Research Article

Anomalous Dispersion of the S_1 Lamb Mode

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Received 30 April 2013; Revised 1 July 2013; Accepted 16 July 2013

Academic Editor: Abdelkrim Khelif

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The S_1 mode of the Lamb spectrum of an isotropic plate exhibits negative group velocity in a narrow frequency domain. This anomalous behavior is explained analytically by examining the slope of each mode first in its initial state and then near its turning points.

1. Introduction

The dispersion relation for symmetric Lamb modes propagating in an infinite isotropic plate of thickness $2h$ is given by the well known Rayleigh-Lamb equation [1]:

$$\frac{\tan(ph)}{\tan(qh)} = -\frac{4pqk^2}{(p^2 - k^2)^2}, \quad (1)$$

where

$$\begin{aligned} p &= \sqrt{\frac{\omega^2}{c_T^2} - k^2}, \\ q &= \sqrt{\frac{\omega^2}{c_L^2} - k^2}. \end{aligned} \quad (2)$$

In (1), c_T and c_L , respectively, denote the phase speeds of the transverse and longitudinal bulk waves in the material. Also, ω and k , respectively, denote the frequency and the wave number of the mode. The phase velocity, c , of a mode is given by

$$c = \frac{\omega}{k}. \quad (3)$$

If c is plotted as a function of the frequency, the spectrum appears as in Figure 1, which depicts the spectrum for a steel plate with $c_T = 3.24$ km/s and $c_L = 5.95$ km/s.

The most striking feature in Figure 1 is the shape of the S_1 mode which has a turning point at $k_T = 2.686$, and the phase velocity becomes double valued for $k_T h$ in $[2.686, 2.873]$. This phenomenon of negative group velocity is of great technical significance and has been observed in a large number of experiments [2–8].

The afore mentioned feature of S_1 mode was first noticed by Tolstoy and Usdin [9] in 1957. In all isotropic materials with $\nu \neq 1/3$, only the S_1 mode has this “anomalous behavior” and other modes behave normally. We will call it the S_1 anomaly. An explanation of this peculiar shape of mode S_1 has posed a challenge since its discovery in 1957.

For the special case of a material with $\nu = 1/3$ that is $c_L = 2c_T$, each of the modes S_{3n+1} , $n = 0, 1, 2, 3, \dots$, exhibits anomalous behavior [10]. Anomalous pairs of modes may also occur for certain special values of the Poisson ratio. We will call it the pair anomaly.

Although (1) governs the behavior of all modes, anomalous or otherwise, no simple theory seems to exist which should provide a satisfactory explanation of why certain modes in the spectrum should possess a bulge while others proceed in a normal manner. However, certain physical explanations of this phenomenon exist. In 1983, Whitaker and Haus [11] noted the fact that “propagation of waves with dispersion of this sort has been experimentally verified [2] but the reason for their appearance is not well understood.” They used the coupled mode theory to argue that, when the fundamental P mode and the first harmonic S mode are

FIGURE 1: Symmetric Lamb modes on a steel plate ($\kappa = 1.83$) showing phase velocity as a function of normalized frequency.

nearly degenerate at cutoff, a coupling effect can occur at the boundaries. Uberall et al. [12] hypothesize that “one observes a repulsion phenomenon between neighboring dispersion curves similar to that encountered in atomic physics for quasidegenerate energy levels of atoms when combining into molecules.” Prada et al. [3] express the same view in the words, “this phenomenon leads to a strong repulsion between the dispersion curves of the neighbouring modes.”

It is clear that all of these authors focused on the pair anomaly only because, in the S_1 anomaly, the S_1 mode remains distinct and it does not coincide with any other mode at cutoff. Mode repulsion cannot explain S_1 anomaly. To the best of our knowledge, the S_1 anomaly still remains an unsolved mystery. Recently Hussain and Ahmad [13] considered ZGV points in the spectrum of Lamb modes in compressible orthotropic plate. It was found that, in addition to modes with a single ZGV point, a large number of modes exist with multiple such points.

In this paper we will examine Rayleigh-Lamb spectrum for the symmetric modes of an isotropic material. We will analyze (1) and derive mathematical expressions which will explain both types of anomalies.

2. The Mode Spectrum

Let k and c , respectively, denote the wave number and phase speed of the mode. Define the dimensionless speed c/c_T by y and the dimensionless frequency $\omega h/c_T$ by u .

