c_j(t) + \omega_j^2 c_j(t) = (F, w_j)_U. $$

(20)

A convolution integral solution of (20) is

$$ c_j(t) = \frac{1}{\omega_j} \int_0^t (F(x, \tau), w_j(x))_U \sin(\omega_j(t - \tau)) d\tau $$

(21)

$$ = \frac{1}{\omega_j} \int_0^t \int_0^1 w_j^T(x)F(x, \tau) \sin(\omega_j(t - \tau)) dxd\tau. $$

6. Uniform Structures with $L_1$ Proportional to $L_4$

A special case occurs when the structures are uniform and operators for the individual structural elements are proportional to one another,

$$ L_i = \mu L_1. $$

(22)

In this case, the component of the stiffness operator due to the elasticity of the structural elements becomes

$$ K = AL_1 = \begin{bmatrix} \mu_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \mu_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \mu_n \end{bmatrix} L_1. $$

(23)

The system may be nondimensionalized such that $M_1 = 1$. Then the differential equation governing the free response
of the first structural element, if isolated from the remainder of the system, is

$$\ddot{\Phi}_1 + L_1 \Phi_1 = 0.$$  \hfill (24)

A normal-mode solution of (22) of the form $\Phi(x, t) = \phi(x)e^{it}$ leads to the eigenvalue-eigenvector problem

$$L_1 \phi = \delta^2 \phi.$$  \hfill (25)

There are an infinite, but countable, number of natural frequencies for the system of (25), $\delta_1, \delta_2, \ldots, \delta_{k-1}, \delta_k, \delta_{k+1}, \ldots$ with corresponding mode shapes $\phi_1, \phi_2, \ldots, \phi_{k-1}, \phi_k, \phi_{k+1}, \ldots$. The mode shapes are normalized by requiring $(\phi_i, \phi_i) = 1$.

Assume a solution to (12) of the form

$$w = a\phi_k(x),$$  \hfill (26)

where $a$ is an $n \times 1$ vector of constants. Substitution of (26) into (12) using (24) and (25) leads to

$$M^{-1}\left(\delta_k^2 A + K_c\right)a = \omega^2a.$$  \hfill (27)

Equation (27) implies that, for this special case, the determination of the natural frequencies of the set of elastically connected structures is reduced to the determination of the eigenvalues of the matrix $M^{-1}(\delta_k^2 A + K_c)$. For each $k$ there are $n$ natural frequencies. The natural frequencies can thus be indexed as $\omega_{kj}$ for $k = 1, 2, \ldots$ and $j = 1, 2, \ldots, n$. The corresponding mode shapes are written as

$$w_{kj}(x) = a_{kj}\phi_k(x),$$  \hfill (28)

where $a_{kj}$ is the eigenvector of $M^{-1}(\delta_k^2 A + K_c)$ corresponding to the natural frequency $\omega_{kj}$.

The function $\phi_k(x)$ represents the $k$th mode shape of a single structure. For each $k$, there are $n$ natural frequencies and $n$ corresponding mode shapes. Such a set of mode shapes, which are referred to as intramodal modes, have the same spatial behavior, but their dependence across the structures varies. Two mode shapes $w_{kj}(x)$ and $w_{pk}(x)$ for which $p \neq k$ are referred to as intermodal modes.

Note that since $M$ and $\delta_k^2 A + K_c$ are both symmetric matrices, the matrix $M^{-1}(\delta_k^2 A + K_c)$ is self-adjoint with respect to a kinetic energy inner product. The intramodal mode shapes satisfy an orthogonality condition on $R^n$ given by

$$(a_{kj}, a_{\ell j})_M = (a_{kj})^T M a_{\ell j} = 0 \text{ for } i \neq j.$$  \hfill (29)

All mode shapes satisfy the orthogonality condition on $U$,

$$(w_{kj}, w_{\ell j})_M = \int_0^1 (a_{kj})^T M a_{\ell j} \phi_k(x) \phi_{\ell j}(x) dx$$

$$= 0 \text{ when } k \neq \ell \text{ or } i \neq j.$$  \hfill (30)

