Research Article

The Bearing Capacity of Circular Footings in Sand: Comparison between Model Tests and Numerical Simulations Based on a Nonlinear Mohr Failure Envelope

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This paper presents the results of a series of triaxial tests with dry sand at confining pressures varying from 1.5 kPa to 100 kPa at relative densities of 0.20, 0.59, and 0.84. The results, which are in reasonable accordance with an equation given by Bolton, show that the friction angle is strongly dependent on the stress level and on the basis of the test results, a nonlinear Mohr failure criterion has been proposed. This yield criterion has been implemented in a finite element program and an analysis of the bearing capacity of a circular shaped model foundation, diameter 100 mm, has been conducted. Comparisons have been made with results from 1g model scale tests with a foundation of similar size and a good agreement between numerical results and test results has been found.

1. Introduction

In conventional design of shallow foundations, earth retaining structures, slopes, and the friction angle of sand are regarded as a constant, being primarily dependent on the relative density of the material. Experimental work (e.g., [1, 2]) has shown that the friction angle is also dependent upon the confining pressure; i.e., the stress level. In cases where the mean effective stress is above 150 kPa the equation developed by Bolton [3] for a sand with a given mineralogy takes this dependency into account in a simple but sufficiently accurate way, but for the lower stress levels typically encountered in connection with simple model tests in a soil laboratory higher accuracy is desirable. This is due to the strong dependency of the bearing capacity factors and soil pressure coefficients on the angle of friction. Also in such cases where high accuracy is needed, grading and grain shape, as shown by Fukushima and Tatsuoka [2], may have an effect on the magnitude of the angle of friction.

One of the main effects in geotechnical structures always stems from body forces and therefore the stress level in

simple laboratory experiments is always much lower than the stress level in real life. For this reason very often, one has to accept large discrepancies between results from these laboratory tests and the equivalent conventional geotechnical calculations. As examples of this Hansen [4] and Zadroga [5] reported that the bearing capacities of model foundations found in tests are considerably higher than the ones found in calculations.

When the friction angle is constant and the response is considered linearly elastic perfectly plastic, the material model is termed Mohr-Coulomb, which is the one used in standard geotechnical designs. For this reason a lot of experience has been gathered regarding the parameters used in the model (Young's modulus E , Poisson's ratio ν and the friction angle, ϕ) and the calculations involving this model. One of the great virtues of the model is the possibility to use hand calculation methods in many practical cases.

The Mohr-Coulomb model predicts a linear relation between the normal stress acting on a slip surface and strength. Experiments, however, have shown that this is not true. When carried out at low stress levels in the

laboratory model experiments predict a far higher friction angle compared to the one obtained at realistic stress levels.

To allow for the nonlinear dependency of the strength on the stress level, physical modelling in connection with centrifuge tests has become widespread over the past thirty to forty years. Also over these years, there has been a great increase in the computational power available, due to bigger and cheaper computers combined with very efficient modelling methods such as the finite element method, and it seems obvious that everyday design of soil structures could now be based on a more advanced model than the Mohr-Coulomb model. In comparison rock mass structures are now routinely being designed using the nonlinear Hoek-Brown criterion [6].

The main aims of the work described in the present paper are threefold. Firstly, it is to determine a simple failure envelope of the type of sand used for simple model tests in the geotechnical laboratory of Esbjerg Institute of Technology in Denmark. Secondly, the model is implemented in an elastoplastic finite element code in order to carry out bearing capacity calculations. Thirdly, these bearing capacity calculations are compared with model tests carried out in the laboratory.

Another purpose of the paper is to advocate the need for a slightly more complicated failure criterion in practical calculations, compared to conventional use of the constant Mohr-Coulomb friction angle. With the modern day availability of inhouse and commercial numerical computations codes, it should now be tractable to carry out routine geotechnical design with more realistic material models. The purpose of this paper is not to propose an all-encompassing constitutive model as it is for example seen in [7–9], as these advanced models are still considered too advanced for everyday use, both with regards to parameter determination and numerical computations.

