Research Article

An Equal-Strain Analytical Solution for the Radial Consolidation of Unsaturated Soils by Vertical Drains considering Drain Resistance

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Developing an analytical solution for the consolidation of unsaturated soils remains a challenging task due to the complexity of coupled governing equations for air and water phases. This paper presents an equal-strain model for the radial consolidation of unsaturated soils by vertical drains, and the effect of drain resistance is also considered. Simplified governing equations are established, and an analytical solution to calculate the excess pore-air and pore-water pressures is derived by using the methods of matrix analysis and eigenfunction expansion. The average degrees of consolidation for air and water phases and the ground surface settlement are also given. The solutions of the equal-strain model are verified by comparing the proposed free-strain model with the equal-strain model, and reasonably good agreement is obtained. Moreover, parametric studies regarding the drain resistance effect are graphically presented.

1. Introduction

As a common phenomenon of civil engineering, consolidation is a process of decreasing soil volume when soil is subjected to an increased stress. Understanding of this phenomenon is vital to designs of soft soil foundations, pavements, and other engineering structures. Therefore, Terzaghi [1] established a classical theory for the analysis of consolidation in saturated soft soil foundation, which is well known in geotechnical engineering. Based on Terzaghi's theory, solutions of one-dimensional consolidation have been derived for multilayered soils [2, 3], non-Darcy flow [4, 5], time-dependent loading [6, 7], arbitrary boundary [8, 9], and nonlinear deformation [10, 11]. In practice, soft soils related to engineering are usually in a state of unsaturation. With the great progress made, some well-developed consolidation theories for unsaturated soft soils have been available [12–15]. Among these theories, Fredlund and Rahardjo's [15] consolidation

theory is well accepted for the consolidation of unsaturated soft soils.

Since the inception of the one-dimensional (1D) consolidation theory of unsaturated soft soils proposed by Fredlund and Hasan [16], the investigations for 1D consolidation have been significantly progressed. By assuming constant soil parameters during consolidation, Qin et al. [17, 18] adopted the Laplace transform and Cayley–Hamilton methods to obtain an analytical solution. Shan et al. [19] and Zhou et al. [20] converted the nonlinear governing flow equations into traditional diffusion equations and proposed alternative solutions satisfying those converted equations. Ho et al. [21], on the other hand, introduced exact solutions using the eigenfunction expansion and Laplace transform techniques. In addition, some general solutions that consider various boundary conditions [9, 19, 20, 22], initial pore-water and -air pressure distributions [9, 20], and complex time-dependent loadings [9, 19, 20, 22–24] were also obtained in a single soil layer. However, the soft soil foundation in practice

is not always singly layered. For this reason, Shan [25] presented an analytical solution of unsaturated multilayered soil with typical boundary conditions.

In subgrade and pavement engineering, vertical drains are one of the most commonly used techniques to accelerate the consolidation process and increase the bearing capacity of the subgrade. It is essential to expand the consolidation theory of unsaturated soft soils from one-dimensional vertical consolidation to radial consolidation. For this purpose, Conte [26] developed a general formulation that can be advantageously used to analyze consolidation under plane-strain conditions. Ho et al. [27] discussed the excess pore-air and pore-water pressure dissipations in the plane-strain consolidation of unsaturated soft soils using eigenfunction expansion and Laplace transform techniques. Shortly afterwards, the solutions of unsaturated soft soils subjected to different time-dependent loadings were also obtained by Ho [28].

However, a lot of attempts have been made to solve the axisymmetric consolidation model of unsaturated soft soils particularly with analytical approaches. Among the pioneered studies, Conte [26] introduced the finite element technique to obtain a solution for the coupled consolidation under the axisymmetric conditions. Qin et al. [29] dealt with the vertical drain consolidation problem in unsaturated soft soils using the modified Bessel functions and the Laplace transformation. On the other hand, Zhou and Tu [30] and Zhou [31] presented the differential quadrature method (DQM) to estimate the axisymmetric consolidation behavior in the unsaturated soft soils. Ho et al. [32] gave an analytical solution for the axisymmetric consolidation of unsaturated soft soils subjected to constant external loading. The analytical procedure employs the variable separation and Laplace transformation techniques while capturing the uniform and linear initial excess pore pressure distributions with depth.

