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Research Article

Control of Bistability in a Delayed Duffing Oscillator

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The effect of a high-frequency excitation on nontrivial solutions and bistability in a delayed Duffing oscillator with a delayed displacement feedback is investigated in this paper. We use the technique of direct partition of motion and the multiple scales method to obtain the slow dynamic of the system and its slow flow. The analysis of the slow flow provides approximations of the Hopf and secondary Hopf bifurcation curves. As a result, this study shows that increasing the delay gain, the system undergoes a secondary Hopf bifurcation. Further, it is indicated that as the frequency of the excitation is increased, the Hopf and secondary Hopf bifurcation curves overlap giving birth in the parameter space to small regions of bistability where a stable trivial steady state and a stable limit cycle coexist. Numerical simulations are carried out to validate the analytical finding.

1. Introduction

This paper concerns the effect of a high-frequency excitation on the nontrivial solutions and on bistability dynamic of a Duffing type oscillator with a delayed displacement feedback in the form

$$\ddot{x} + \eta\dot{x} + \omega^2 x + \beta x^3 + \lambda x(t-\tau) = 0, \tag{1}$$

where η is the damping, ω is the natural frequency, β the nonlinearity, and λ, τ are the amplitude and the time delay, respectively. The parameters η, β and λ are assumed to be small and positive. This equation may serve as the simplest model for describing the dynamic of various controlled physical and engineering systems; see [1–3]. Other works have been devoted to study the dynamic of a Duffing oscillator under a delayed feedback control [4–6]. In addition, (1) has been considered in [7] as a simple model for a vibration problem in turning machine for modelling the nonlinear generative effect in metal cutting [1] or for exploring the control of a flexible beam in a simple mode approach [8]. On the other hand, numerous physical applications focusing on bistability dynamic can be found in the literature [9, 10]. In [1, 11–13], attention has been paid to the linear stability analysis of the trivial steady state of delayed oscillators of type (1). While the investigation of stability of nontrivial solutions in (1) has received little attention from analytical view point, the influence of high-frequency excitation on nontrivial steady state has not been tackled. In [12], for instance, a detailed and systematic study on the dynamic of a delayed Duffing oscillator was confined to the linear stability analysis.

In this paper, we explore the bifurcation of a nontrivial solution of (1) created in Hopf bifurcation, and we analyze the influence of a high-frequency excitation on the bifurcation diagram of this nontrivial state. We show analytically and numerically that as the frequency of the excitation is increased, bistability dynamic appears in small regions in the parameter space causing the response of the system to undergo possible jumps between two steady states.

2. Trivial Solution

We begin with a brief review of the stability chart of the trivial solution $x = 0$ of (1) by considering its linear version

$$\ddot{x} + \eta\dot{x} + \omega^2 x + \lambda x(t-\tau) = 0. \tag{2}$$

The stability analysis of the trivial solution is obtained using the corresponding transcendental characteristic equation

$$s^2 + \eta s + \omega^2 + \lambda e^{-\tau s} = 0. \tag{3}$$

This equation possesses infinitely many finite roots for $\lambda \neq 0$ and $\tau \neq 0$. The stability occurs when two dominant roots of

(3) are placed on the imaginary axis at the desired resonant frequency, while other roots remain in the stable left half of the complex plane. The imaginary characteristic roots are $s = \pm i\omega_c$, where ω_c is the resonance frequency and $i = \sqrt{-1}$. The subscript c implies the crossing of the root loci on the imaginary axis. We substitute $s = \pm i\omega_c$ into (3), separate the real and imaginary parts, eliminate the trigonometric terms and we solve for the control parameters λ and τ. This yields the stability diagram corresponding to the Hopf bifurcation curves in the (λ, τ) parameter plane with the parametric representation

$$\lambda = \sqrt{\left(\left(\omega^2 - \omega_c^2\right)^2 + \left(\eta\omega_c\right)^2\right)},$$
$$\tau_d = \frac{1}{\omega_c}\left\{\arctan\left[\frac{\eta\omega_c}{\omega^2 - \omega_c^2}\right] + 2(\ell - 1)\pi\right\}, \quad \ell = 1, 2, 3, \ldots, \tag{4}$$

where ℓ corresponds to the ℓth lobe from the left in the stability diagram illustrated in Figure 1. The family of Hopf curves $\ell = 1, 2, \ldots$ are shown for the given parameters $\eta_1 = 0.067$ and $a_1 = 0.02$. In the dashed region bellow the Hopf curve, the trivial solution is stable. Above this region, the trivial equilibrium is unstable.

