Research Article

Consistency Degrees of Theories in Łukasiewicz Fuzzy and n -Valued Propositional Logic Systems

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By means of theory of truth degrees of formulas, according to deduction theorems and completeness theorems, the new concepts of consistency degrees and polar index for general theories in Łukasiewicz fuzzy and n -valued propositional logic systems are introduced. Moreover, sufficient and necessary conditions for a theory Γ to be consistent, inconsistent, and fully divergent are obtained. Finally, some important properties of truth degrees of formulas are proposed.

1. Introduction

Whether a theory (i.e., a set of formulas) is consistent or not (a theory Γ is inconsistent if $\Gamma \vdash \bar{0}$, otherwise it is consistent, where $\bar{0}$ is a contradiction [1]) is one of the crucial questions in any logic system. The reason is that, in classical logic, a contradictory theory (i.e., a theory which is not consistent) turns into a useless theory in which everything is provable. Quite surprising is that the same result holds also in fuzzy and many-valued logic systems (it was proved in [1] that Γ is inconsistent if and only if $\Gamma \vdash A$ for each A). It is clear that theories Γ containing a contradiction are inconsistent, but inconsistent theories Γ need not contain a contradiction. For example, let $\Gamma = \{p, \neg p\}$, where p is any atomic formula of S , since $p \rightarrow (\neg p \rightarrow \bar{0})$ is a theorem, then Γ is an inconsistent theory not containing any contradiction. In [4], a systematic and exhaustive treatment of fuzzy logic (FL n) is proposed where both semantics and syntax are evaluated gradationally and even axioms there need not be fully true. Hence to propose a graded theory for expounding consistent degrees of fuzzy theories seems not easy to be accomplished as it is pointed out in [4] that “even the attempt to introduce some kind of degrees of consistency mostly does not work.” The concept of degree of inconsistency of a fuzzy theory Γ has been introduced in [4] as follows:

$$\text{Incons}(\Gamma) = \sup\{a \mid \Gamma \vdash_a A \wedge \neg A \text{ and } A \in \mathcal{F}_{\mathcal{L}(\Gamma)}\}, \quad (1)$$

where $\vdash_a$ is an inference sign (see [5]). It was proven that a fuzzy theory is consistent if and only if $\text{Incons}(\Gamma) \leq 1/2$. As was pointed out in [2], the truth valuation $D(A \wedge \neg A)$ of a formula $A \wedge \neg A$ need not be zero in multivalued logic even though it looks like a contradiction. Hence, it seems that to define $\text{Incons}(\Gamma)$ by means of $\Gamma \vdash_a A \wedge \neg A$ is not as good as to analyse the disparities of formulas of Γ -conclusions.

How to measure the extent to which a theory is consistent is also one of the crucial questions in logic systems. For trying to grade the extent of consistency of different theories, many authors have proposed different methods in fuzzy (continuous-valued) logic systems and have obtained many good results [2, 3]. The concept of consistency degree of a fuzzy theory Γ has been introduced in [2, 3] as follows:

$$\begin{aligned} \xi(\Gamma) &= \lim_{n \rightarrow \infty} \left[1 - \text{div}(\Gamma) \left(1 - \frac{1}{2} [\alpha_n, \Gamma] \right) \right], \\ \xi^*(\Gamma) &= 1 - \frac{1}{2} \text{div}(\Gamma)(1 + i(\Gamma)), \end{aligned} \quad (2)$$

where $\alpha_{n,\Gamma} = (d(\Gamma^{(n+2)}) - d(\Gamma^{(n)}))/d(\Gamma^{(n+2)})$, $\text{div}(\Gamma) = \sup\{\rho(A, B) \mid A, B \in D(\Gamma)\}$, $\Gamma^{(n)} = \{A \in \mathcal{F} \mid \text{there exists a deduction of } A \text{ from } \Gamma \text{ of length } n\}$, $d(\Gamma) = \sup\{\rho(A, B) \mid A, B \in \Gamma\}$, $i(\Gamma) = \max\{\lfloor \rho(A, B) \rfloor \mid A, B \in D(\Gamma)\}$, $\lfloor x \rfloor$ is the smallest integer at least as large as x (for any real number x), and $\rho(A, B)$ and $D(\Gamma)$ were defined by Definitions 5 and 7.