Most of the models mentioned above were obtained based on the free-strain assumption. However, the solutions of the free-strain model are complex and inconvenient to use in practice. In the research of saturated soft soils, Barron [33] firstly proposed the consolidation model for vertical drains under free-strain and equal-strain models. The mathematical expressions are as follows:

$$\begin{aligned} c_r \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} \right) + c_v \frac{\partial^2 u}{\partial z^2} &= \frac{\partial u}{\partial t} \quad (\text{free-strain model}), \\ c_r \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} \right) + c_v \frac{\partial^2 \bar{u}}{\partial z^2} &= \frac{\partial \bar{u}}{\partial t} \quad (\text{equal-strain model}), \end{aligned} \quad (1)$$

where u and $\bar{u}$ are the excess pore-water pressure and average excess pore-water pressure, respectively; c_r and c_v are the coefficients of consolidation in the radial and vertical directions, respectively; r is the distance from the origin of the vertical drain in the radial direction; z is the distance from the surface of the foundation in the vertical direction; and t is the time. Barron [33] pointed out that there is negligible difference between the free-strain and equal-strain models. After this, equal-strain assumption is frequently used in most consolidation theories for vertical drains [34–36].

FIGURE 1: Calculation diagram of vertical drain in an unsaturated soil stratum.

At present, there is no reference about the equal-strain model in the research of unsaturated soft soils. The objective of this paper is to propose an equal-strain consolidation model to investigate the consolidation behavior of unsaturated soft soils. In addition, drain resistance will also be considered in this study, and the accuracy of the equal-strain model will be verified by comparing with the free-strain model proposed by Qin et al. [29]. Moreover, parametric studies regarding the drain resistance effect are also presented in this paper.

2. Problem Statements

2.1. Model Description. Figure 1 describes the details of the representative vertical drain within an unsaturated soft soil foundation. Dimensions of the soil system include the depth H and the radius of the influence zone r_e . k_r^w and k_r^a are the coefficients of water and air permeability of the foundation in the radial direction, respectively. The circular vertical drain with the radius r_w is installed at the center of the influence zone. k_v^w and k_v^a are the coefficients of water and air permeability of the vertical drain in the vertical direction, respectively. A constant surcharge q is instantaneously applied on the top surface of soft soil foundation.

2.2. Basic Assumptions. The basic assumptions are the same as those of Fredlund's consolidation theory for unsaturated soft soils and Barron's [33] consolidation theory for vertical drains. The other three assumptions are listed as follows:

- (1) All the soil parameters are constants during consolidation.
- (2) The strains at the same depth of the foundation are equal (equal-strain assumption).
- (3) The distribution of excess pore-air pressure in unsaturated soft soils is uniform.

2.3. Derivation of Consolidation Governing Equations. The governing equations for the consolidation of unsaturated soft soils with vertical drains consist of water phase and air phase equations under equal-strain conditions. In the polar coordinate system, considering a representative element of unsaturated soft soil foundation, with water and air flow in and out in the radial direction during consolidation, the derivation process is given as follows.

2.3.1. Water Phase. The net flux of water through the element is computed from the volume of water entering and leaving the element within a period of time. By utilizing Darcy's law, the net flux of water per unit volume of the soil can be expressed in the polar coordinate system as

$$\frac{\partial(V_w/V_0)}{\partial t} = \frac{k_r^w}{\gamma_w} \left(\frac{\partial^2 u_w}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_w}{\partial r} \right), \quad (2)$$

where V_0 is the initial total volume of the soil element, V_w is the volume of water in the soil element, γ_w is the unit weight of water, and u_w is the excess pore-water pressure.

The net flux of water per unit volume of the soil can be obtained by differentiating the water phase constitutive relation with respect to time:

$$\frac{\partial(V_w/V_0)}{\partial t} = m_{1k}^w \frac{\partial(q - \bar{u}_a)}{\partial t} + m_2^w \frac{\partial(\bar{u}_a - \bar{u}_w)}{\partial t}, \quad (3)$$

where m_{1k}^w and m_2^w describe the coefficients of water volume change with respect to a change in the net normal stress $d(q - \bar{u}_a)$ and matric suction $d(\bar{u}_a - \bar{u}_w)$, respectively; and $\bar{u}_w$ and $\bar{u}_a$ are the average excess pore-water and pore-air pressures throughout the entire soil mass, respectively, which can be expressed as

$$\begin{aligned} \bar{u}_w &= \frac{1}{\pi(r_e^2 - r_w^2)} \int_{r_w}^{r_e} 2\pi r u_w dr, \\ \bar{u}_a &= \frac{1}{\pi(r_e^2 - r_w^2)} \int_{r_w}^{r_e} 2\pi r u_a dr, \end{aligned} \quad (4)$$

where u_a is the excess pore-air pressure.