As a validation of the model, results from numerical simulations of the bearing capacity of a circular model footing are compared with the results of simple model tests carried out in the geotechnical laboratory of Esbjerg Institute of Technology in Denmark. The footing has a diameter of 10 cm and tests are carried out on sand at different relative densities. The numerical simulations are based on a nonlinear Mohr failure envelope of which the parameters have been determined through triaxial tests.

2. Type of Sand Tested

The sand used in the experiments is Esbjerg sand, which is an alluvial, medium grained sized quartz sand of subangular shape with the characteristics given in Table 1. To find the strength properties of the sand, triaxial tests were carried out.

3. Triaxial Tests, Specimen Preparation, and Testing Equipment

The triaxial tests were carried out with dry sand in a triaxial testing apparatus. The confining pressure was applied by lowering the air pressure inside the test specimen. The

TABLE 1: Properties of Esbjerg sand.

Parameter	Value
D_{10} (mm)	0.25
D_{60} (mm)	0.58
$C_u = D_{60}/D_{10}$	2.32
D_{50} (mm)	0.50
Specific density	2.621
Maximum void ratio $e_{\max}$	0.733
Minimum void ratio $e_{\min}$	0.449
Relative density in tests	0.20, 0.59, 0.84
Dry unit weight in tests (kN/m ³)	15.64 16.74 17.54

diameter and height of the test specimens were 70 mm. A schematic drawing of the load test set up is shown in Figure 1(a).

The deviatoric stress was applied using a hydraulic cylinder operated with a hand pump, and the force was recorded by means of an electronic load cell which was placed inside the cell to obtain as accurate values as possible. The confining pressure was provided by a vacuum pump connected to the specimen through the loading plates at either end of the specimen and the amount of vacuum was controlled by a valve operated by hand and recorded using a pressure transducer. The vertical displacements were measured by means of two displacement transducers mounted outside the cell and the displacements were taken as average values of these two transducer readings. The volumetric changes of the specimens were recorded by a displacement transducer which could register the movements of the water table in a burette connected to the water filled cell. The diameter of both loading plates is 90 mm, and the thickness is 50 mm. The membrane, made from 0.30 mm rubber, was fastened to the base plate and sealed with an O-ring and held by vacuum to the inner surface of a cylindrical split mould. Both the bases made from aluminium and the nylon cap were provided with 2 layers of lubricated 0.30 mm latex sheets.

The specimens were prepared by preweighing the specific amount of sand to obtain the desired relative density and the sand was placed through a funnel in an appropriate number of steps. Between each of the steps the sand was compacted by tamping to obtain the desired density. For the loose samples (relative density, $D_r = 0.20$), no compacting at all was necessary; for the medium dense samples ($D_r = 0.59$), the sand was placed in three steps and for the dense samples, ($D_r = 0.84$) the sand was placed in five steps. When the surface of the sand had been scraped carefully with a metal plate, the cap was placed and the upper part of the membrane was sealed to the cap with an O-ring. A vacuum of approximately 20 kPa was applied and the split mould was removed and the specimen was placed in the cell. Following this, the load cell was put in position, the lid of the cell, and the burette was placed and the cell was filled with water, until the water in the burette reached the desired level which was 0.53 m above the center of the specimen. In the tests with a confining pressure of 1.5 kPa, the cell was not completely

FIGURE 1: Schematic drawing and photograph of the triaxial test setup.

filled with water, so it was not possible to record the volume changes.

After the three displacement transducers and the hydraulic cylinder had been put in position, the vacuum was removed and all transducers were zeroed. Hereafter, the specimen was isotropically consolidated by application of vacuum to the desired level of the effective confining pressure and after a few minutes when it was clear from the transducer readings, that there were no further volume changes, all transducers were zeroed again and the load was applied, so as to produce a rate of displacement of approximately 5 mm pr minute. For the two lowest values of the confining pressure the confining was provided solely by the water pressure.

The additional radial stress due to the stiffness of the membrane has been taken account by the equation [2]:

$$\Delta\sigma_r = \frac{2E_m t \varepsilon_\theta}{d}, \quad (1)$$

where E_m = modulus of elasticity of the membrane (1.7 MPa), t = thickness of membrane (0.30 mm), ε_θ = tangential strain of the membrane = the radial strain of the membrane, and d = diameter of sample (which varies during the test and = 70 mm at the beginning).