Consider the case when $K_c$ is singular ($\lambda_0 = 0$ and $\lambda_n = 0$). Then zero is the smallest eigenvalue of $K_c$ and $b$ is its corresponding eigenvector, $b = c\left[\begin{array}{c}1 \\ 1 \\ \vdots \\ 1\end{array}\right]$, where $c$ is a normalization constant. Suppose that $M^{-1}A = I$, then (27) becomes

$$\left(\delta_k^2 I + K_c\right)a = \omega^2a.$$  \hfill (31)

Note that the solution of (31) corresponds to $\omega = \delta_k$ and $a = b$. Thus, for this special case, the lowest natural frequency for each set of intramodal modes is equal to the natural frequency for the spatial mode of one structural element and its corresponding mode shape is the null space of $K_c$.

To examine the most general case when $K_c$ is singular, take the standard inner product for $R^n$ of both sides of (27) with $b$. Using properties of inner products and noting that $A$, $K_c$, and $M$ are symmetric leads to

$$\sum_{i=1}^n a_i (\delta_k^2 M_{ii} - \omega^2 \beta_i) = 0.$$  \hfill (32)

Application of the orthogonality condition of (32) leads to

$$\sum_{i=1}^n a_i (\delta_k^2 M_{ii} - \omega^2 \beta_i) = 0.$$  \hfill (33)

The expansion theorem is used to assume a forced response of the form

$$W(x, t) = \sum_{k=1}^{\infty} \sum_{j=1}^n c_{kj}(t) a_{kj} \phi_k(x).$$  \hfill (34)

Use of (34) in (3) leads to differential equations of the form

$$\ddot{c}_{kj} + \omega_k^2 c_{kj} = \left(F, a_{kj} \phi_k(x)\right)_U.$$  \hfill (35)

A special case occurs when uniform structures are identical such that $A = I$ and $M = I$. Then, (27) becomes

$$\left(\delta_k^2 I + K_c\right)a = \omega^2a.$$  \hfill (36)

Equation (36) can be rewritten as

$$K_c a = (\omega^2 - \delta_k^2)a.$$  \hfill (37)

Let $\kappa_j = 1, 2, \ldots, n$ be the eigenvalues of $K_c$. Then,

$$\omega_k^2 = \delta_k^2 + \kappa_j.$$  \hfill (38)

In addition, since the same set of eigenvalues is used to calculate the natural frequencies for each intramodal sets, the intramodal mode shape vectors are the same for each set and are the eigenvectors of $K_c$, that is

$$a_{kj} = \alpha_{kj} = v_j k, \quad k = 1, 2, \ldots, j = 1, 2, \ldots, n.$$  \hfill (39)

where $v_j$, $j = 1, 2, \ldots, n$ are the eigenvectors of $K_c$.  

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**Table 1:** First five sets of intramodal natural frequencies of four elastically connected fixed free shafts, $\omega_{k,j}$.

<table>
<thead>
<tr>
<th>j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1.5708</td>
<td>4.7124</td>
<td>7.8540</td>
<td>10.9956</td>
<td>14.1372</td>
</tr>
<tr>
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<td>1.5763</td>
<td>4.7142</td>
<td>7.8551</td>
<td>10.9964</td>
<td>14.1378</td>
</tr>
<tr>
<td>3</td>
<td>1.5965</td>
<td>4.7210</td>
<td>7.8592</td>
<td>10.9993</td>
<td>14.1400</td>
</tr>
<tr>
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<td>4.8247</td>
<td>7.9219</td>
<td>11.0442</td>
<td>14.1750</td>
</tr>
</tbody>
</table>
7. Nonuniform Structures

The case when one or more of the structures is nonuniform is more difficult in that the differential equations of (12) have variable coefficients. Consider the case when the structures are nonuniform, but identical. In this case, the coupled set of differential equations can be written as

$$ILw + K_c w - \omega^2 IMw = 0, \quad (40)$$

where $L$ and $M$ are the stiffness and inertia operators, respectively, for any of the structures.