The consistency degrees formula of theories of Łukasiewicz fuzzy propositional logic systems is obtained only in

the condition of finite theories Γ (Γ is a finite set of formulas) in [2]; it was pointed out in the conclusion of [2] that “How to define a reasonable concept of consistency degrees for infinite theories of Łuk and how to extend the result obtained in the present paper to the more general fuzzy logic with graded syntax given in [5] would be attractive research topics.” In the condition of infinite theories, the consistency degrees formula is obtained by the definitions of divergence degree in the references [2, 3]. But the character of divergence degrees is different from the character of consistency degree. It is natural for us to find another measure to describe consistency degree. In this paper, from logical point of view and based on deduction theorems, completeness theorems, and the concept of truth degrees of formulas, we introduce a more natural and reasonable definition of consistency degrees of theories in Łukasiewicz n -valued and fuzzy propositional logic systems.

2. Preliminaries

2.1. Logic Systems: $\mathbb{L}_n$ and $\mathbb{L}_{\text{Łuk}}$. Suppose that $S = \{p_1, p_2, \dots\}$ is a countable set, let $\neg$ and $\rightarrow$ be unary and binary operations, respectively, and let $F(S)$ be the free algebra of type $(\neg, \rightarrow)$ generated by S , that is, $F(S)$ is the smallest set containing S and closed under the operation $\neg$ and $\rightarrow$. Elements of $F(S)$ are called propositions or formulas and those of S are called atomic propositions or atomic formulas. In Łukasiewicz fuzzy propositional logic (briefly, Łuk), there are four axiom schemes as follows: $A \rightarrow (B \rightarrow A)$, $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$, $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$, $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$. The axioms of Łukasiewicz n -valued propositional logic (briefly, $\mathbb{L}_n$) are those of Łuk plus $(A \rightarrow^n B) \rightarrow (A \rightarrow^{n-1} B)$, $(A \equiv (A \rightarrow^{s-2} \neg B)) \rightarrow^{n-1} A$ for any $2 \leq s \leq n-1$ such that s is not a divisor of $n-1$. Where $A \equiv B$ instead of $(A \rightarrow B) \& (B \rightarrow A)$, $A \& B$ is $\neg(A \rightarrow \neg B)$ (see [6]).

Here are two kinds of valuation sets: $L_n = \{0, 1/(n-1), 2/(n-1), \dots, (n-2)/(n-1), 1\}$ and $L = [0, 1]$. Define on L_n and L a unary operator and a binary operator as follows: $\neg x = 1 - x$, $x \rightarrow y = (1 - x + y) \wedge 1$. These two sets of valuation are just the standard MV_n -chain and the standard MV -algebra $(L, \neg, \rightarrow)$, respectively.

A homomorphism $v : F(S) \rightarrow L_n$ (or L) of type $(\neg, \rightarrow)$ from $F(S)$ into the valuation set L_n (or L), that is, $v(\neg A) = \neg v(A)$, $v(A \rightarrow B) = v(A) \rightarrow v(B)$, is called a valuation of $F(S)$. The set of all valuations will be denoted by Ω .

Łukasiewicz propositional logic system is called the Łukasiewicz n -valued propositional logic system if the set of valuation is $L_n = \{0, 1/(n-1), 2/(n-1), \dots, (n-2)/(n-1), 1\}$. Łukasiewicz propositional logic system is called the Łukasiewicz Fuzzy propositional logic system if the set of valuation is $L = [0, 1]$. It has been proved that the semantics and syntax of $\mathbb{L}_n$ and $\mathbb{L}_{\text{Łuk}}$ are in perfect harmony, that is, the standard completeness theorem holds in $\mathbb{L}_n$ and $\mathbb{L}_{\text{Łuk}}$, respectively.

The deduction rule is Modus Ponens (briefly, MP) of which B can be deduced from A and $A \rightarrow B$. As it is well

known, the following abbreviations introduced in Łuk and $\mathbb{L}_n$ are popular [1]:

$$\begin{aligned} A \vee B &:= (B \rightarrow A) \rightarrow A, \\ A \wedge B &:= \neg(\neg A \vee \neg B), \\ A^k &:= \underbrace{A \& A \& \dots \& A}_{k\text{-times}}. \end{aligned} \quad (3)$$

2.2. Truth Degrees of Formulas in $\mathbb{L}_n$

Definition 1 (see [7]). Suppose that (X_k, A_k, μ_k) ($k = 1, 2, \dots$) are probability measure spaces; let $X = \prod_{k=1}^{\infty} X_k$, then $\prod_{k=1}^{\infty} A_k$ generates on X a σ -algebra $\mathcal{A}$. There exists on X a measure μ satisfying the following conditions.