Substituting (2) into (3), the governing equation for the water phase can be expressed as

$$\frac{\partial \bar{u}_w}{\partial t} + C_a^w \frac{\partial \bar{u}_a}{\partial t} + C_r^w \left(\frac{\partial^2 u_w}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_w}{\partial r} \right) = 0, \quad (5)$$

where C_r^w is the coefficient of consolidation with respect to the water phase in the radial direction (i.e., $k_r^w/(\gamma_w m_2^w)$) and C_a^w is the interactive constant associated with the water phase (i.e., $(m_{1k}^w - m_2^w)/m_2^w$).

2.3.2. Air Phase. The net flux of air through the element is computed from the volume of air entering and leaving the element within a period of time. The flux of air per unit volume of the soil can be obtained by Fick's law as

$$\frac{\partial(\rho_a V_a/V_0)}{\partial t} = \frac{k_r^a}{g} \left(\frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_a}{\partial r} \right), \quad (6)$$

where ρ_a is the density of air, g is the gravitational acceleration, and V_a is the volume of air phase.

Based on assumption 3, the distribution of excess pore-air pressure in unsaturated soft soils is uniform. The density of air is a function of air pressure in accordance with the ideal gas law. It can be expressed as

$$\rho_a = \frac{\omega_a}{RT_{at}} \bar{u}_a, \quad (7)$$

where ω_a is the molecular mass of air, R is the molar gas constant, T_{at} is the absolute temperature, $\bar{u}_a$ is the absolute excess pore-air pressure (i.e., $\bar{u}_a + u_{atm}$), and u_{atm} is the atmospheric pressure.

Replacing the air density, ρ_a , in (6) with (7) gives

$$\frac{\partial(V_a/V_0)}{\partial t} = \frac{k_r^a RT_{at}}{\bar{u}_a \omega_a g} \left(\frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_a}{\partial r} \right) - \frac{(1 - S_s)n_s}{\bar{u}_a} \cdot \frac{\partial \bar{u}_a}{\partial t}, \quad (8)$$

where S_s is the degree of saturation and n_s is the porosity.

The flux of air per unit volume of the soil due to changes in the net normal stress, $d(q - \bar{u}_a)$, and the matric suction, $d(\bar{u}_a - \bar{u}_w)$, can be expressed by the following equation:

$$\frac{\partial(V_a/V_0)}{\partial t} = m_{1k}^a \frac{\partial(q - \bar{u}_a)}{\partial t} + m_2^a \frac{\partial(\bar{u}_a - \bar{u}_w)}{\partial t}, \quad (9)$$

where m_{1k}^a and m_2^a describe the coefficients of air volume change with respect to a change in the net normal stress $d(q - \bar{u}_a)$ and matric suction $d(\bar{u}_a - \bar{u}_w)$, respectively.

By substituting (8) into (9), the governing equation for the air phase can be given as

$$C_w^a \frac{\partial \bar{u}_w}{\partial t} + \frac{\partial \bar{u}_a}{\partial t} + C_r^a \left(\frac{\partial^2 u_a}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u_a}{\partial r} \right) = 0, \quad (10)$$

where C_r^a is the coefficient of consolidation with respect to the air phase in the radial direction (i.e., $k_r^a RT_{at}/\{\omega_a g [\bar{u}_a (m_{1k}^a - m_2^a) - (1 - S_s)n_s]\}$) and C_w^a is the interactive constant associated with the air phase (i.e., $m_2^a \bar{u}_a/[\bar{u}_a (m_{1k}^a - m_2^a) - (1 - S_s)n_s]$).