Corrections to the deviatoric stress due to the increase in the cross-sectional area of the sample during testing have been made by computing the deviatoric stress from the equation:

$$\Delta\sigma_a = \frac{P}{A_0} \frac{1 - \varepsilon_a}{1 - \varepsilon_v}, \quad (2)$$

where P is the applied load, A_0 is the initial cross sectional area of the sample, and ε_a is the axial strain and ε_v the volumetric strain. For the tests carried out at a confining stress of 1.5 kPa it was not possible to measure the volumetric strain and instead an estimated value for ε_v has been used.

4. Testing Programme and Results

Triaxial tests were carried out for relative densities of 0.20, 0.59, and 0.84 and for each relative density, the following values of the initial confining stress were applied: 1.5 kPa, 5.3 kPa, 20 kPa, 50 kPa, and 100 kPa. The results are shown in Figures 2, 3, and 4 and summarized in Table 2. The peak friction angles in column 3 are calculated according to the equation:

$$\left(\frac{\sigma_1}{\sigma_3} \right)_{\max} = \frac{1 + \sin \varphi_{\text{peak}}}{1 - \sin \varphi_{\text{peak}}}. \quad (3)$$

FIGURE 2: Test results for Esbjerg sand— $D_r = 0.20$. FIGURE 3: Test results for Esbjerg sand— $D_r = 0.59$.

In this equation, which is derived from the linear Mohr-Coulomb failure criterion, σ_3 is the confining pressure, σ_1 is the maximum stress, and φ_{peak} is the peak friction angle, which varies with the confining pressure σ_3 and because of this becomes a secant angle.

The angle of dilation ψ is defined in plane strain by the equation originally suggested by Hansen [4]:

$$\sin(\psi_{\text{max}}) = \left(-\frac{d\varepsilon_v}{d\gamma_{\text{max}}} \right)_{\text{max}} = \left(-\frac{d\varepsilon_1 + d\varepsilon_3}{d\varepsilon_1 - d\varepsilon_3} \right)_{\text{max}}. \quad (4)$$

In triaxial compression at least two different definitions exist (see e.g. [10, 11]):

$$\sin \psi_{\text{max}} = \left(-\frac{d\varepsilon_1 + 2d\varepsilon_3}{d\varepsilon_1 - d\varepsilon_3} \right)_{\text{max}}, \quad (5)$$

$$\sin \psi_{\text{max}} = \left(-\frac{d\varepsilon_1 + 2d\varepsilon_3}{d\varepsilon_1 - 2d\varepsilon_3} \right)_{\text{max}}. \quad (6)$$

The commonly used numerical code Plaxis suggest ψ be estimated from (6) and also this definition has been used in the present study.

FIGURE 4: Test results for Esbjerg sand— $D_r = 0.84$.

It is a well-known fact that dilation plays an important role in the study of the strength of soil [12] and also in general, the angle of dilation is considerably smaller than the angle of friction. Collapse loads for materials with a nonassociated flow rule are smaller than those obtained for the same material when an associated flow rule is assumed. From experience it is known that in cases where there is a substantial difference between φ and ψ numerical problems arise in the solution of nonlinear finite element equations. To overcome these difficulties [13] proposed using a modified friction angle making it possible to deal with the material, as if it was obeying the normality condition. According to [14], the modified friction angle φ_{mod} can be found from the equation:

$$\tan \varphi_{\text{mod}} = \frac{\sin \varphi \cos \psi}{1 - \sin \varphi \sin \psi}. \quad (7)$$

The values of φ and ψ on the right hand side of the equation are given in column 3 and 4 in Table 2 and the values of φ_{mod} are given in column 5 in Table 2.