Let $\kappa_1, \kappa_2, \ldots, \kappa_n$ be the eigenvalues of $K_c$ and let $z_1, z_2, \ldots, z_n$ be their corresponding eigenvectors normalized with respect to the standard inner product on $R^n$ ($z_i^T z_i = 1$). Let $P$ be the matrix whose columns are the normalized eigenvectors. Since $K_c$ is symmetric, it can be shown that $P^T P = I$ and $P^T K_c P = \Delta$, where $\Delta$ is a diagonal matrix with $\Delta_{ii} = \kappa_i$. Defining

$$w = Pq \quad (41)$$

and substituting into (40) and then premultiplying by $P^T$ leads to

$$ILq + \Delta q - \omega^2 IMq = 0. \quad (42)$$

Equation (42) represents a set of uncoupled differential equations, each of the form

$$Lq_j + \kappa_j q_j - \omega^2 Mq_j = 0 \quad j = 1, 2, \ldots, n. \quad (43)$$

Equation (43) represents, along with appropriate homogeneous boundary conditions, an eigenvalue problem to determine the natural frequencies. For each $j$, there are an infinite, but countable, number of natural frequencies. Thus, the natural frequencies can be indexed by $\omega_{j,k} \ j = 1, 2, \ldots, n \ k = 1, 2, \ldots$.

It still may not be possible to solve (43) in closed form; however, it is now known how to index the natural frequencies. For a set of identical structures, there are an infinite number of natural frequencies corresponding to each eigenvalue of $K_c$. The term intramodal is not appropriate for this set of natural frequencies as they do not correspond to the same mode. Indeed, there are not necessarily intramodal frequencies for the nonuniform case.

If $K_c$ is singular and thus has zero as its lowest eigenvalue, then (43) shows that one set of natural frequencies is identical to the natural frequencies of the individual structures.

8. Examples

8.1. Shafts. Consider $n$ concentric shafts of equal length connected by elastic layers. The stiffness and inertia operators for uniform shafts are, respectively,

$$L_i \theta_i = -\mu_i \frac{\partial^2 \theta_i}{\partial x^2},$$

$$M_i \theta_i = \beta_i \theta_i. \quad (44)$$

The stiffness operators are of the form of those considered in Section 6. Hence, the natural frequencies and mode shapes can be determined from solving a matrix eigenvalue problem. If all shafts are made of the same material then $\beta_i = \mu_i$ leading to $M^{-1}K = I$, and thus if $K_c$ is singular then the lowest natural frequency for each set of intramodal modes is the same as the modal natural frequency for the innermost shaft, and the mode shapes corresponding to each frequency are the eigenvector of $K_c$ corresponding to its zero eigenvalue times, the spatial mode shape for the shaft.

The eigenvalue problem for the innermost shaft is

$$\phi''(x) + \delta^2 \phi(x) = 0, \quad (45)$$

subject to appropriate boundary conditions. The values for $\delta_k k = 1, 2, \ldots, 5$ and the corresponding normalized mode shapes $\phi_k(x)$ are listed in Table 1 for various end conditions.

The matrix eigenvalue problem for a set of intramodal frequencies is of the form

$$Ka = \omega^2 Ma, \quad (46)$$

where $K$ is a tridiagonal matrix whose elements are

$$k_{ij} = \mu_i \delta^2 + \lambda_{j-1} + \lambda_j \quad i = 1, 2, \ldots, n,$$

$$k_{i,i-1} = -\lambda_{i-1} \quad i = 2, 3, \ldots, n, \quad (47)$$

$$k_{i,i+1} = -\lambda_i \quad i = 1, 2, \ldots, n - 1,$$

and $M$ is a diagonal matrix with $m_{ii} = \beta_i$.

8.2. Natural Frequencies of Four Concentric Fixed-Free Shafts. As a numerical example, consider four concentric fixed-free shafts connected by layers of torsional stiffness. Solving (45) subject to $\phi_k(0) = 0$ and $\phi_k'(1) = 0$,

$$\delta_k = \frac{(2k - 1)\pi}{2} \quad \phi_k(x) = \sqrt{2} \sin \left[ \frac{(2k - 1)\pi}{2} x \right]. \quad (48)$$