Then

$$ph = h\sqrt{\frac{\omega^2}{c_T^2} - k^2} \quad (4)$$

$$= hk\sqrt{y^2 - 1} = \frac{hkc/c_T}{c/c_T}\sqrt{y^2 - 1} = \frac{u}{y}\sqrt{y^2 - 1},$$

$$qh = \frac{u}{y}\sqrt{\frac{y^2}{\kappa^2} - 1}. \quad (5)$$

With respect to the variables u and y , (1) becomes

$$\begin{aligned} f(u, y) &= (y^2 - 2)^2 \sin\left(\sqrt{y^2 - 1}\frac{u}{y}\right) \cos\left(\sqrt{\frac{y^2}{\kappa^2} - 1}\frac{u}{y}\right) \\ &\quad + 4\sqrt{y^2 - 1}\sqrt{\frac{y^2}{\kappa^2} - 1} \sin\left(\sqrt{\frac{y^2}{\kappa^2} - 1}\frac{u}{y}\right) \\ &\quad \times \cos\left(\sqrt{y^2 - 1}\frac{u}{y}\right) = 0. \end{aligned} \quad (6)$$

In this section, we will consider $\kappa \neq 2$. We will calculate the derivative dy/du at two positions of the spectrum. We write

$$\begin{aligned} D_{n,\kappa} &= \left. \frac{dy}{du} \right|_{u=u_n, y=\kappa}, \\ D_{n,L} &= \left. \frac{dy}{du} \right|_{u=u_n, y=L \gg \kappa}, \end{aligned} \quad (7)$$

where the subscript n refers to the mode under consideration. For $2/\sqrt{3} < \kappa < 3.39$ ($-1 < \nu < 0.452$), we will show that $D_{n,\kappa} < 0$ for all modes. On the other hand, $D_{1,L} > 0$ for the S_1 mode while $D_{n,L} < 0$ for all other modes. Since the derivative for the S_1 mode changes from positive to negative, it must exhibit a bulge before $y = \kappa$. No other mode undergoes a reversal of the slope; hence, all other modes continue their downward journey until they asymptotically approach the line $y = 1$.

In Appendix A, we give expressions for $\partial f/\partial u$ and $\partial f/\partial y$. The derivative dy/du is found as

$$\frac{dy}{du} = -\frac{\partial f/\partial u}{\partial f/\partial y}. \quad (8)$$

When $y = \kappa$, (6) becomes

$$\sin\left(\frac{u\sqrt{\kappa^2 - 1}}{\kappa}\right) = 0. \quad (9)$$

Therefore,

$$u_n = \frac{n\pi\kappa}{\sqrt{\kappa^2 - 1}}. \quad (10)$$

Equation (10) shows that the line $y = \kappa$ intersects the modes at infinitely many points. At $y = \kappa$ and $u = u_n$, partial derivatives (A.1) become

$$\begin{aligned} \left. \frac{\partial f}{\partial u} \right|_{u=u_n, y=\kappa} &= \frac{(\kappa^2 - 2)^2 \sqrt{\kappa^2 - 1}}{\kappa}, \\ \left. \frac{\partial f}{\partial y} \right|_{u=u_n, y=\kappa} &= \left[\frac{(\kappa^2 - 2)^2}{\kappa^2 \sqrt{\kappa^2 - 1}} + 4 \frac{\sqrt{\kappa^2 - 1}}{\kappa^2} \right] u_n. \end{aligned} \quad (11)$$

Hence,

$$D_{n,\kappa} = -\frac{(\kappa^2 - 1)^{3/2}(\kappa^2 - 2)^2}{[(\kappa^2 - 2)^2 + 4(\kappa^2 - 1)]n\pi}. \quad (12)$$

It is clear that $D_{n,\kappa} < 0$ for all modes. Also $|D_{n,\kappa}|$ becomes progressively smaller as n increases and $D_{n,\kappa} \rightarrow 0$ as $n \rightarrow \infty$. This means that, for $y = \kappa$, there is a plateau region and this plateau is flatter for higher modes.