The test results have been compared with values found with the Bolton equation [3] for triaxial compression:

$$\varphi_{\text{peak}} = \varphi_{\text{cv}} + 3(D_r(Q - \ln p') - R). \quad (8)$$

In this equation, φ_{peak} is the friction angle at peak, φ_{cv} is the friction angle at constant volume, D_r the relative density, and Q and R are parameters which depend on the mineralogy of the grain material. For the quartz sand used in this study $Q = 10$ and $R = 1$ are used. The mean normal stress is denoted p' and can be found as:

$$p' = \frac{1}{3}(\sigma_1 + 2\sigma_3), \quad (9)$$

FIGURE 5: Friction angles for Esbjerg Sand.

under triaxial conditions. Inserting (3) and (9) in (8) and solving for σ_3 yields the following equation:

$$\sigma_3 = \frac{3(1 - \sin \varphi_{\text{peak}})}{3 - \sin \varphi_{\text{peak}}} \exp\left(\frac{\varphi_{\text{cv}} - \varphi_{\text{peak}} + 3D_r Q - 3R}{3D_r}\right). \quad (10)$$

The variation of the friction angle with confining pressure found in the triaxial tests is shown in Figure 5 together with the values of the friction angle according to (10). The value of φ_{cv} has been found according to the method proposed by Cornforth [15] and the average of ten tests yielded a value of 32.7° .

5. Discussion of Test Results

The tests have shown that the angle of friction varies significantly with the confining pressure and that the greater the confining stress, the smaller the angle of friction. This variation is greater for smaller values of the confining stress. The values of the friction angle according to Bolton's equation are, in general, smaller than the test values. These differences are greater for smaller values of the relative density and the confining pressure and this tendency has been confirmed by the tests carried out by Ponce and Bell, while the tests by Fukushima and Tatsouka have shown no variation of the friction angle with confining pressures below 50 kPa. From Figures 2, 3, and 4, it can be seen that the sand exhibits strain softening and increases its volume in almost all the tests and this characteristic is more pronounced for the larger relative densities and the smaller confining pressures.

6. The Nonlinear Failure Envelope

The linear Mohr-Coulomb failure envelope is given by the equation:

$$\sigma_1 = \frac{1 + \sin \varphi}{1 - \sin \varphi} \sigma_3 + 2c \sqrt{\frac{1 + \sin \varphi}{1 - \sin \varphi}} \quad (11)$$

$$= k\sigma_3 + \sigma_c,$$

where c is the cohesion, k and σ_c are a friction parameter and the uniaxial compression strength, respectively, given by:

$$k = \frac{1 + \sin \varphi}{1 - \sin \varphi}, \quad \sigma_c = 2c \sqrt{\frac{1 + \sin \varphi}{1 - \sin \varphi}} = 2c\sqrt{k}. \quad (12)$$

For a purely frictional material, such as sand, we have $c = \sigma_c = 0$.

As discussed above the linear Mohr-Coulomb envelope is a poor fit to the test results at small stress levels where the dependency of φ on the stress level is more pronounced.

Several authors have suggested yield functions that take the φ -dependency of stress level into account, for example: De Mello [16], Charles and Watts [17], Collins et al. [18], Simonini [19], Baker [20], and Baker and Awiadat [21]. In the present study, it has been found appropriate to use an expression of the form:

$$\sigma_1 = k_0\sigma_3 + s_{c0} \left(1 - \exp \left(-a \frac{\sigma_3}{s_{c0}} \right) \right), \quad (13)$$

which is a curved envelope that passes through the origin and tends toward the asymptote:

$$\sigma_1 = k_0\sigma_3 + s_{c0} \quad \text{for } \sigma_3 \rightarrow \infty. \quad (14)$$

The parameters k_0 and s_{c0} define the asymptote slope and intersection with the σ_1 axis, respectively, and a adjusts the curvature. The parameters k_0 , s_{c0} , and a are determined from nonlinear regression analysis based on the modified friction angle given in Table 2, column 5, for the three relative densities of the experiments. The values are given in Table 3. In Figure 6, the nonlinear failure envelope, modified test results, and the linear asymptote for $D_r = 0.84$ are shown.

7. Finite Element Calculations

The failure criterion of (13) is implemented into an in house finite element code using the software package Matlab. The material is considered as a linearly elastic—perfectly plastic material and for the elastic modulus E and the poisson ratio ν values of 10 MPa and 0.30 were used. The main aim of the calculations is to determine the failure loads, which is independent of the chosen values of E and ν . For the plastic stress update, a method analogous to the one for a Hoek-Brown material is employed, see Clausen and Damkilde [22]. The footing is assumed to be rigid and perfectly rough. A downwards displacement is applied to the footing nodes in steps until the failure load is reached. The footing load is then calculated by summing up the vertical reaction forces of the footing nodes and dividing with the footing area.