Each shaft is made of the same material. The inner shaft is solid of radius $r$. The outer shafts are each of thickness $t$. The thickness of each elastic layer is negligible. This leads to $\mu_1 = 1$ $\mu_2 = 15$ $\mu_3 = 65$, $\mu_4 = 175$, $\beta_1 = 1$ $\beta_2 = 15$ $\beta_3 = 65$, $\beta_4 = 175$. The torsional stiffness of each elastic layer is the same and is taken such that $\lambda_1 = 1$. The inner shaft is solid, thus, $\lambda_0 = 0$. The outer radius of the outer shaft is unrestrained from rotation, hence, $\lambda_4 = 0$. The matrix eigenvalue problems become

$$Ka = \omega^2 Ma, \quad (46)$$

where $K$ is a tridiagonal matrix whose elements are

$$k_{ij} = \mu_i \delta^2 + \lambda_{j-1} + \lambda_j \quad i = 1, 2, \ldots, n,$$

$$k_{i,i-1} = -\lambda_{i-1} \quad i = 2, 3, \ldots, n, \quad (47)$$

$$k_{i,i+1} = -\lambda_i \quad i = 1, 2, \ldots, n - 1,$$

and $M$ is a diagonal matrix with $m_{ii} = \beta_i$.
The natural frequencies for \( k = 1, 2, \ldots, 5 \) are given in Table 1. Since \( \mathbf{K} \) is singular and \( \mathbf{M}^{-1}\mathbf{K}_b = \mathbf{I} \), the lowest natural frequency in each intramodal set is \( \delta_k \). Each mode shape in a set of intramodal mode shapes corresponds to the same spatial mode \( \phi_k(x) \). The difference in intramodal mode shapes is in the relative magnitude and signs of the displacements of the individual shafts. The normalized mode shapes of Figure 2 correspond to the first mode shape in the intramodal set for the first spatial mode and illustrate the mode in which the shafts rotate as if they are rigidly connected. The mode shapes of Figure 3 correspond to the third intramodal mode for the first spatial mode and illustrate that when the rotations of the first, second, and fourth shafts are counterclockwise, the rotation of the third shaft is clockwise. Figures 4 and 5 illustrate mode shapes corresponding to the third spatial mode. All mode shapes in the intramodal set for this mode have two nodes across the length of the shaft. Note that for the second intramodal frequency there is a change in the direction of rotation of the shafts between the third and fourth shafts. Thus, there is a cylindrical surface of nodes between these shafts. There are two changes in the direction of rotation for the third intramodal frequency, between the second and third shafts and between the third and fourth shafts, leading to two cylindrical surfaces of nodes.

### 8.3. Forced Response of Four Concentric Shafts

Suppose that the midspan of the outer shaft is subject to a constant torque, \( T_0 \), such that the nondimensional applied torques are \( T_1(x, t) = T_2(x, t) = T_3(x, t) = 0, T_4(x, t) = \delta(x - 1/2) \). The forced response of the system is calculated by using a convolution integral solution of the form of (21) leading to

\[
c_{k,i}(t) = \frac{1}{\omega_{k,j}} \int_0^t \int_0^1 \mathbf{a}_{k,j}^T \mathbf{F}(x, \tau) \phi_k(x) \sin \left( \omega_{k,j}(t - \tau) \right) dx d\tau
\]

\[
= \left( \mathbf{a}_{k,j} \right) \frac{\omega_{k,j}^2(1/2)}{\omega_{k,j}^2} \left[ 1 - \cos \left( \omega_{k,j}t \right) \right].
\]

### Figure 2: Set of intramodal mode shapes of elastically connected fixed-free torsional shafts with \( k = 1 \) and \( j = 1 \). The mode shapes correspond to rigid-body motion across the set of shafts.

Substitution of (50) into (18) leads to

\[
\mathbf{W}(x, t) = \sum_{k=1}^{\infty} \sum_{j=1}^{4} \frac{\mathbf{a}_{k,j}}{\omega_{k,j}} \left[ 1 - \cos \left( \omega_{k,j}t \right) \right] \phi_k \left( \frac{1}{2} \right) \phi_k(x).
\]

Equation (51) is evaluated leading to the time dependence of the response at \( x = 1/2 \) and \( x = 1 \) illustrated in Figures 6 and 7.