By rearranging the governing equations of water and air phases, the consolidation equations for water and air phases can be written in matrix form as

$$\mathbf{K}_r \left(\mathbf{u}_{,rr} + \frac{1}{r} \mathbf{u}_{,r} \right) + \mathbf{C} \bar{\mathbf{u}}_{,t} = 0, \quad (11)$$

where $\bar{\mathbf{u}} = \begin{Bmatrix} \bar{u}_w \\ \bar{u}_a \end{Bmatrix}$, $\mathbf{u} = \begin{Bmatrix} u_w \\ u_a \end{Bmatrix}$, $\mathbf{C} = \begin{bmatrix} 1 & C_a^w \\ C_a^w & 1 \end{bmatrix}$, $\mathbf{K}_r = \begin{bmatrix} C_r^w & 0 \\ 0 & C_r^a \end{bmatrix}$, and $(\cdot)_{,r}$ and $(\cdot)_{,t}$ denote the derivatives with respect to r and t , respectively.

2.4. Initial and Boundary Conditions. The external radius, r_e , is impervious or, because of symmetry, no water and air flow passes this boundary; that is,

$$\mathbf{u}_{,r}|_{r=r_e} = 0. \quad (12)$$

Excess pore pressures at the interface of soil stratum and vertical drain satisfy flow continuity condition. The following boundary condition is considered:

$$\left(\mathbf{u}_{,r} + \frac{r_w}{2} \mathbf{B} \mathbf{u}_{,zz} \right) \Big|_{r=r_w} = 0, \quad (13)$$

$$\text{where } \mathbf{B} = \begin{bmatrix} k^w/k_r^w & 0 \\ 0 & k^a/k_r^a \end{bmatrix}.$$

The vertical boundary conditions are

$$\begin{aligned} \mathbf{u}|_{z=0} &= 0, \\ \mathbf{u}_{,z}|_{z=H} &= 0, \end{aligned} \quad (14)$$

which indicate that the top surface of the soft soil foundation is permeable to water and air, while the bottom surface is impermeable to water and air.

The initial excess pore pressures are uniform throughout the soil mass when $t = 0$. They can be calculated according to the method given by Fredlund and Rahardjo [15]. The initial condition is given as follows:

$$\bar{\mathbf{u}}|_{t=0} = \mathbf{u}_0, \quad (15)$$

$$\text{where } \mathbf{u}_0 = \begin{Bmatrix} u_0^w \\ u_0^a \end{Bmatrix}.$$

3. Solution of the Problem

3.1. General Solution. According to the boundary condition (12), $\mathbf{u}_{,r}$ has to vanish as r goes to r_e . An integration of (11) with respect to r from r to r_e gives

$$\mathbf{u}_{,r} = \varphi(r) \mathbf{K}_r^{-1} \mathbf{C} \bar{\mathbf{u}}_{,t}, \quad (16)$$

where $\varphi(r) = 1/2((r_e^2/r) - r)$.

Then, an integration of (16) with respect to r in interval $[r_w, r]$ results in

$$\mathbf{u} = \mathbf{u}|_{r=r_w} + \phi(r) \mathbf{K}_r^{-1} \mathbf{C} \bar{\mathbf{u}}_{,t}, \quad (17)$$

where $\phi(r) = [(1/2)r_e^2 \ln(r/r_w)] - [(1/4)(r^2 - r_w^2)]$.

Incorporating (17) into (4) to replace $\mathbf{u}$ for $\bar{\mathbf{u}}$, we have

$$\bar{\mathbf{u}} = \mathbf{u}|_{r=r_w} + \psi \mathbf{K}_r^{-1} \mathbf{C} \bar{\mathbf{u}}_{,t}, \quad (18)$$

where $\psi = (r_w^2/n^2 - 1)((1/2)n^4 \ln n - (3/8)n^4 + (1/2)n^2 - (1/8))$.

The boundary condition of $\mathbf{u}_{,zz}$ at r_w can be expressed as follows by combining (13) and (16):

$$\mathbf{u}_{,zz}|_{r=r_w} = -\frac{2\varphi(r_w)}{r_w} \mathbf{B}^{-1} \mathbf{K}_r^{-1} \mathbf{C} \bar{\mathbf{u}}_{,t}. \quad (19)$$

Substituting (19) into (18) results in

$$\mathbf{u}_{,zz}|_{r=r_w} = \frac{1}{\kappa^2} \mathbf{B}^{-1} (\mathbf{u}|_{r=r_w} - \bar{\mathbf{u}}), \quad (20)$$

where $\kappa = (r_w/n^2 - 1)\sqrt{(1/2)n^4 \ln n - (3/8)n^4 + (1/2)n^2 - (1/8)}$.