- (i) $\mathcal{A}$ is the set consisting of all μ -measurable sets.
- (ii) For any measurable subset E of $\prod_{k=1}^m X_k$, $E \times \prod_{k=m+1}^{\infty} X_k$ is μ -measurable and

$$\mu\left(E \times \prod_{n=m+1}^{\infty} X_n\right) = (\mu_1 \times \mu_2 \times \dots \times \mu_m)(E), \quad m = 1, 2, \dots, \quad (4)$$

μ is called the infinite product measure of $\mu_1, \mu_2, \dots$, and $(X, \mathcal{A}, \mu)$ is called the infinite product of $\{(X_k, A_k, \mu_k)\}_{k=1}^{\infty}$. $(X, \mathcal{A}, \mu)$ can be abbreviated as X if no confusion arises.

Suppose that n is a fixed natural number, $n \geq 2$, and (Y, B, η) is an evenly distributed probability measure space, where $Y = \{y_1, y_2, \dots, y_n\}$, that is, $\eta(\emptyset) = 0$, $\eta(Y) = 1$, and $\eta(\{y_i\}) = 1/n$ ($i = 1, 2, \dots, n$). Let $(X_k, A_k, \mu_k) = (Y, B, \eta)$ ($k = 1, 2, \dots$), and let $(X, \mathcal{A}, \mu)$ be the infinite product of $\{(X_k, A_k, \mu_k)\}_{k=1}^{\infty}$.

In the following, Y will be written as $Y = L_n = \{0, 1/(n-1), 2/(n-1), \dots, (n-2)/(n-1), 1\}$; hence we have

$$\begin{aligned} X_k &= L_n \\ &= \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\right\}, \quad k = 1, 2, \dots, \end{aligned} \quad (5)$$

and $(X, \mathcal{A}, \mu)$ can also be written as L_n^{∞} .

Let $v \in \Omega$, then v is decided by its restriction $v|_S$ because S generates the free algebra $F(S)$. Assume that $v(p_k) = v_k$ ($k = 1, 2, \dots$), then an infinite-dimensional vector $\vec{v} = (v_1, v_2, \dots)$ in L_n^{∞} is obtained. Conversely, let $\vec{v} = (v_1, v_2, \dots)$ be any element of L_n^{∞} , then there exists a unique $v \in \Omega$ such that $v(p_k) = v_k$ ($k = 1, 2, \dots$). Hence there exists a bijection $\varphi : \Omega \rightarrow L_n^{\infty}$ defined by $\varphi(v) = \vec{v}$.

Definition 2 (see [8]). The abovementioned mapping is called the measured mapping of Ω .

Definition 3 (see [8]). Suppose that $A \in F(S)$, $n \geq 2$ and define

$$[A]_{i/(n-1)} = \left\{ \vec{v} \in L_n^\infty \mid \phi^{-1}(\vec{v})(A) = \frac{i}{n-1} \right\},$$

$$i = 0, 1, \dots, n-1,$$

$$T(A) = \sum_{i=0}^{n-1} \frac{i}{n-1} \mu([A]_{i/(n-1)}) \quad (6)$$

$$= \sum_{i=1}^{n-1} \frac{i}{n-1} \mu([A]_{i/(n-1)}),$$

then $T(A)$ is called the n -valued truth degree of A .

Example 1. Let $L_8 = \{0, 1/7, 2/7, \dots, 6/7, 1\}$. Consider the truth degree $T(p \rightarrow q)$. Notice that $v(p \rightarrow q) = v(p) \rightarrow v(q) = (1 - v(p) + v(q)) \wedge 1$. Let $v(p) = j/7, v(q) = k/7$, and $v(p \rightarrow q) = i/7$, then $(1 - (j/7) + (k/7)) \wedge 1 = i/7, i = 1, 2, \dots, 7 \dots (*)$. If $i = 7$, then $0 \leq j \leq k \leq 7$ by $(*)$; there are a total of $(8 \times 9)/2$ pairs $(v(p), v(q))$ such that $v(p \rightarrow q) = 1$. If $1 \leq i \leq 6$, then $j = 7 - i + k$ by $(*)$; there are a total of $i + 1$ pairs (j, k) such that $j = 7 - i + k$. Thus,

$$\mu([p \rightarrow q]_{i/7}) = \begin{cases} \frac{8 \times 9}{2 \times 64}, & i = 7, \\ \frac{i+1}{64}, & 1 \leq i \leq 6, \end{cases} \quad (7)$$

$$T(p \rightarrow q) = \sum_{i=1}^6 \frac{i}{7} \frac{i+1}{64} + \frac{8 \times 9}{2 \times 64} = \frac{39}{48}.$$