Next, we find $D_{n,L}$. For $y \gg \kappa$, the expressions for $\partial f/\partial u$ and $\partial f/\partial y$ are given by

$$\frac{\partial f}{\partial u} \approx y^4 \left[\cos u \cos \frac{u}{\kappa} - \frac{1}{\kappa} \sin u \sin \frac{u}{\kappa} \right] \quad (13)$$

$$\begin{aligned} \frac{\partial f}{\partial y} \approx & uy \cos u \cos \frac{u}{\kappa} + 4y^3 \cos \frac{u}{\kappa} \sin u \\ & + \frac{8y}{\kappa} \cos u \sin \frac{u}{\kappa} - \kappa uy \sin u \sin \frac{u}{\kappa}. \end{aligned} \quad (14)$$

Thus,

$$\begin{aligned} \frac{dy}{du} \\ \approx -\frac{y^4 - (y^4/\kappa) \tan u \tan (u/\kappa)}{uy + 4y^3 \tan u + (8y/\kappa) \tan (u/\kappa) - \kappa uy \tan u \tan (u/\kappa)}. \end{aligned} \quad (15)$$

For $y \gg \kappa$, (6) gives

$$y^2 \approx -\frac{4 \tan (u/\kappa)}{\kappa \tan u}. \quad (16)$$

Also, from (15) and (16), we have

$$D_{n,L} \approx -\frac{y^3 + (y^5/4) \tan^2 u_n}{u_n + y^2 \tan u_n (2 + (\kappa^2/4) u_n \tan u_n)}, \quad (17)$$

or equivalently

$$D_{n,L} \approx -\frac{y^3 + (y^5/4) \tan^2 u_n}{u_n - (4/\kappa) \tan (u_n/\kappa) (2 + (\kappa^2/4) u_n \tan u_n)}. \quad (18)$$

We have replaced u by u_n in (17) and (18), since, for a fixed y , (16) yields infinitely many roots u_n , $n = 1, 2, 3, \dots$

To fix ideas, we consider the case of a steel plate for which $\kappa = 1.83$. The general case follows on similar lines.

From (16), we see that, for the S_1 mode, u_1/κ should be slightly less than $\pi/2$, so that u_1/κ is in the first quadrant and the corresponding u_1 is in the second quadrant to yield a large positive y^2 .

Hence, for the mode S_1 , we have

$$\begin{aligned} D_{1,L} \\ \approx -\frac{y^3 + (y^5/4) \tan^2 (\kappa\pi/2)}{(\kappa\pi/2) + y^2 \tan (\kappa\pi/2) (2 + (\kappa^2/4) \cdot (\kappa\pi/2) \tan (\kappa\pi/2))}. \end{aligned} \quad (19)$$

For $\kappa = 1.83$ and for large y , $D_{1,L} > 0$.

TABLE 1: Approximate and exact values of u_n when $y = 20$ for the first five modes.

Modes	Approximate u_n	Exact value at $y = 20$
S_1	2.875	2.856
S_2	3.142	3.180
S_3	6.283	6.289
S_4	8.624	8.650
S_5	9.425	9.449

For $2/\sqrt{3} < \kappa < 2$ and large y , approximate values of u_n from (16) for the first few modes are $\kappa\pi/2$, π , $3\kappa\pi/2$, 2π , and $5\kappa\pi/2$. For the steel plate, these values are compared in Table 1 with the exact values found from (6) when $y = 20$.

We have seen previously that $D_{1,L} > 0$ for the S_1 mode. Now for the S_2 mode $u_2 \approx \pi$ and from (18) we get

$$D_{2,L} \approx -\frac{y^3}{\pi - (8/\kappa) \tan (\pi/\kappa)}. \quad (20)$$

Here, $\kappa = 1.83$ and $\tan(\pi/\kappa) < 0$; thus $D_{2,L} < 0$. In a similar fashion by successively using (17) and (18), we can show that $D_{n,L} < 0$ for all $n \geq 2$.

From the previous, we conclude that for steel, the S_1 is the only mode which reverses its slope while going from large values of y to $y = \kappa$. The previous analysis applies to all materials with $\kappa < 2$.

If $\kappa > 2$, the S_1 mode occurs, when $u_1 \approx \pi$, and from (18), the S_1 mode will have positive slope as long as

$$\pi - \frac{8}{\kappa} \tan \frac{\pi}{\kappa} < 0, \quad (21)$$

or $2 < \kappa < 3.39$ which corresponds to $1/3 < \nu < 0.452$.

Thus, we have established that the S_1 mode will be anomalous for all materials falling in the range $2/\sqrt{3} < \kappa < 3.39$. Since the slope of S_1 becomes negative for all y when $\kappa > 3.39$, the mode will lose its anomalous character beyond $\kappa = 3.39$ which corresponds to $\nu = 0.452$.

3. The Exceptional Case $\kappa = 2$

The case $\kappa = 2$ merits special treatment. The spectrum of symmetric modes appears as in Figure 2.