General solution for $\mathbf{u}|_{r=r_w}$ can be written as products of functions with respect to dimensions z and time t . In addition, based on the homogeneous boundary conditions for the depth given in (14), the eigenfunction of $\mathbf{u}|_{r=r_w}$ is $\sin(Mz/H)$. Hence,

$$\mathbf{u}|_{r=r_w} = \sum_{m=1}^{\infty} \frac{2}{M} \sin \frac{Mz}{H} \left[\left(\frac{\kappa M}{H} \right)^2 \mathbf{B} + \mathbf{I} \right]^{-1} \mathbf{T}, \quad (21)$$

where $\mathbf{I}$ is the identity matrix and $M = [(2m-1)/2]\pi$.

The average excess pore pressure $\bar{\mathbf{u}}$ can be written as follows by substituting (21) into (20):

$$\bar{\mathbf{u}} = \sum_{m=1}^{\infty} \frac{2}{M} \sin \frac{Mz}{H} \mathbf{T}. \quad (22)$$

By substituting (21) and (22) into (18), the characteristic equation can be obtained using the orthogonality of sine function, as shown below:

$$\mathbf{T}_{,t} = \frac{1}{\psi} \mathbf{C}^{-1} \mathbf{K}_r \left\{ \mathbf{I} - \left[\left(\frac{\kappa M}{H} \right)^2 \mathbf{B} + \mathbf{I} \right]^{-1} \right\} \mathbf{T}. \quad (23)$$

Based on the orthogonality of the sine function, the initial condition of $\mathbf{T}$ can be obtained by the initial condition of $\bar{\mathbf{u}}$ as

$$\mathbf{T}|_{t=0} = \mathbf{u}_0. \quad (24)$$

According to the initial condition (24), (23) can be solved as

$$\mathbf{T} = e^{-\mathbf{D}t} \mathbf{u}_0, \quad (25)$$

where $\mathbf{D} = (1/\psi) \mathbf{C}^{-1} \mathbf{K}_r \left\{ \left[(\kappa M/H)^2 \mathbf{B} + \mathbf{I} \right]^{-1} - \mathbf{I} \right\}$.

Combining (21), (22), and (25) and substituting (21) and (22) into (17) give the solutions of $\bar{\mathbf{u}}$ and $\mathbf{u}$ as shown below:

$$\bar{\mathbf{u}} = \sum_{m=1}^{\infty} \frac{2}{M} \sin \frac{Mz}{H} e^{-\mathbf{D}t} \mathbf{u}_0, \quad (26)$$

$$\begin{aligned} \mathbf{u} &= \sum_{m=1}^{\infty} \frac{2}{M} \sin \frac{Mz}{H} \\ &\times \left\{ \left[1 - \frac{\phi(r)}{\psi} \right] \left[\left(\frac{\kappa M}{H} \right)^2 \mathbf{B} + \mathbf{I} \right]^{-1} + \frac{\phi(r)}{\psi} \mathbf{I} \right\} e^{-\mathbf{D}t} \mathbf{u}_0. \end{aligned} \quad (27)$$

After solving $\bar{\mathbf{u}}$, the average degree of consolidation for water and air phases (i.e., $\mathbf{U}_e = \begin{pmatrix} U_{we} \\ U_{ae} \end{pmatrix}$) can be obtained as follows:

$$\mathbf{U}_e = \mathbf{F} - \frac{1}{H} \mathbf{E}^{-1} \int_0^H \bar{\mathbf{u}} dz = \mathbf{F} - \mathbf{E}^{-1} \sum_{m=1}^{\infty} \frac{2}{M^2} e^{-\mathbf{D}t} \mathbf{u}_0, \quad (28)$$

where $\mathbf{E} = \begin{bmatrix} u_0^w & 0 \\ 0 & u_0^a \end{bmatrix}$ and $\mathbf{F} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$.