### 8.4. Nonuniform Shafts

Now consider the same set of shafts, except that each has a taper, such that the differential equation for the innermost shaft when isolated from the system is

\[
\frac{d}{dx} \left[ (1 - 0.1x)^2 \frac{d\theta}{dx} \right] + \omega^2 (1 - 0.1x)^2 \theta = 0.
\]
Along with the boundary for a fixed-free shaft, (52) has a Bessel function solution leading to the characteristic equation for the shaft’s natural frequencies as

\[ j_0(9\omega) y_0(10\omega) - j'_0(9\omega) j_0(10\omega) = 0, \]  

(53)

where \( j_n(x) \) and \( y_n(x) \) are spherical Bessel functions of the first and second kinds of order \( n \) and argument \( x \).

Even though the shafts are not identical, the differential equations in (55) may still be decoupled because the stiffness and inertia matrices are the same. The eigenvalues of \( M^{-1}K \)
are \( \kappa_1 = 0, \kappa_2 = 0.0172, \kappa_3 = 0.0815, \) and \( \kappa_4 = 1.0712. \) The corresponding matrix of eigenvectors is

\[
P = \begin{bmatrix}
0.0625 & 0.1191 & 0.2292 & 0.9640 \\
0.0625 & 0.1171 & 0.2106 & -0.0686 \\
0.0625 & 0.0848 & -0.0654 & 0.0010 \\
0.0625 & -0.0422 & 0.0049 & -5.44 \times 10^{-6}
\end{bmatrix}.
\]

The columns of \( P \) have been normalized such that \( P^T M P = I \) and \( P^T K_2 P = \Delta. \) Following the same procedure as in the derivation of (43), the uncoupled differential equations become

\[
\begin{align*}
\frac{d}{dx} \left( (1 - 0.1x)^2 \frac{d q_1}{dx} \right) + \omega^2 (1 - 0.1x)^2 q_1 &= 0, \\
\frac{d}{dx} \left( (1 - 0.1x)^2 \frac{d q_2}{dx} \right) + \left[ \omega^2 (1 - 0.1x)^2 - 0.0172 \right] q_2 &= 0, \\
\frac{d}{dx} \left( (1 - 0.1x)^2 \frac{d q_3}{dx} \right) + \left[ \omega^2 (1 - 0.1x)^2 - 0.0815 \right] q_3 &= 0, \\
\frac{d}{dx} \left( (1 - 0.1x)^2 \frac{d q_4}{dx} \right) + \left[ \omega^2 (1 - 0.1x)^2 - 1.0712 \right] q_4 &= 0.
\end{align*}
\]

The solutions of (57) are

\[
\begin{align*}
q_1(x) &= C_1 j_0(\omega(1 - 0.1x)) + C_2 y_0(\omega(1 - 0.1x)), \\
q_2(x) &= C_1 j_{0.904}(\omega(1 - 0.1x)) + C_2 y_{0.904}(\omega(1 - 0.1x)), \\
q_3(x) &= C_1 j_{2.398}(\omega(1 - 0.1x)) + C_2 y_{2.398}(\omega(1 - 0.1x)), \\
q_4(x) &= C_1 j_{8.862}(\omega(1 - 0.1x)) + C_2 y_{8.862}(\omega(1 - 0.1x)).
\end{align*}
\]

The characteristic equations to determine the natural frequencies are

\[
\begin{align*}
&j_{0}(10\omega_{k,1}) y_{1}(9\omega_{k,1}) - y_{0}(10\omega_{1}) j_{1}(9\omega_{k,1}) = 0, \\
j_{0.904}(10\omega_{k,2}) \begin{bmatrix} 0.904 & y_{0.904}(9\omega_{k,2}) \\ 0 & \omega_{k,2} \end{bmatrix} \begin{bmatrix} y_{1.904}(9\omega_{k,2}) \\ \omega_{k,2} \end{bmatrix} = 0, \\
&j_{2.398}(10\omega_{k,3}) \begin{bmatrix} 2.398 & y_{2.398}(9\omega_{k,3}) \\ 0 & \omega_{k,3} \end{bmatrix} \begin{bmatrix} y_{3.398}(9\omega_{k,3}) \\ \omega_{k,3} \end{bmatrix} = 0, \\
j_{8.862}(10\omega_{k,4}) \begin{bmatrix} 9.862 & y_{8.862}(9\omega_{k,4}) \\ 0 & \omega_{k,4} \end{bmatrix} \begin{bmatrix} y_{10.862}(9\omega_{k,4}) \\ \omega_{k,4} \end{bmatrix} = 0.
\end{align*}
\]

The characteristic equation for the first set of frequencies is identical to (53). The first five frequencies for each \( j \) are given in Table 2.

