Pavel Praks and Dejan Brkić (2018): Advanced Iterative Procedures for Solving the Implicit Colebrook Equation for Fluid Flow Friction. Advances in Civil Engineering vol. 2018, Article 5451034, https://doi.org/10.1155/2018/5451034

List of Equations used in this paper:

$$\frac{1}{\sqrt{\lambda}} = \underbrace{-2 \cdot \log_{10}\left(\frac{2.51}{Re \cdot \sqrt{\lambda}} + \frac{\varepsilon}{3.7 \cdot D}\right)}_{Colebrook} \tag{1}$$

$$x_0 = \frac{1}{\sqrt{\lambda_0}} = \underbrace{-2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D}\right)}_{rough\ part\ of\ Colebrook} \tag{2}$$

$$\frac{1}{\sqrt{\lambda}} = -2 \cdot \log_{10}\left(\frac{2 \cdot 2.51 \cdot W(y)}{Re \cdot \ln(10)} + \frac{\varepsilon}{3.7 \cdot D}\right) = -2 \cdot \log_{10}\left(10^{\frac{-W(y)}{\ln(10)}} + \frac{\varepsilon}{3.7 \cdot D}\right) \tag{3}$$

where $y = \frac{Re \cdot \ln(10)}{2 \cdot 2.51} \approx \frac{Re}{2.18}$

$$\lambda = \left(\frac{2}{\ln(10)} \cdot W(e^{\alpha}) - \frac{Re \cdot \frac{\varepsilon}{D}}{3.7 \cdot 2.51}\right)^{-2} \tag{4}$$

Where $\alpha = \left(\frac{Re \cdot \frac{\varepsilon}{D} \cdot \ln(10)}{2 \cdot 2.51 \cdot 3.7} - \ln \frac{2 \cdot 3.7}{Re \cdot \ln(10)}\right)$

$$f(\lambda) = \underbrace{\frac{1}{\sqrt{|\lambda|}} + 2 \cdot \log_{10}\left(\frac{2.51}{Re \cdot \sqrt{|\lambda|}} + \frac{\varepsilon}{3.7 \cdot D}\right) = 0}_{Colebrook:\ f(\lambda)=0} \tag{5}$$

$$f'(\lambda) = \frac{d}{d\lambda} f(\lambda) = \underbrace{-\frac{1}{2} \cdot \left(\frac{1}{\sqrt{|\lambda|}}\right)^3 \cdot \left(1 + \frac{2 \cdot 2.51}{\ln(10) \cdot Re \cdot \left(\frac{2.51}{Re \cdot \sqrt{|\lambda|}} + \frac{\varepsilon}{3.7 \cdot D}\right)}\right)}_{1st\ derivative\ f'(\lambda)-Analytical} \tag{6}$$

$$\lambda_1 = \underbrace{\lambda_0 - \frac{f(\lambda_0)}{f'(\lambda_0)}}_{Newton-Raphson} = \lambda_0 - \frac{\frac{1}{\sqrt{|\lambda_0|}} + 2 \cdot \log_{10}\left(\frac{2.51}{Re \cdot \sqrt{|\lambda_0|}} + \frac{\varepsilon}{3.7 \cdot D}\right)}{-\frac{1}{2} \cdot \left(\frac{1}{\sqrt{|\lambda_0|}}\right)^3 \cdot \left(1 + \frac{2.18}{Re \cdot \left(\frac{2.51}{Re \cdot \sqrt{|\lambda_0|}} + \frac{\varepsilon}{3.7 \cdot D}\right)}\right)} \tag{7}$$

$$f(x) = \underbrace{x + 2 \cdot \log_{10}\left(\frac{2.51 \cdot x}{Re} + \frac{\varepsilon}{3.7 \cdot D}\right) = 0}_{Colebrook:\ f(x)=0} \tag{8}$$