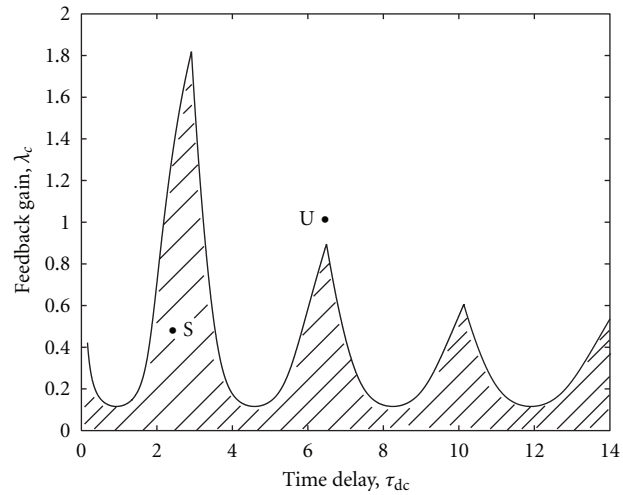

Figure 1: Hopf bifurcation curves, taken from [8], for $\ell = 1, \ldots, 4$, $\Omega = 100$, $\eta_1 = 0.067$, and $a_1 = 0.02$.

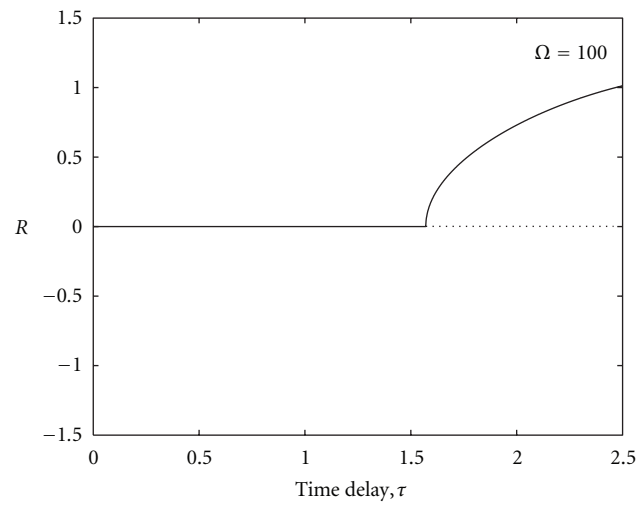

Figure 2: Amplitude of the periodic oscillation versus the time delay as given by (10), $\beta = 1.25$, $\eta = 0.067$, $\lambda = 1.2$, and $a_1 = 0.02$.

3. Nontrivial Solutions

Following [8], (1) can be viewed as the one mode model of a hinged-clamped beam. Assume that the beam is subjected to an axial high-frequency excitation of the form $a\Omega^2 \cos \Omega t$, where a is a nondimensional amplitude of excitation and Ω is the excitation frequency; the quantity $a\Omega$ denotes the excitation strength. Applying the standard method of direct partition of motion [8, 14, 15], we can separate the dynamic of (1) into a slow dynamic (at the time-scale of free system oscillations) and the fast motions (at the rate of the fast excitation). Since the slow motions, denoted by the variable z, are of primary concern, the equation describing the slow dynamic of the oscillator (1) reads

$$\ddot{z} + \eta\dot{z} + \omega^2 z + \beta z^3 + \lambda z(t - \tau) = 0, \tag{5}$$

where the frequency is now depends on the excitation strength $a\Omega$ and given by [8]

$$\omega = \sqrt{1 + \frac{\pi^4}{2}(a\Omega)^2}. \tag{6}$$

Now the method of of multiple scales [16] is applied to explore the existence of nontrivial steady state. We assume that damping, nonlinearity, and delay are small, and scaling by introducing a small book-keeping parameter μ, (5) can be recast as

$$\ddot{z} + \omega^2 z + \mu\left(\eta\dot{z} + \beta z^3 + \lambda z(t - \tau)\right) = 0. \tag{7}$$

A first-order uniform expansion of the solution to (7) is sought in the form

$$z(t;\mu) = z_1(T_1, T_2) + \mu z_2(T_1, T_2) + O(\mu^2), \tag{8}$$

where the independent time scales are defined as $T_1 = t$ and $T_2 = \mu t$. It follows that the derivatives become $d/dt = D_1 + \mu D_2$ and $d^2/dt^2 = D_1^2 + 2\mu D_1 D_2 + \mu^2 D_2$ where $D_n^j = \partial^j/\partial T_n^j$. We follow, as usual, the classical steps of the multiple scale method by substituting (8) into (7), using the notation $D_n^j = \partial^j/\partial T_n^j$, equating coefficients of like powers of μ, and eliminating secular terms. The modulation equations of amplitude R and phase θ of the periodic solutions are given at first-order approximation by the system

$$\frac{dR}{dt} = -\mu\frac{1}{2}\eta R + \mu\frac{\lambda}{2\omega}R\sin(\omega\tau), \tag{9a}$$

$$R\frac{d\theta}{dt} = \mu\frac{3\beta}{8\omega}R^3 + \mu\frac{\lambda}{2\omega}R\cos(\omega\tau). \tag{9b}$$