Based on Fredlund's theory, the volume change is given by the following constitutive equation for unsaturated soft soils [15]:

$$\frac{\partial \varepsilon_v}{\partial t} = m_{1k}^s \frac{\partial (q - \bar{u}_a)}{\partial t} + m_2^s \frac{\partial (\bar{u}_a - \bar{u}_w)}{\partial t}, \quad (29)$$

where ε_v is the volumetric strain, $m_{1k}^s = m_{1k}^w + m_{1k}^a$ is the coefficient of volume change with respect to a change in net normal stress, and $m_2^s = m_2^w + m_2^a$ is the coefficient of volume change with respect to a change in matric suction.

The ground surface settlement S_e of the unsaturated soft soil can be computed as

$$S_e = \int_0^H \int_0^t \frac{\partial \varepsilon_v}{\partial t} dt dz = HG \left(\mathbf{I} - \sum_{m=1}^{\infty} \frac{2}{M^2} e^{-Dt} \right) \mathbf{u}_0, \quad (30)$$

where $\mathbf{G} = \{m_2^s \quad m_{1k}^s - m_2^s\}$.

The normalized settlement, denoted as S_{ne} , can be determined based on the volumetric strain ε_v provided in (30):

$$S_{ne} = \frac{S_e}{S_e^{\infty}} = \frac{1}{\mathbf{G}\mathbf{u}_0} \mathbf{G} \left(\mathbf{I} - \sum_{m=1}^{\infty} \frac{2}{M^2} e^{-Dt} \right) \mathbf{u}_0, \quad (31)$$

where S_e^{∞} is the ultimate ground surface settlement when the pore pressures dissipated entirely (i.e., $S_e^{\infty} = H[m_{1k}^s u_0^a + m_2^s (u_0^w - u_0^a)]$). As can be seen from (31), S_{ne} is also the total degree of consolidation which is dominated by both air and water phases.

3.2. A Special Case. When the effect of drain resistance is neglected, the permeability of drain well is endless, videlicet k^w and k^a tend towards infinity, which yields $[(\kappa M/H)^2 \mathbf{B} + \mathbf{I}]^{-1} \rightarrow 0$. Parameter $\mathbf{D}$ can be rewritten as

$$\mathbf{D} = -\frac{1}{\psi} \mathbf{C}^{-1} \mathbf{K}_r. \quad (32)$$

Based on the theory of series, the following can be obtained:

$$\sum_{m=1}^{\infty} \frac{2}{M} \sin \frac{Mz}{H} = \sum_{m=1}^{\infty} \frac{2}{M^2} = 1 \quad (0 < z \leq H). \quad (33)$$

By substituting (32) and (33) into (26–28) and (30) and (31), the solutions can be simplified to

$$\bar{\mathbf{u}} = e^{(1/\psi)\mathbf{C}^{-1}\mathbf{K}_r t} \mathbf{u}_0, \quad (34)$$

$$\mathbf{u} = \frac{\phi(r)}{\psi} e^{(1/\psi)\mathbf{C}^{-1}\mathbf{K}_r t} \mathbf{u}_0, \quad (35)$$

$$\mathbf{U}_e = \mathbf{F} - \mathbf{E}^{-1} e^{(1/\psi)\mathbf{C}^{-1}\mathbf{K}_r t} \mathbf{u}_0, \quad (36)$$

$$S_e = HG \left(\mathbf{I} - e^{(1/\psi)\mathbf{C}^{-1}\mathbf{K}_r t} \right) \mathbf{u}_0, \quad (37)$$

$$S_{ne} = \frac{1}{\mathbf{G}\mathbf{u}_0} \mathbf{G} \left(\mathbf{I} - e^{(1/\psi)\mathbf{C}^{-1}\mathbf{K}_r t} \right) \mathbf{u}_0. \quad (38)$$

The equations from (34) to (38) are the simplified solutions in which the effect of drain resistance is neglected.