As pointed out by Siddiquee et al. [23] and Tatsouka et al. [24], there may be some effect on the bearing capacity from strain softening causing progressive failure, anisotropy of the sand, and the mean particle size D_{50} . The model applied in this study is not able to take these effects into account. As to strain softening, it can be said that progressive failure is not as pronounced with axisymmetric footings as with strip footings, because of the tougher behavior of the sand with the former. From Figures 3 and 4, it can be seen that the strength reduction due to softening becomes of some importance only at an axial strain in the region of 6–8%, and therefore there may be some justification in considering softening and hence progressive failure being of minor importance. As to anisotropy, according to Kimura et al. [25], the effect of anisotropy is more marked for the stiffness than for the bearing capacity, and therefore it has not been considered. Kusakabe [26] found the value of $D_{50}/B = 1/100$ to be the limit, where the effect of the particle size on the bearing capacity becomes less marked and as the sand used in the tests has a value of $D_{50}/B = 0.5/100 = 1/200$ being well below the limit, this effect was ignored.

8. Tests with Model Foundations

To verify the validity of the proposed nonlinear yield function, 1g model tests in axisymmetric conditions were carried out. The sand used for the tests was the Esbjerg sand described above, and the tests were conducted at relative densities $D_r = 0.59$ and $D_r = 0.84$. A circular footing with diameter $B = 10$ cm was used. Its base was covered with a rough material, to make it perfectly rough. In all tests, the footing was resting at the surface, resulting in no overburden pressure. The dry sand was placed in a cylindrical plastic container with a diameter = 55 cm and height = 35 cm in layers of approximately 5 cm. Each layer was tamped a certain number of times to give the desired relative density and the total volume of sand was weighed. To minimize the effect of the sand not having a completely uniform density throughout the container, ten identical tests at each relative density were performed.

The load was applied to the footing by a hydraulic jack mounted on a steel beam, which was fastened to the concrete floor in the lab using 28 mm threaded steel bars. During

TABLE 2: Test results for Esbjerg sand, *estimated value.

Confining pressure σ_3 [kPa]	Relative density D_r	Peak angle of friction φ_{peak} [deg.]	Angle of dilation ψ_{max} [deg.]	Modified peak angle of friction φ_{mod} [deg.]
1.5	0.20	44.3	7.0*	37.2
5.3	0.20	39.0	5.2	33.6
20	0.20	35.2	3.5	30.8
50	0.20	33.0	2.7	29.2
100	0.20	31.8	1.5	28.1
1.5	0.59	47.6	17.0*	42.0
5.3	0.59	44.8	13.4	39.3
20	0.59	42.4	10.8	37.2
50	0.59	40.0	11.0	35.7
100	0.59	38.9	9.4	34.6
1.5	0.84	53.3	23.0*	47.0
5.3	0.84	48.6	18.3	43.0
20	0.84	46.1	14.8	40.1
50	0.84	42.4	15.5	38.7
100	0.84	41.3	13.7	37.6

TABLE 3: Parameters in nonlinear yield function.

Relative density	k_0	s_{c0} [kPa]	a
0.20	2.6613	12.0619	0.9082
0.59	3.4409	18.8313	1.0887
0.84	3.9355	19.7073	1.4330

the test, the load was recorded using a load transducer, HBM S9, and the vertical displacements were recorded by a displacement transducer HBM WA/50 mm. Both load and displacement transducers were calibrated before the tests. The load was raised continuously, and the rate of displacement was app. 5 mm/minute. The test values were recorded by means of a datalogger Spider 8 from HBM. A photo of the load test setup is shown in Figure 7.

Bulging was observed both for the medium dense and the dense sand. The bulge never reached the edge of the testing container, indicating that the size was adequately large, in order to have only minor influence on the test results. Especially for the dense sand, heave of the surface was pronounced, meaning that the outermost slip line, separating the soil body at yield from the soil at rest, was easily identified.