A fixed point in this slow flow corresponds to a periodic motion in the original system (7). Solving for the fixed points

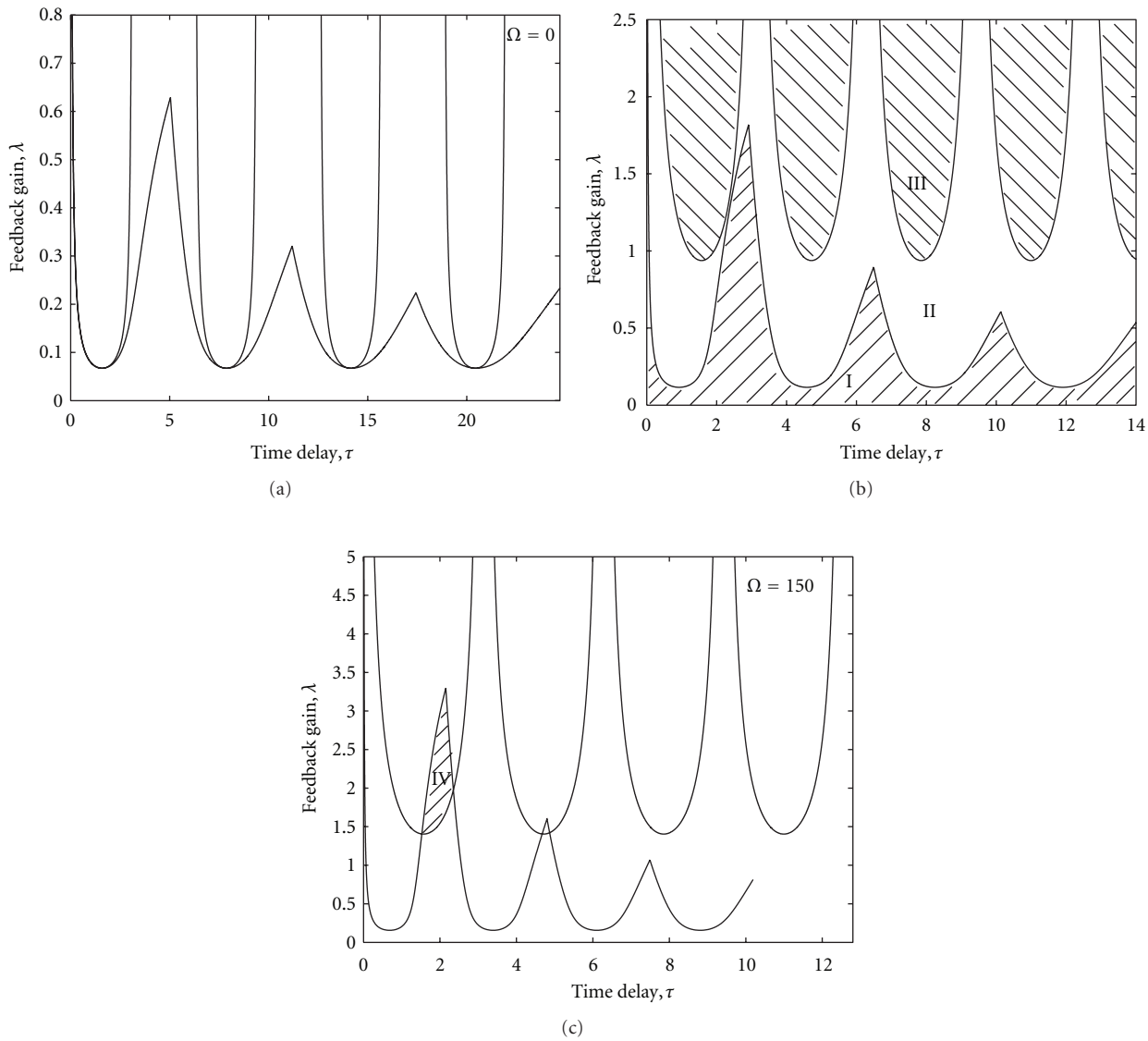

Figure 3: Hopf and secondary Hopf bifurcation curves for different values of Ω; $\beta = 1.25$, $\eta = 0.067$, and $a_1 = 0.02$. (a) $\Omega = 0$, Hopf and secondary Hopf curves; (b) $\Omega = 100$, region I: stable trivial solution, region II: stable limit cycle, and region III: quasiperiodic solution; (c) $\Omega = 150$, region IV: multistability domain.