Assume that $A = A(p_1, p_2, \dots, p_m)$ is a formula generated by atomic formulas $p_1, p_2, \dots, p_m$ through connectives $\neg$ and $\rightarrow$. Substitute x_i for p_i in $A (i = 1, 2, \dots, m)$ and keep the logic connectives in A unchanged but explain them as the corresponding operators defined on the valuation lattice L_n . Then we get a function $A^* : L_n^m \rightarrow L_n$ and call $A^*(x_1, \dots, x_m)$ the truth degree function of A .

Lemma 1 (see [8]). Let $A = A(p_1, \dots, p_m) \in F(S), n \geq 2$. Then

$$T(A) = \sum_{i=0}^{n-1} \frac{i}{n-1} \frac{|(A^*)^{-1}(i/(n-1))|}{n^m}, \quad (8)$$

where $A^*(x_1, \dots, x_m) = i/(n-1)$ if and only if $x_j = v(p_j), j = 1, 2, \dots, m$, and $v(A(p_1, \dots, p_m)) = i/(n-1), i = 0, 1, \dots, n-1$. Obviously, $L_n^m = \cup \{(A^*)^{-1}(i/(n-1)) \mid i = 0, 1, \dots, n-1\}$, $|L_n^m| = n^m$, and $\mu([A]_{i/(n-1)}) = |(A^*)^{-1}(i/(n-1))|/n^m$.

2.3. Truth Degrees of Formulas in Luk . Let $A = A(p_1, p_2, \dots, p_m)$ be a formula obtained by connecting the atomic formulas $p_1, p_2, \dots, p_m$ with the connectives $\neg$ and $\rightarrow$, then A determines a function $\bar{A} : [0, 1]^m \rightarrow [0, 1]$, where $\bar{A}(x_1, \dots, x_m)$ is obtained by connecting the variables $x_1, \dots, x_m$ in $[0, 1]$ with the operators $\neg$ and $\rightarrow$ on $[0, 1]$

in the same way as A is constructed from $p_1, \dots, p_m$. For example, if $A(p_1, p_2, p_3) = p_1 \rightarrow (\neg p_2 \rightarrow p_3)$, then $\bar{A}(x_1, x_2, x_3) = x_1 \rightarrow (\neg x_2 \rightarrow x_3) = (1 - x_1 + (x_2 + x_3) \wedge 1) \wedge 1$. It is clear that $v(A(p_1, p_2, p_3)) = \bar{A}(v(p_1), v(p_2), v(p_3))$ and in general $v(A(p_1, \dots, p_m)) = \bar{A}(v(p_1), v(p_2), \dots, v(p_m)) (A \in F(S), v \in \Omega)$ [9, 10], where $\bar{A}$ is a McNaughton function [11].

Definition 4 (see [9, 10]). Let $A(p_1, p_2, \dots, p_m)$ be a formula of $F(S)$ and

$$T(A) = \underbrace{\int_0^1 \cdots \int_0^1}_{m} \bar{A}(x_1, \dots, x_m) dx_1 dx_2 \cdots dx_m. \quad (9)$$

Then $T(A)$ is called the truth degree of A .

Remark 1. (1) It is easy to verify that $T(A) = 1$ if and only if A is a (logical) theorem, and $T(A) = 0$ if and only if A is a contradiction (see [9]).

(2) Notice that $\bar{A}$ can also be considered as a function of $m + k$ variables, where

$$\bar{A}(x_1, \dots, x_{m+k}) = \bar{A}(x_1, \dots, x_m)(x_{m+1}, \dots, x_{m+k} \in [0, 1]),$$

$$T(A) = \underbrace{\int_0^1 \cdots \int_0^1}_{m+k} \bar{A}(x_1, \dots, x_{m+k}) dx_1 dx_2 \cdots dx_{m+k}. \quad (10)$$

Hence if we are given two formulas $A(p_1, p_2, \dots, p_n)$ and $B(q_1, q_2, \dots, q_k)$, we can choose an m large enough such that both $\bar{A}$ and $\bar{B}$ can be written as $\bar{A}(x_1, \dots, x_m)$ and $\bar{B}(x_1, \dots, x_m)$, respectively.

2.4. Pseudometric of $F(S)$ in L_n and Luk

Definition 5. (i) A subset Γ of $F(S)$ is called a theory.