$$f'(x) = \frac{d}{dx} f(x) = \underbrace{1 + 2 \cdot \frac{\frac{2.51}{Re \cdot \ln(10)}}{\frac{\varepsilon}{3.7 \cdot D} + \frac{2.51}{Re} \cdot x}}_{1st\ derivative\ f'(x)-Analytical} \tag{9}$$

$$x_1 = \underbrace{x_0 - \frac{f(x_0)}{f'(x_0)}}_{Newton-Raphson} = x_0 - \frac{x_0 + 2 \cdot \log_{10}\left(\frac{2.51 \cdot x_0}{Re} + \frac{\varepsilon}{3.7 \cdot D}\right)}{1 + 2 \cdot \frac{\frac{2.51}{Re \cdot \ln(10)}}{\frac{\varepsilon}{3.7 \cdot D} + \frac{2.51}{Re} \cdot x_0}} \tag{10}$$

$$f'(x) = \frac{d}{dx} f(x) = \underbrace{\frac{5.02}{\mathrm{Re} \cdot \ln(10) \cdot \left(\frac{10}{37}\frac{\varepsilon}{\mathrm{D}} + \frac{251 \cdot \mathrm{x}}{100 \cdot \mathrm{Re}}\right)} + 1 = \frac{9287 \cdot \ln(10) \cdot x + 1000 \cdot \ln(10) \cdot \frac{\varepsilon}{D} \cdot Re + 18574}{\ln(10) \cdot \left(9287 \cdot x + 1000 \cdot \frac{\varepsilon}{D} \cdot Re\right)}}_{1st\ derivative\ f'(\mathrm{x})-\mathrm{MATLAB}} \tag{11}$$

$$x_1 = \underbrace{x_0 - \frac{f(x_0)}{f'(x_0)}}_{Newton-Raphson} = x_0 - \frac{x_0 + 2\cdot\log_{10}\left(\frac{2.51\cdot x_0}{Re} + \frac{\varepsilon}{3.7\cdot D}\right)}{\frac{21384.11\cdot x_0 + 2302.58\cdot\frac{\varepsilon}{D}\cdot Re + 18574}{21384.11\cdot x_0 + 2302.58\cdot\frac{\varepsilon}{D}\cdot Re}} \tag{12}$$

$$x_1 = \underbrace{x_0 - \frac{\frac{f(x_0)}{f'(x_0)}}{1 - \frac{f(x_0)\cdot f''(x_0)}{2\cdot(f'(x_0))^2}}}_{Halley} = \underbrace{x_0 - \frac{2\cdot f(x_0)\cdot f'(x_0)}{2\cdot(f'(x_0))^2 - f(x_0)\cdot f''(x_0)}}_{Halley} \tag{13}$$

$$f''(x) = \frac{d}{dx}f'(x) = \underbrace{\frac{-12.6}{\mathrm{Re}^2\cdot\ln(10)\cdot\left(\frac{10}{37}\cdot\frac{\varepsilon}{\mathrm{D}} + \frac{251\cdot x}{100\cdot\mathrm{Re}}\right)^2} = \frac{-172496738}{\ln(10)\cdot\left(9287\cdot x + 1000\cdot\frac{\varepsilon}{D}\cdot Re\right)^2}}_{\text{2nd derivative } f''(\mathrm{x}) - \mathrm{MATLAB}} \tag{14}$$

$$x_1 = \underbrace{x_0 - \frac{6\cdot f(x_0)\cdot(f'(x_0))^2 - 3\cdot(f(x_0))^2\cdot f''(x_0)}{6\cdot(f'(x_0))^3 - 6\cdot f(x_0)\cdot f'(x_0)\cdot f''(x_0) + (f(x_0))^2\cdot f'''(x_0)}}_{\text{3rd order Householder}} \tag{15}$$