9. Analytical Calculation of the Bearing Capacity of a Shallow Footing

The ultimate bearing capacity of a rough, shallow, circular foundation resting on a cohesionless material is traditionally found from the following equation, given in several textbooks:

$$p_f = 0.5\gamma B N_\gamma s_\gamma + q N_q s_q, \quad s_\gamma = 0.6, s_q = 1.2, \quad (15)$$

FIGURE 6: Failure envelopes and test results, $D_r = 0.84$.

where p_f = bearing capacity pressure, γ = effective unit weight of soil, B = diameter of foundation, N_γ and N_q = bearing capacity factors which are functions of the soil friction angle φ , s_γ and s_q = shape factors, and q = effective overburden pressure.

In the tests in this project, the footings are initially resting on the surface of the sand; that is the contribution from the overburden pressure to the ultimate load is zero at the beginning of the test, but as the load increased, the footing sinks into the ground introducing a vertical effective stress at the foundation level, which cannot be ignored, as it accounts for a significant part of the bearing capacity of the foundation.

FIGURE 7: Load test arrangement.

The bearing capacity factors N_q and N_γ can be found from the below equations, the expression for N_q presented by Reissner [27] and for N_γ by Caquot and Kerisel [28]:

$$N_q = \exp(\pi \tan \varphi) \tan^2 \left(\frac{\pi}{4} + \frac{\varphi}{2} \right), \quad N_\gamma = 2(N_q + 1) \tan \varphi. \quad (16)$$

The overburden pressure at the foundation level can be found as $q = \delta \gamma$ where δ is the vertical displacement of the foundation and γ the density of the sand. Inserting (16) in (15) yields the following equation for the bearing capacity:

$$p_f = \gamma B \tan \varphi s_\gamma + \exp(\pi \tan \varphi) \tan^2 \left(\frac{\pi}{4} + \frac{\varphi}{2} \right) \times (\gamma B \tan \varphi s_\gamma + \delta \gamma s_q). \quad (17)$$

With known values of p_f and the vertical displacement δ , the value of the mobilized φ can be found from (17); the current values of N_q and N_γ can be found from (16) and finally the contribution from the overburden pressure can be deducted from p_f in (15) to produce the carrying capacity due to the selfweight of the soil. An example of the magnitudes of these corrections is given in Figure 8.

10. Definition of Failure Load

For shallow foundations, three failure modes have been described by Vesić [29], amongst others. The relevant failure modes in the present study are general shear failure and local shear failure. The general shear failure mode is expected to take place for the tests with relative density $D_r = 0.84$ and the local shear failure mode for $D_r = 0.59$. Typical load-displacement graphs for the two different modes are shown in Figure 8.

When the footing fails in general shear, there is no doubt as to the magnitude of the failure load, as the load-displacement curve displays a pronounced peak, as shown by the red curve in Figure 8. When local shear occurs, the black curve in Figure 8, it becomes more difficult to point out the magnitude of the failure load, as the load continues to increase. Cerato and Lutenege [30] suggested the failure load to be defined as the load producing a settlement of 10% of the width of the foundation, basically because this simple

FIGURE 8: Definition of failure loads.

but rather arbitrary rule is easy to remember and makes the failure point trivial to identify. In Figure 8, it is seen that the initial part of the test results representing local shear is curved and at a certain point, the curve changes into a straight line. Vesić [29] has defined the point of failure as the point, where the rate of displacement reaches its maximum value. This definition has been applied in the present study and all the test results can be seen in Table 4 and a summary together with the results from the finite element simulations is given in Table 5. It can be seen from Figure 8, that the point of failure according to Cerato and Lutenege and according to Vesić happens to coincide, as the vertical settlement in this particular case is approximately equal to 10 mm which is 10% of the diameter of the model footing.

For tests where the relative density is not equal to the reference values of 0.59 or 0.84, a modification of the measured value has been made. This modification is necessary in order to be able to calculate the average failure load for $D_r = 0.59$ and $D_r = 0.84$ and is carried out in the following way. From (8) it can be seen that a small change ΔD_r of D_r leads to a change $\Delta \varphi$ of φ which can be found as

$$\Delta \varphi = 3(Q - \ln p') \Delta D_r, \quad (18)$$

where $Q = 10$ for quartz sand and p' is the mean effective stress, which can be found from the equation [31]:

$$p' = p_f \frac{1 - \sin \varphi_{\text{peak}}}{4 \cos^2 \varphi_{\text{peak}}}, \quad (19)$$

where p_f is the ultimate bearing capacity and φ_{peak} is the friction angle at peak.