4. Analysis of Consolidation Behavior

In this study, the accuracy of the proposed equal-strain consolidation model in the unsaturated soft soils is

investigated by comparing with the free-strain model. The consolidation behavior and influence factors are also discussed in this section. Following Fredlund and Rahardjo [15] and Conte [26], the consolidation parameters adopted in this study are listed as follows:

(i) Mechanical parameters

$n_s = 0.5$, $S_s = 0.8$, $m_{1k}^w = -5 \times 10^{-8} \cdot \text{Pa}^{-1}$, $m_2^w = -2 \times 10^{-7} \cdot \text{Pa}^{-1}$, $m_{1k}^a = -2 \times 10^{-7} \cdot \text{Pa}^{-1}$, $m_2^a = 10^{-7} \cdot \text{Pa}^{-1}$, $k_r^w = 10^{-10} \text{ m/s}$, $k_r^a = 10^{-7} \text{ m/s}$, $k^w = 10^{-7} \text{ m/s}$, and $k^a = 10^{-4} \text{ m/s}$.

(ii) Fundamental constants of physics

$T_{at} = 298.15 \text{ K}$, $R = 8.314 \text{ J/mol} \cdot \text{K}$, $u_{atm} = 101.3 \text{ kPa}$, $\omega_a = 29 \text{ g/mol}$, $g = 9.8 \text{ m/s}^2$, and $\gamma_w = 10^4 \text{ N/m}^3$.

(iii) Geometrical parameters

$r_w = 0.2 \text{ m}$, $r_e = 1.8 \text{ m}$, and $H = 10 \text{ m}$.

(iv) Other parameters

$q = 100 \text{ kPa}$, $u_0^w = 40 \text{ kPa}$, and $u_0^a = 20 \text{ kPa}$.

The above parameters are assumed to be constant during the consolidation process. An instantaneous compression induced by the external applied load q generates initial excess pore-water and pore-air pressures (i.e., u_0^w and u_0^a , resp.). Considering that the soil is loaded three-dimensionally under isotropic conditions, changes in excess pore pressures can be determined using a method given by Fredlund and Hasan [16].

4.1. Verification. The validity of the equal-strain model had been verified in the saturated soft soils. In this section, the validity of the proposed equal-strain model in the unsaturated soft soils will be verified as well. For this purpose, the special consolidation case without considering drain resistance is analyzed by using the proposed equal-strain solution and one available free-strain solution [29].

Figures 2(a) and 2(b), respectively, demonstrate changes in the average degrees of consolidation of air and water phases with different ratios of k_r^a/k_r^w under the equal-strain and free-strain conditions. It is worth noting that the air permeability k_r^a is varying while the water permeability k_r^w is kept constant at 10^{-10} m/s . In addition, the time factor is defined as $T = C_r^w t/H^2$. It can be observed from the figures that the results obtained from these two different models are in good agreement with each other. The slightly large difference between the two models is mainly found at the earlier stage of the consolidation process. Moreover, it can be seen that with an increase of k_r^a/k_r^w , the average degrees of consolidation increase gradually for both air phase and water phase. With an increase of consolidation time T , the curve of average degree of consolidation for water phase gradually tends to be consistent. This is because the pore-air pressure almost completely dissipates at the early stage of consolidation.

Figures 3(a) and 3(b) represent the difference of average consolidation degree between equal-strain model and free-strain model varying with T in different k_r^a/k_r^w values. As observed from Figure 3(a), the maximum difference (slightly more than 3%) between two models for air phase is mainly at

FIGURE 2: Average degrees of consolidation of (a) air phase and (b) water phase varying with different ratios of k_r^a/k_r^w .

the earlier stage of consolidation before the excess pore-water pressure begins to dissipate. Inversely, the maximum difference in the water phase is mainly near the end of the air dissipation, as shown in Figure 3(b). In brief, there is negligible influence on the difference between equal-strain condition and free-strain condition with different ratios of k_r^a/k_r^w . It shows that the equal-strain model is efficient in solving the unsaturated soil consolidation problem. Also, the solution is convenient for engineering application.