(ii) Let Γ be a theory; $A \in F(S)$. A deduction of A from Γ , in symbols, $\Gamma \vdash A$, is a finite sequence of formulas $A_1, \dots, A_n = A$ such that, for each $1 \leq i \leq n$, A_i is an axiom, or $A_i \in \Gamma$, or there are $j, k \in \{1, \dots, i-1\}$ such that A_i follows from A_j and A_k by MP. Equivalently, we say that A is a conclusion of Γ (or Γ -conclusion). The set of all conclusions of Γ is denoted by $D(\Gamma)$. By a proof of A we will henceforth mean a deduction of A from the empty set. We will also write A in place of $\emptyset \vdash A$ and call A a theorem.

Definition 6 (see [9, 10]). Suppose that $A, B \in F(S)$, then

$$\xi(A, B) = T((A \rightarrow B) \wedge (B \rightarrow A)) \quad (11)$$

is called the resemblance degree between A and B .

Definition 7 (see [9, 10]). Suppose that $A, B \in F(S)$, and define

$$\rho(A, B) = 1 - \xi(A, B). \quad (12)$$

Then ρ is a pseudometric on $F(S)$, and $(F(S), \rho)$ is called the logic metric space.

Definition 8 (see [9, 10]). Suppose that Γ is a theory; define

$$\text{div}(\Gamma) = \sup\{\rho(A, B) \mid A, B \in D(\Gamma)\}. \quad (13)$$

Then $\text{div}(\Gamma)$ is called the divergence degree of Γ . Γ is said to be fully divergent if $\text{div}(\Gamma) = 1$.

2.5. Generalized Deduction Theorems. Deduction theorem is one of the most important theorems in classical mathematical logic, and it says that

$$\Gamma \cup \{A\} \vdash B \quad \text{iff} \quad \Gamma \vdash A \rightarrow B. \quad (14)$$

The deduction theorem is no longer valid in fuzzy logic systems in general. Fortunately, it has been proved that there exist different weak forms of the deduction theorems (called Generalized deduction theorems) in different logic systems, respectively [1, 12].

Theorem 1. Suppose that Γ is a theory, and $A, B \in F(S)$, then

- (i) in Luk , $\Gamma \cup \{A\} \vdash B$ if and only if $\exists m \in N$, s.t. $\Gamma \vdash A^m \rightarrow B$;
- (ii) in Ln , $\Gamma \cup \{A\} \vdash B$ if and only if $\Gamma \vdash A^{n-1} \rightarrow B$.

Remark 2. It is easy to verify that $(A \& B) \rightarrow C$ and $A \rightarrow (B \rightarrow C)$ are provably equivalent; hence by the definition of deduction, the generalized deduction theorem in Luk and Ln can be equivalently described as follows

$$\begin{aligned} \Gamma \vdash A \quad & \text{iff} \quad \exists A_1, \dots, A_s \in \Gamma, \quad m_1, \dots, m_s \in N, \\ & \text{s.t.} \quad \vdash A_1^{m_1} \& \dots \& A_s^{m_s} \rightarrow A, \text{ hold.} \end{aligned} \quad (15)$$

3. Main Results

It is easy for the reader to check the following lemma.

Lemma 2. Suppose that $A = A(p_{i_1}, p_{i_2}, \dots, p_{i_n}) \in F(S)$ and $n \geq 2$, then $T(A) = 1$ if and only if A is a tautology in Ln .

Lemma 3. In Ln and Luk , if $A \rightarrow B$ is a logical theorem, then $T(A) \leq T(B)$.

Lemma 4. Let $A, B, C \in F(S)$ in Ln and Luk , then the following properties hold.

- (i) If A is a theorem, then $T(A \rightarrow B) = T(B)$.
- (ii) one has $T(A \rightarrow \bar{0}) = 1 - T(A)$.

Now we are ready to define the concept of consistency degree for finite and infinite theories. Let us first take an analysis on the inconsistency of a theory in Luk and Ln . Suppose that Γ is a theory and Γ is inconsistent, then the contradiction $\bar{0}$ is a conclusion of Γ ; that is to say, $\Gamma \vdash \bar{0}$ holds. It follows from Remark 2 that there exists a finite string of formulas $A_1, A_2, \dots, A_s \in \Gamma$ and $m_1, m_2, \dots, m_s \in N$ such that $\vdash A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}$ holds. By standard completeness theorem, $A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}$ is a tautology; then $T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}) = 1$ in Ln and Luk , respectively. Conversely, if there are a finite sequence of formulas $A_1, A_2, \dots, A_s \in \Gamma$ and $m_1, m_2, \dots, m_s \in N$