It can be seen from (18) that for a constant value of p' , there is a linear relationship between $\Delta \varphi$ and ΔD_r . In order to establish a relationship in terms of the bearing capacity between φ and D_r , the bearing capacity of the 10 cm diameter footing has been found, assuming a linear Mohr Coulomb failure envelope for friction angles in the interval 34° to 42° and the results of this FEM analysis are given in Table 6.

TABLE 4: Failure loads from tests.

Test no.	Relative density	Displacement at failure [mm]	Load at failure in test [kPa]	Modified failure load [kPa]
A1	0.59	12.6	51	51
A2	0.59	9.8	50	50
A3	0.59	8.7	67	67
A4	0.59	14.8	60	60
A5	0.56	26.7	44	50
A6	0.61	9.0	62	56
A7	0.58	8.0	56	58
A8	0.58	10.0	59	62
A9	0.58	9.5	59	62
A10	0.61	9.5	61	56
B1	0.75	4.5	110	158
B2	0.80	5.6	132	155
B3	0.84	6.3	116	116
B4	0.84	7.5	158	158
B5	0.80	6.7	120	141
B6	0.84	7.3	144	144
B7	0.80	6.5	138	161
B8	0.82	6.5	136	147
B9	0.75	8.0	104	152
B10	0.80	6.5	126	147

TABLE 5: Summary of results from tests and FEM calculations.

Relative density	Diameter of footing [mm]	Failure load from FEM analysis [kPa]	Failure load from tests [kPa]	Standard deviation of failure loads [kPa]
0.59	100	71	57	5.7
0.84	100	151	148	13.0

TABLE 6: Results from linear Mohr-Coulomb analysis.

Friction angle [°]	35	36	37	38	39	40	41	42
Failure load [kPa]	37	46	57	71	89.0	111	141	174

By comparing the values of the failure loads in Table 6, with the values for the failure loads in the third column of Table 5, it can be found that the failure load for a relative density 0.59 assuming the nonlinear failure envelope is equal to the failure load at a constant friction angle of 38° assuming a linear Mohr Coulomb failure envelope. For the relative density of 0.84, the equivalent value of the constant friction angle is found by linear interpolation to be 41.3°. The values of p' found from (19) for the two relative densities are 11 and 23 kPa.

As an example, the modification of the result for test A7 is demonstrated. $D_r = 0.58$, $p_f = 56$ kPa; that is, the equivalent φ is in the interval 36-37° and from (18);

$$\begin{aligned}\Delta\varphi &= 3(Q - \ln p') \Delta D_r = 3(10 - \ln 11) \\ &\quad \times (0.59 - 0.58) = 0.2^\circ \\ \Delta p_f &= \frac{57 \text{ kPa} - 46 \text{ kPa}}{37^\circ - 36^\circ} \cdot 0.2^\circ = 2.2 \text{ kPa}.\end{aligned}\quad (20)$$

That is the modified result for A7 is 58.2 kPa, which is rounded off to 58 kPa.

11. Conclusions

Triaxial tests carried out on medium and dense Esbjerg sand at low stress levels show that the triaxial angle at peak depends strongly on the stress level. For confining pressures above 20 kPa, there is a reasonable match with results obtained from the equation proposed by Bolton. Because the friction angle is dependent on the confining pressure, the linear Mohr-Coulomb yield criterion is ill-suited for the determination of the failure load of 1g model scale footings. Therefore, on the basis of the results from the triaxial tests, a nonlinear Mohr failure criterion has been proposed and implemented in a finite element program. To overcome numerical difficulties due to the nonassociative behaviour of sand the associative flow rule is used, but with yield parameters modified with the equation given by Davis. Results from the finite element analysis of the bearing capacity of 100 mm diameter footings show a good agreement with results obtained from simple model-scale footing tests on medium and dense sand.

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