of $d\theta/dt = 0$ in (9a) and (9b) we obtain the amplitude of the periodic motion (limit cycle)

$$R = 2\sqrt{-\frac{\lambda}{3\beta}\cos\omega\tau}. \tag{10}$$

Figure 2 plots the variation of this amplitude versus the time delay τ. As it can be seen from this figure, the bifurcation value of this limit cycle is $\tau = \pi/2$.

Instead of employing the system (9a) and (9b), to obtain the relation between the amplitude of the periodic solution, R, and time delay, τ, as done in [12], we will take advantage from this modulation system to determine the region of existence of this periodic motion born by Hopf bifurcation.

The condition for the nontrivial solution (10) to be real is

$$\frac{(4n-3)\pi}{2\omega} \leqslant \tau \leqslant \frac{(4n-1)\pi}{2\omega}, \quad n = 1, 2, \ldots. \tag{11}$$

Substituting ω by its value given by (6), the condition (11) becomes

$$\frac{(4n-3)\pi}{2\sqrt{1+\pi^4(a\Omega)^2/2}} \leqslant \tau \leqslant \frac{(4n-1)\pi}{2\sqrt{1+\pi^4(a\Omega)^2/2}}, \quad n = 1, 2 \ldots. \tag{12}$$

By differentiating once (9a), the stability of the nontrivial solution can be discussed using the corresponding characteristic equation

$$s^2 + \left(\frac{1}{2}\eta - \frac{\lambda}{2\omega}\sin\omega\tau\right)s = 0 \tag{13}$$