The pairs S_1 - S_2 , S_4 - S_5 , S_7 - S_8 ... appear to merge for large y and then bifurcate as they descend to lower values of the phase speed. On the other hand, S_3 , S_6 , S_9 ,... appear to behave normally.

This phenomenon was first reported by Mindlin [14]. Each of the modes S_1 , S_4 , S_7 ,... shows anomalous dispersion.

With $\kappa = 2$, (6) becomes, for $y \gg 1$,

$$y^4 \cos \frac{u}{2} \sin u + y^2 \sin \frac{u}{2} \cos u = 0, \quad (22)$$

which is satisfied for $u_n = 2n\pi$, $n = 1, 2, 3, \dots$. It is shown in Appendix B that a more accurate solution of (6) is

$$u_n = n\pi \left(2 + \frac{1}{y^2} \right). \quad (23)$$

FIGURE 2: Symmetric Lamb modes on a nickel plate ($\kappa = 2$).

In addition to (23), (16) has roots given by

$$-\frac{1}{2}y^2 \approx \frac{\tan(u/2)}{\tan u} = \frac{1}{2} \left(1 - \tan^2 \frac{u}{2}\right). \quad (24)$$

Ignoring 1 in comparison with y^2 , we get

$$\tan^2 \frac{u}{2} \approx y^2, \quad (25)$$

or

$$\cot^2 \frac{u}{2} \approx \frac{1}{y^2}. \quad (26)$$

Let

$$\frac{u}{2} = \frac{(2n+1)\pi}{2} + \epsilon. \quad (27)$$

Equation (26) becomes

$$\cot^2 \left(\frac{\pi}{2} + \epsilon \right) \approx \frac{1}{y^2}, \quad (28)$$

or

$$\tan^2 \epsilon \approx \frac{1}{y^2}, \quad (29)$$

which leads to

$$\epsilon \approx \pm \frac{1}{y}. \quad (30)$$

Thus, for large y , $u_1 = \pi - 2\epsilon$, $u_2 = \pi + 2\epsilon$, $u_4 = 3\pi - 2\epsilon$, $u_5 = 3\pi + 2\epsilon$, and so forth.

Since $\epsilon \rightarrow 0$ as $y \rightarrow \infty$, the modes $u_1, u_2, u_4, u_5, u_7, u_8, \dots$ appear to coalesce for large y . Now, with $\kappa = 2$, (18) gives $D_{1,L} > 0$, $D_{4,L} > 0$, and $D_{7,L} > 0$ while the slopes for all other modes are negative. This argument establishes the anomalous dispersion of the modes S_{3n+1} , $n = 0, 1, 2, 3, \dots$. As $u_3, u_6, u_9, \dots$ occur slightly above $2\pi, 4\pi, 6\pi, \dots$, (17) gives a negative value for the slope of each of these modes. Hence, these modes behave in a normal manner.

4. Conclusion

We have explained analytically the anomalous behavior of Lamb modes for an isotropic material by looking at the slope of each mode for large as well as small y . For small y , slope is found at $y = \kappa$. This simple technique explains, in an analytic manner, theoretical results given by several authors about the anomalous dispersion of the S_1 mode.

Appendices

A. The Partial Derivatives

Expressions for the partial derivatives are as follows:

$$\begin{aligned} \frac{\partial f}{\partial u} &= \left((y^2 - 2)^2 \sqrt{y^2 - 1} \cos \frac{u \sqrt{y^2 - 1}}{y} \right. \\ &\quad \times \left. \cos \frac{u \sqrt{y^2/\kappa^2 - 1}}{y} \right) (y)^{-1} \\ &\quad + \left(4 \left(\frac{y^2}{\kappa^2} - 1 \right) \sqrt{y^2 - 1} \cos \frac{u \sqrt{y^2 - 1}}{y} \right. \\ &\quad \times \left. \cos \frac{u \sqrt{y^2/\kappa^2 - 1}}{y} \right) (y)^{-1} \\ &\quad - \left((y^2 - 2)^2 \sqrt{\frac{y^2}{\kappa^2} - 1} \sin \frac{u \sqrt{y^2 - 1}}{y} \right. \\ &\quad \times \left. \sin \frac{u \sqrt{y^2/\kappa^2 - 1}}{y} \right) (y)^{-1} \\ &\quad - \left(4 (y^2 - 1) \sqrt{\frac{y^2}{\kappa^2} - 1} \sin \frac{u \sqrt{y^2 - 1}}{y} \right. \\ &\quad \times \left. \sin \frac{u \sqrt{y^2/\kappa^2 - 1}}{y} \right) (y)^{-1}, \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= (y^2 - 2)^2 \left(\frac{u}{\sqrt{y^2 - 1}} - \frac{u \sqrt{y^2 - 1}}{y^2} \right) \end{aligned}$$