**Figure 6:** Forced response of elastically connected torsional shafts at \( x = 0.5 \) due to constant concentrated torque applied to outer shaft at \( x = 0.5. \)

**Figure 7:** Forced response of elastically connected torsional shafts at \( x = 1 \) due to constant concentrated torque applied to outer shaft at \( x = 0.5. \)

8.5. Euler-Bernoulli Beams. Consider a set of \( n \) parallel Euler-Bernoulli beams connected by elastic layers. For uniform beams, the mass and stiffness operators are \( L_i = \mu_i (\partial^3/\partial x^3) \) and \( M_i = \beta_i. \) The differential eigenvalue problem for the first beam in the set is

\[
\frac{d^4\phi}{dx^4} - \beta_1 \phi = 0.
\]
The transverse displacements of Euler-Bernoulli beams is another example of the special case discussed in Section 6.

If the beams are identical (μ = 1 and β = 1) and if Kc is singular, then the lowest natural frequency for the kth set of intramodal modes is δk with the mode shape, such that each beam has the same displacement and the springs are unstretched. Otherwise, the mode shape for the lowest natural frequency of each intramodal set satisfies the orthogonality condition of (31).

As a numerical example, consider a set of five fixed-free elastically connected Euler-Bernoulli beams. The solution of Equation (60) subject to the boundary conditions ϕ(0) = 0, $\phi'(0) = 0$, $\phi''(0) = 0$, and $\phi'''(0) = 0$ leads to

$$\phi_k(x) = \cosh(\sqrt{\delta_k}x) - \cos(\sqrt{\delta_k}x) - \frac{\cos(\sqrt{\delta_k}) + \cosh(\sqrt{\delta_k})}{\sin(\sqrt{\delta_k}) + \sinh(\sqrt{\delta_k})} \left[ \sinh(\sqrt{\delta_k}x) - \sin(\sqrt{\delta_k}x) \right],$$

(61)

where $\delta_k$ is the kth solution of $\cos(\sqrt{\delta_k})\cosh(\sqrt{\delta_k}) = -1$. Numerical values used in the computations are $\mu_1 = 1$, $\mu_2 = 2$, $\mu_3 = 0.5$, $\mu_4 = 1$, $\mu_5 = 0.25$, $\beta_1 = 1$, $\beta_2 = 1.5$, $\beta_3 = 0.75$, $\beta_4 = 1$, $\beta_5 = 0.5$, $\lambda_0 = 0$, $\lambda_1 = 100$, $\lambda_2 = 50$, $\lambda_3 = 50$, $\lambda_4 = 20$, and $\lambda_5 = 0$.

Using these numerical values, the matrix eigenvalue problem for a set of intramodal frequencies and mode shapes is

$$
\begin{bmatrix}
\delta_k^2 + 100 & -100 & 0 & 0 & 0 \\
-100 & 2\delta_k^2 + 150 & -50 & 0 & 0 \\
0 & -50 & 0.5\delta_k^2 + 100 & -50 & 0 \\
0 & 0 & -50 & \delta_k^2 + 70 & -20 \\
0 & 0 & 0 & -20 & 0.25\delta_k^2 + 20
\end{bmatrix}
$$

×

$$
\begin{bmatrix}
\omega^2 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1.5 & 0 & 0 & 0 \\
0 & 0 & 0.75 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix}
$$

(62)

The sets of intramodal frequencies are listed in Table 3. Recall that δk is the natural frequency of the first beam. The natural frequency of an Euler-Bernoulli beam increases with stiffness. Since the first beam is stiffer than several other beams in the set, some intramodal frequencies are lower than δk. The mode shapes for k = 2 and j = 4 illustrated in Figure 8 show one spatial node and three cylindrical surfaces of nodes.