$$f'''(x) = \frac{d}{dx}f''(x) = \underbrace{\frac{63.253}{\mathrm{Re}^3\cdot\ln(10)\cdot\left(\frac{10}{37}\cdot\frac{\varepsilon}{\mathrm{D}} + \frac{251\cdot x}{100\cdot\mathrm{Re}}\right)^3} = \frac{3203954411612}{\ln(10)\cdot\left(9287\cdot x + 1000\cdot\frac{\varepsilon}{D}\cdot Re\right)^3}}_{\text{3rd derivative } f'''(\mathrm{x}) - \mathrm{MATLAB}} \tag{16}$$

$$x_1 = \underbrace{x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f''(x_0)\cdot(f(x_0))^2}{2\cdot(f'(x_0))^3}}_{Schröder} \tag{17}$$

$$\lambda_1 = \lambda_0 - \frac{f(\lambda_0)}{\frac{f(\lambda_{-1}) - f(\lambda_0)}{\lambda_{-1} - \lambda_0}} \tag{18}$$

$$x_1 = x_0 - \frac{f(x_0)}{\frac{f(x_{-1}) - f(x_0)}{x_{-1} - x_0}} \tag{19}$$

$$\left.\begin{array}{c}
\mathrm{x}_0 = 7.273124147 \\
f(x_0) = x_0 + 2\cdot\log_{10}\left(\frac{2.51\cdot x_0}{Re} + \frac{\varepsilon}{3.7\cdot D}\right) = -2.692152546 \\
f'(x_0) = \frac{9287\cdot\ln(10)\cdot x_0 + 1000\cdot\ln(10)\cdot\frac{\varepsilon}{D}\cdot Re + 18574}{\ln(10)\cdot\left(9287\cdot x_0 + 1000\cdot\frac{\varepsilon}{D}\cdot Re\right)} = \frac{5.02}{\mathrm{Re}\cdot\ln(10)\cdot\left(\frac{10}{37}\cdot\frac{\varepsilon}{\mathrm{D}} + \frac{2.51\cdot x_0}{\mathrm{Re}}\right)} + 1 = 1.041894438 \\
y_0 = x_0 - \frac{f(x_0)}{f'(x_0)} = 7.273124147 - \frac{-2.692152546}{1.041894} = 9.857025593360860 \\
f(y_0) = y_0 + 2\cdot\log_{10}\left(\frac{2.51\cdot y_0}{Re} + \frac{\varepsilon}{3.7\cdot D}\right) = -0.006232787 \\
z_0 = y_0 - \frac{f(x_0)}{f(x_0) - 2\cdot f(y_0)}\cdot\frac{f(y_0)}{f'(x_0)} = 9.863035589 \\
f(z_0) = z_0 + 2\cdot\log_{10}\left(\frac{2.51\cdot z_0}{Re} + \frac{\varepsilon}{3.7\cdot D}\right) = -0.006232787 \\
x_1 = z_0 - \frac{f(z_0)}{f'(x_0)\cdot\left[1 - 2\cdot\frac{f(y_0)}{f(x_0)} - \left(\frac{f(y_0)}{f(x_0)}\right)^2\right]\cdot\left[1 - \frac{f(z_0)}{f(y_0)}\right]\cdot\left[1 - 2\cdot\frac{f(z_0)}{f(x_0)}\right]} = 9.863034564 \\
x_1 = 9.863034564 \rightarrow \lambda_1 = 0.010279663295529
\end{array}\right\} \tag{20}$$

$$f(z) = z\cdot e^z - y = 0 \tag{21}$$

$$f'(z) = e^z\cdot(z+1) \tag{22}$$

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)} = \underbrace{z_0 - \frac{z_0 \cdot e^{z_0} - y}{e^{z_0} \cdot (z_0+1)}}_{Newton-Raphson} = z_0 - \frac{z_0 \cdot e^{z_0} - \frac{Re}{2.18}}{e^{z_0} \cdot (z_0+1)} \tag{23}$$