$$\begin{aligned}
& \times \cos \frac{u\sqrt{y^2-1}}{y} \cos \frac{u\sqrt{y^2/\kappa^2-1}}{y} \\
& + 4\sqrt{y^2-1}\sqrt{y^2/\kappa^2-1} \\
& \times \left(\frac{u}{\kappa^2\sqrt{y^2/\kappa^2-1}} - \frac{u\sqrt{y^2/\kappa^2-1}}{y^2} \right) \\
& \times \cos \frac{u\sqrt{y^2-1}}{y} \cos \frac{u\sqrt{y^2/\kappa^2-1}}{y} \\
& + 4y(y^2-2) \sin \frac{u\sqrt{y^2-1}}{y} \cos \frac{u\sqrt{y^2/\kappa^2-1}}{y} \\
& + \left(4y\sqrt{y^2-1} \cos \frac{u\sqrt{y^2-1}}{y} \right. \\
& \quad \left. \times \sin \frac{u\sqrt{y^2/\kappa^2-1}}{y} \right) \left(\kappa^2\sqrt{y^2/\kappa^2-1} \right)^{-1} \\
& + \left(4y\sqrt{y^2/\kappa^2-1} \cos \frac{u\sqrt{y^2-1}}{y} \right. \\
& \quad \left. \times \sin \frac{u\sqrt{y^2/\kappa^2-1}}{y} \right) \left(\sqrt{y^2-1} \right)^{-1} \\
& - 4\sqrt{y^2-1}\sqrt{y^2/\kappa^2-1} \left(\frac{u}{\sqrt{y^2-1}} - \frac{u\sqrt{y^2-1}}{y^2} \right) \\
& \times \sin \frac{u\sqrt{y^2-1}}{y} \sin \frac{u\sqrt{y^2/\kappa^2-1}}{y} \\
& - (y^2-2)^2 \left(\frac{u}{\kappa^2\sqrt{y^2/\kappa^2-1}} - \frac{u\sqrt{y^2/\kappa^2-1}}{y^2} \right) \\
& \times \sin \frac{u\sqrt{y^2-1}}{y} \sin \frac{u\sqrt{y^2/\kappa^2-1}}{y}.
\end{aligned} \tag{A.1}$$

B. The Mode S_{3n} for $\kappa = 2$

For $y \gg 1$, $\sqrt{y^2-1}/y \simeq 1 - 1/(2y^2)$ and $\sqrt{(y^2/4)-1}/y \simeq 1/2 - 1/y^2$.

Let $u_n \simeq 2n\pi + (\alpha_n/y^2)$ and ignore the terms of order $1/y^4$ or higher. Then,

$$\begin{aligned}
\cos \left(\frac{\sqrt{y^2-1}}{y} u_n \right) & \simeq 1, \\
\cos \left(\frac{\sqrt{y^2/4-1}}{y} u_n \right) & \simeq -1, \\
\sin \left(\frac{\sqrt{y^2-1}}{y} u_n \right) & \simeq \sin \frac{\alpha_n - n\pi}{y^2} \simeq \frac{\alpha_n - n\pi}{y^2}, \\
\sin \left(\frac{\sqrt{y^2/4-1}}{y} u_n \right) & \simeq \mp \sin \frac{\alpha_n - 2n\pi}{y^2} \simeq \mp \frac{\alpha_n - 2n\pi}{y^2}.
\end{aligned} \tag{B.1}$$

Putting these expressions in (6), we have

$$-y^4 \frac{\alpha_n - n\pi}{y^2} \mp 2(\alpha_n - 2n\pi) = 0, \tag{B.2}$$

or

$$(y^2 \pm 2) \alpha_n = n\pi y^2 \mp 4n\pi, \tag{B.3}$$

or

$$\alpha_n = \frac{n\pi y^2 \mp 4n\pi}{y^2 \pm 2} \simeq n\pi, \tag{B.4}$$

for large y .

This result shows that the modes S_{3n} , $n = 1, 2, 3, \dots$, intersect the line $y = y_0$, for large y_0 , at points slightly to the right of $u_n = 2n\pi$, $n = 1, 2, 3, \dots$

Acknowledgment

Faiz Ahmad is grateful to the Higher Education Commission of Pakistan for financial support.

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