### Table 2: First five sets of frequencies for set of linearly tapered shafts.

<table>
<thead>
<tr>
<th>j</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.639</td>
<td>4.736</td>
<td>7.868</td>
<td>11.006</td>
<td>14.145</td>
</tr>
<tr>
<td>2</td>
<td>1.645</td>
<td>4.738</td>
<td>7.869</td>
<td>11.007</td>
<td>14.146</td>
</tr>
<tr>
<td>3</td>
<td>1.667</td>
<td>4.745</td>
<td>7.874</td>
<td>11.010</td>
<td>14.148</td>
</tr>
<tr>
<td>4</td>
<td>1.981</td>
<td>4.861</td>
<td>7.944</td>
<td>11.010</td>
<td>14.187</td>
</tr>
</tbody>
</table>

### Table 3: First four sets of intramodal natural frequencies for a set of five elastically connected fixed-fixed Euler-Bernoulli beams.

<table>
<thead>
<tr>
<th>j</th>
<th>3.5100</th>
<th>22.0300</th>
<th>61.7000</th>
<th>120.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.4630</td>
<td>16.7195</td>
<td>44.0798</td>
<td>85.7222</td>
</tr>
<tr>
<td>2</td>
<td>6.6167</td>
<td>20.6044</td>
<td>51.6490</td>
<td>99.3829</td>
</tr>
<tr>
<td>3</td>
<td>8.5529</td>
<td>23.4906</td>
<td>62.2901</td>
<td>121.1925</td>
</tr>
<tr>
<td>4</td>
<td>12.7319</td>
<td>24.1708</td>
<td>62.4633</td>
<td>121.3072</td>
</tr>
<tr>
<td>5</td>
<td>14.8317</td>
<td>28.0564</td>
<td>71.9859</td>
<td>139.9667</td>
</tr>
</tbody>
</table>

### Figure 8: Intramodal mode shapes for a set of five elastically connected Euler-Bernoulli beams.
(iv) A normal-mode solution for the free response leads to the formulation of an eigenvalue problem defined for a matrix of operators.

(v) The operator is self-adjoint with respect to the energy inner products leading to the development of an orthogonality condition.

(vi) The expansion theorem is used to develop a modal analysis for the forced response.

(vii) The case where the structures are uniform and the individual stiffness operators are proportional is a special case in which the determination of natural frequencies and mode shapes can be reduced to eigenvalue problems for matrices on $\mathbb{R}^n$.

(viii) When the stiffness operators are proportional, the natural frequencies and mode shapes are indexed with two indices, the first representing the spatial mode shape, the second representing the intramodal mode shapes.

(ix) If the uniform structures are identical, then a simple formula can be derived for the sets of intramodal natural frequencies using the eigenvalues of the coupling stiffness matrix. The intramodal mode shapes for each spatial mode are the eigenvectors of the coupling stiffness matrix.

(x) An iterative solution must be applied to determine the natural frequencies for the most general case of the uniform structure.

(xi) The differential equations for the coupling of identical structures, uniform, or nonuniform can be uncoupled through diagonalization of the coupling stiffness matrix.

(xii) Elastically connected uniform strings and elastically connected uniform concentric shafts are applications in which the stiffness operators are proportional.

(xiii) The differential equations for the concentric shafts, even though they are not identical, can be decoupled when each individual stiffness operator is the same as the individual mass operator.

(xiv) The individual stiffness operators for uniform Euler-Bernoulli beams are proportional implying that their natural frequencies can be indexed as an infinite number of sets of intramodal frequencies.

The general method is applied here only for undamped systems. However, it can be applied to certain damped systems as well. If the structures are undamped but the Winkler layers have viscous damping, the same $\delta_k$ and $\phi_k$ for the undamped system may be used, but an eigenvalue problem is obtained involving complex numbers. If the structures are damped but the Winkler layers are undamped, the choice of $\delta_k$ and $\phi_k$ is modified to include viscous damping, but again a complex eigenvalue problem is obtained. If the entire system (both the individual structures and the Winkler layers) is subject to proportional damping, the eigenvalues and the eigenvectors of the undamped system can be used to uncouple the forced vibrations equations.

References