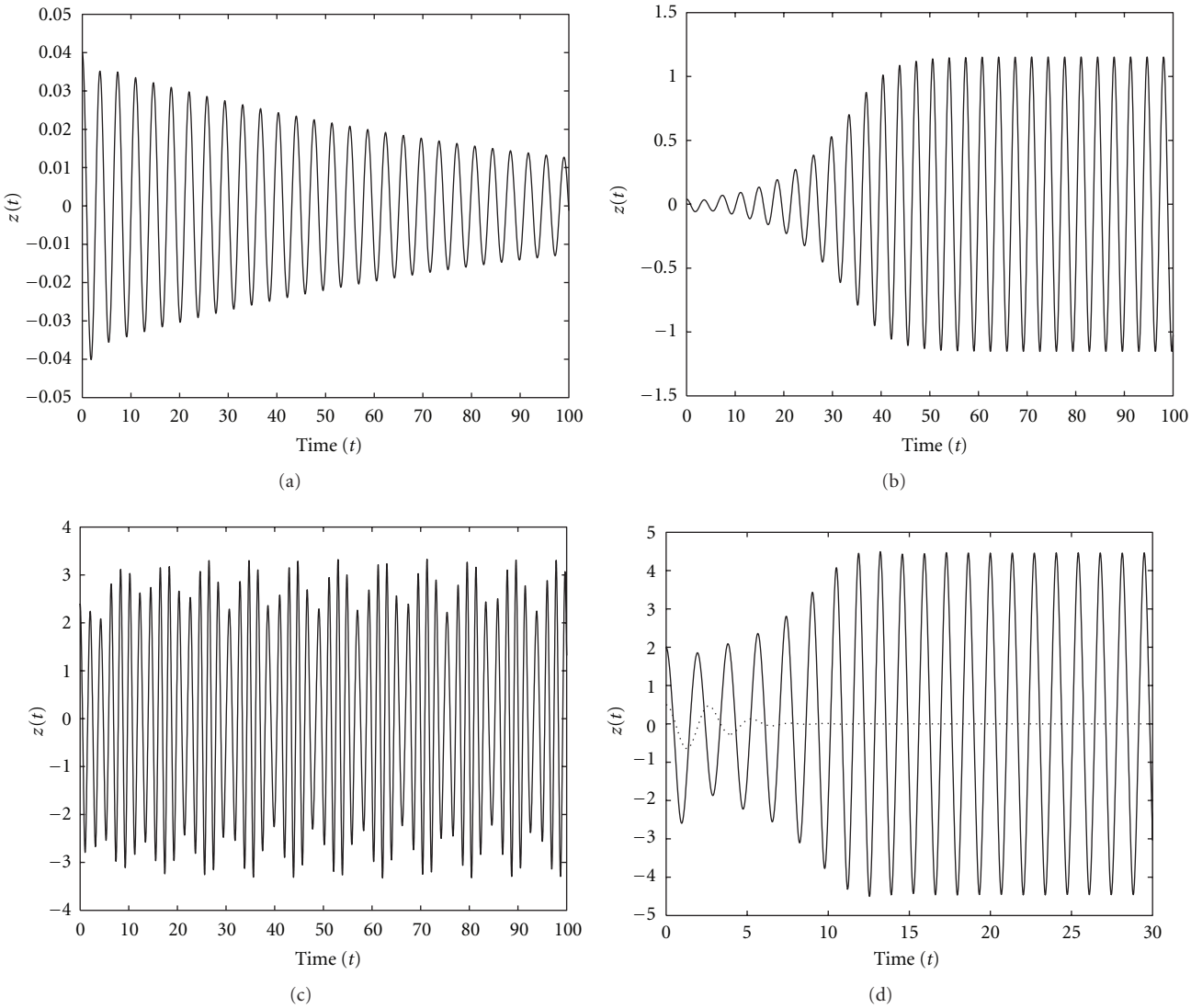

Figure 4: Time trace of $z(t)$; $\beta = 1.25$, $a_1 = 0.02$, $\tau = 5$, (a) region I, $\lambda = 0.1$, stable trivial solution, (b) region II, $\lambda = 0.8$, stable limit cycle, (c) region III, $\lambda = 1.2$, quasiperiodic solution, and (d) region IV, $\lambda = 2$, $\tau = 2$, multistability solutions.

and the critical value for tested sign of the nontrivial eigenvalue

$$\lambda_{\mathrm{cr}} = \frac{\eta\omega}{\sin\omega\tau}. \tag{14}$$

This eigenvalue is negative if $\lambda < \lambda_{\mathrm{cr}}$ for any time delay τ and negative or positive on the rest of the (λ, τ) plane. The bifurcation curves of the nontrivial steady states given by (12) and (14), corresponding to the secondary Hopf bifurcation, are illustrated in Figure 3(a) for $\Omega = 0$. The Hopf curves of Figure 1 are also plotted in this figure. In Figures 3(b) and 3(c) are shown the Hopf and the secondary Hopf curves for $\Omega = 100$ and $\Omega = 150$, respectively. Three regions can be distinguished in Figure 3(b). The region I (dashed zone) located bellow the Hopf curves corresponds to the domain of stability of the trivial steady state $z = 0$. The region II (white zone) corresponds to the existence domain of a stable limit cycle born by Hopf bifurcation when crossing from region I to region II. In the region III (antidashed zone), quasiperiodic oscillations resulting from a secondary Hopf bifurcation take place when crossing from region II to region III. It is worthy to notice that a similar equation to (7) was studied numerically, and it was shown that as the delay gain is increased, the system undergoes a secondary Hopf bifurcations [12, 13, 17]. Figure 3(c) indicates that by increasing the frequency Ω, the Hopf and the secondary Hopf bifurcation curves overlap giving birth to regions (dashed region IV in Figure 3(c)) on which a stable trivial steady state and a stable limit cycle coexist. To validate the analytical finding, we show in Figure 4 numerical time traces integration of (7) corresponding to the different regions I, II, III, and IV of Figures 3(b) and 3(c). Figure 4(a) shows that the stable trivial equilibrium in region I loses its stability, and a stable periodic solution is born by Hopf bifurcation as illustrated in Figure 4(b). Figure 4(c) indicates the existence of a quasiperiodic solution

born by a secondary Hopf bifurcation (region III). Finally, it can be seen from Figure 4(d) (corresponding to region IV) that the trivial stable solution (dotted line) coexists with a stable large amplitude limit cycle (solid line) indicating that multistability can occur in small regions in the parameter plane (λ, τ).

4. Conclusion

We have investigated the effect of a high-frequency excitation on nontrivial steady-state solutions and bistability in a delayed Duffing oscillator. The technique of direct partition of motion and the multiple scales method were applied to obtain the equation governing the slow dynamic of the oscillator and the corresponding slow flow. The nontrivial solutions of the slow flow were studied, and the secondary Hopf bifurcation curves were obtained. It was shown that a high-frequency excitation causes the Hopf and the secondary Hopf diagrams to overlap giving rise to small regions in the parameter space control (gain versus time delay) where a stable equilibrium and a stable large amplitude limit cycle may coexist. This coexistence may produce possible jumps between the two steady states. This bistability regime can be either desirable or undesirable depending on the application under consideration. The analytical result of this work has been confirmed using numerical simulations.

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