4.2. Influence of Drain Resistance. Figure 4 represents the degrees of consolidation of air and water phases and normalized settlement varying with T under different values of r_w . The influence radius $r_e = 1.8$ m is adopted in this investigation. Figure 4(a) shows the degrees of consolidation of air phase varying with different values of r_w as time elapses. It is obvious that the consolidation proceeds more quickly with the increase of r_w . As the radius of vertical drain increases, the radial drainage area will be indirectly increased, which accelerates the process of consolidation. Figure 4(b) shows the degrees of consolidation of the water phase with different values of r_w . As observed, the rate of consolidation of water phase at the initial stage proceeds more quickly because of increasing r_w . After the excess pore-air pressure

FIGURE 3: Difference of degrees of consolidation in (a) air phase and (b) water phase varying with different ratios of k_r^a/k_r^w .

dissipated almost completely, a plateau period may occur in the excess pore-water pressure patterns. At the later stage, the rate of water phase consolidation is similar to the initial stage. Besides, it can also be found that the excess pore-air pressure dissipated faster than that of the excess pore-water pressure because the permeability of air phase is greater than water phase both in the vertical drain and unsaturated soft soil foundation. Figure 4(c) presents the normalized settlement against time factor T with different values of r_w . The settlement process is similar to the consolidation process of water phase.

Figure 5 demonstrates changes in excess pore pressures and normalized settlement with different ratios of k^a/k^w . Figure 5(a) illustrates the dissipation curves of excess pore-air pressure varying with T in different ratios of k^a/k^w . As observed, when k^a/k^w increases, the excess pore-air pressure tends to dissipate faster. When k^a/k^w is very high, for instance, $k^a/k^w = 10^3$, the excess pore-air pressure may dissipate instantaneously. Figure 5(b) shows the dissipation curves of excess pore-water pressure varying with k^a/k^w when time elapses. It is evident that the dissipation at the initial stage of consolidation proceeds more quickly as

FIGURE 4: Degrees of consolidation of (a) air phase, (b) water phase, and (c) normalized settlement varying with different values of r_w .

a result of increasing k^a/k^w . After the excess pore-air pressure diminished almost fully, a plateau period may occur in the excess pore-water pressure patterns when $k^a/k^w > 1$. It should be noted that the greater the ratio of k^a/k^w is, the longer the plateau gets. Lastly, the curves of excess pore-water pressure dissipation converge to a single curve, and complete dissipation occurs at approximately the same time. Figure 5(c) shows the variations of the normalized settlement against time factor T with different ratios of k^a/k^w . It is worth noting that S_{ne} patterns consist of double inverse S curves when $k^a/k^w > 0.1$, similar to the excess pore-water pressure dissipation process. The initial stage of consolidation is governed by the simultaneous dissipation of both excess pore-air and pore-water pressures, and once the

FIGURE 5: Degrees of consolidation of (a) air phase, (b) water phase, and (c) normalized settlement varying with different ratios of k^a/k^w .

excess pore-air pressure is fully dissipated in the soft soil system, S_{ne} patterns converge to only one curve and gradually approach 1 at the later stages. More remarkable, after the plateau period, the consolidation behavior is dominated by single phase (water phase) and the consolidation resembles to the classical Terzaghi's consolidation [1].

By comparing Figure 4 with Figure 5, the variation of vertical drain radius makes the normalized settlement S_{ne} a great change in the whole period of consolidation. In contrast, the difference of normalized settlement S_{ne} between different air permeabilities k^a is mainly observed before the plateau period. This is because the change of the radius of vertical drain influenced all period of consolidation process. However, the change of the air permeability of vertical drain

only had significant influence before the plateau period. After the plateau period, the consolidation process is dominated by water phase.

5. Summary and Conclusions

The equal-strain model established in this paper has a negligible difference for average degree of consolidation compared with the free-strain model. Parametric studies regarding well-resistance effect are graphically presented and discussed, and the key findings are summarized as follows:

- (1) The solutions calculated from the equal-strain model show good agreement with that obtained from the free-strain model. The main difference between the two models is mainly found at the earlier stage of the consolidation.
- (2) The higher the ratio of k_r^a/k_r^w is, the faster the consolidation is. With an increase of consolidation time T , the curve of average degree of consolidation for water phase gradually tends to be consistent.
- (3) With the increase of k^a/k^w , the excess pore-air and -water pressures tend to dissipate faster. After the excess pore-air pressure dissipated almost completely, a plateau period occurs in the excess pore-water pressure patterns when $k^a/k^w > 1$. It should be noted that the greater the ratio of k^a/k^w is, the longer the plateau gets.
- (4) Both the water and air phases of consolidation proceed more quickly with the increase of r_w , and the change of the radius of vertical drain influenced all period of consolidation process.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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