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0) - \frac{f(z_0) \cdot f''(z_0)}{2 \cdot f'(z_0)}} = \underbrace{z_0 - \frac{z_0 \cdot e^{z_0} - y}{e^{z_0} \cdot (z_0+1) - \frac{(z_0 \cdot e^{z_0} - y) \cdot (z_0+2)}{2 \cdot (z_0+1)}}}_{Halley} \tag{24}$$

$$f''(z) = e^z \cdot (z+2) \tag{25}$$

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)} - \frac{f''(z_0) \cdot \left(f(z_0)\right)^2}{2 \cdot \left(f'(z_0)\right)^3} = \underbrace{z_0 - \frac{z \cdot e^z - y}{e^z \cdot (z+1)} - \frac{e^z \cdot (z+2) \cdot (z \cdot e^z - y)^2}{2 \cdot \left(e^z \cdot (z+1)\right)^3}}_{\text{Schröder}} \tag{26}$$

$$\frac{1}{\sqrt{\lambda_0}} \approx \underbrace{8 - \frac{2 \cdot A}{2 - A \cdot B}}_{Halley} \approx \underbrace{8 - A - \frac{A^2 \cdot B}{2}}_{Schröder} \approx \underbrace{8 - \frac{6 \cdot A - 3 \cdot A^2 \cdot B}{6 - 6 \cdot A \cdot B + A^2 \cdot C}}_{3^{rd}\ order} \tag{27}$$

$$\left.\begin{aligned}
&\frac{1}{\sqrt{\lambda_1}} \approx \underbrace{-2 \cdot \log_{10}\left(\frac{2.51}{Re} \cdot \frac{1}{\sqrt{\lambda_0}} + \frac{\varepsilon}{3.7 \cdot D}\right)}_{1st\ Colebrook's\ acceleration} \\
&\frac{1}{\sqrt{\lambda_2}} \approx \underbrace{-2 \cdot \log_{10}\left(\frac{2.51}{Re} \cdot \frac{1}{\sqrt{\lambda_1}} + \frac{\varepsilon}{3.7 \cdot D}\right)}_{2st\ Colebrook's\ acceleration} = \underbrace{-2 \cdot \log_{10}\left(\frac{2.51}{Re} \cdot \left(\underbrace{-2 \cdot \log_{10}\left(\frac{2.51}{Re} \cdot \frac{1}{\sqrt{\lambda_0}} + \frac{\varepsilon}{3.7 \cdot D}\right)}_{1st\ Colebrook's\ acceleration}\right) + \frac{\varepsilon}{3.7 \cdot D}\right)}_{2st\ Colebrook's\ acceleration} \\
&\vdots \\
&\frac{1}{\sqrt{\lambda_{i+1}}} \approx \underbrace{-2 \cdot \log_{10}\left(\frac{2.51}{Re} \cdot \frac{1}{\sqrt{\lambda_i}} + \frac{\varepsilon}{3.7 \cdot D}\right)}_{Colebrook-acceleration}
\end{aligned}\right\} \tag{28}$$

$$A \approx 8 + 2 \cdot \log_{10}\left(\frac{16}{Re} + \frac{\varepsilon}{3.7 \cdot D}\right) \tag{29}$$

$$B \approx \frac{-74914381.46}{\nabla^2} \tag{30}$$

$$C \approx \frac{1391459721232.67}{\nabla^3} \tag{31}$$

$$\nabla \approx 74205.5 + 1000 \cdot \frac{\varepsilon}{D} \cdot Re \tag{32}$$

$$\left.\begin{aligned}
\frac{1}{\sqrt{\lambda}} &\approx -2 \cdot \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} - \frac{5.02}{Re} \cdot \alpha_1\right) \\
\alpha_1 &\approx \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} - \frac{5.02}{Re} \cdot \alpha_2\right) \\
\alpha_2 &\approx \log_{10}\left(\frac{\varepsilon}{3.7 \cdot D} - \frac{13}{Re}\right)
\end{aligned}\right\} \tag{33}$$