such that $T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}) = 1$, by Lemma 2 and Remark 1, $A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}$ is a theorem; hence by the generalized deduction theorem, $\Gamma \vdash \bar{0}$ holds, and so Γ is inconsistent. From the above analysis, in order to judge whether a given theory Γ is inconsistent or not, it suffices to calculate $T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0})$ of all possible formulas of the form $A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}$, where $A_1, A_2, \dots, A_s \in \Gamma$ and $m_1, m_2, \dots, m_s \in N$. Moreover, the larger $T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0})$ of such formulas is, the more closer Γ that is to be inconsistent (see Theorem 6). Note that there may be an infinite number of such formulas for a given theory Γ even though Γ is finite; hence it is natural and reasonable for us using the supremum of $T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0})$ of all formulas with the form $A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}$, where $A_1, \dots, A_s \in \Gamma$ and $m_1, \dots, m_s \in N$ to measure the inconsistency of Γ .

Definition 9. Suppose that Γ is a theory, and $\bar{0}$ is a contradiction. Define

$$\begin{aligned} j(\Gamma) &= \sup\{T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}) \mid A_1, \dots, A_s \in \Gamma, \\ &\quad s \in N, (m_1, \dots, m_s) \in N^s\}, \end{aligned} \quad (16)$$

$$\begin{aligned} k(\Gamma) &= \max\{[T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0})] \mid A_1, \dots, A_s \in \Gamma, \\ &\quad s \in N, (m_1, \dots, m_s) \in N^s\}, \end{aligned} \quad (17)$$

where

$$[x] = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases} \quad (18)$$

Then $j(\Gamma)$ is called the first polar index of Γ and $k(\Gamma)$ is called the second polar index of Γ in Luk and Ln , respectively.

Definition 10. Suppose that Γ is a theory and define

$$\text{Consist}(\Gamma) = 1 - \frac{1}{2}(j(\Gamma) + k(\Gamma)), \quad (19)$$

where $\text{Consist}(\Gamma)$ is called the consistency degree of Γ in Luk and Ln , respectively.

Remark 3. (i) Since, for all $A, B, C \in F(S)$, $A \& B \rightarrow C$ and $B \& A \rightarrow C$ are logically equivalent, the definition of $A_1^{m_1} \& \dots \& A_s^{m_s} \rightarrow \bar{0}$ does not depend on the order of $A_i^{m_i}$'s.

(ii) It is easy to verify that, for all $A, B, C \in F(S)$, $T(A \& B \rightarrow C) \geq T(A \rightarrow C)$; hence if $s_1 \leq s_2$, then $T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_{s_1}^{m_{s_1}} \rightarrow \bar{0}) \leq T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_{s_2}^{m_{s_2}} \rightarrow \bar{0})$.

Theorem 2. Suppose that Γ is a theory, then, in Ln ,

$$j(\Gamma) = \sup\{T(A_1^{n-1} \& A_2^{n-1} \& \dots \& A_s^{n-1} \rightarrow \bar{0}) \mid A_1, A_2, \dots, A_s \in \Gamma, s \in N\}. \quad (20)$$

Proof. For every $A \in F(s)$, if $m \geq n - 1$, then it is easy to check that $A^m \sim A^{n-1}$. If $m < n - 1$, then $\vdash A^{n-1} \rightarrow A^m$ holds by axioms. Since $\vdash (A^{n-1} \rightarrow A^m) \rightarrow ((A^m \rightarrow \bar{0}) \rightarrow (A^{n-1} \rightarrow \bar{0}))$, thus, for every $m \in N$, $\vdash ((A^m \rightarrow \bar{0}) \rightarrow (A^{n-1} \rightarrow \bar{0}))$ holds; then by Lemma 3, $T(A^m \rightarrow \bar{0}) \leq T(A^{n-1} \rightarrow \bar{0})$. For the same reason, for any $A_1, A_2, \dots, A_s \in \Gamma$ and any $m_1, m_2, \dots, m_s \in N$, we have $\vdash A_1^{n-1} \& \dots \& A_s^{n-1} \rightarrow A_1^{m_1} \& \dots \& A_s^{m_s}$ and $T(A_1^{n-1} \& \dots \& A_s^{n-1} \rightarrow \bar{0}) \geq T(A_1^{m_1} \& \dots \& A_s^{m_s} \rightarrow \bar{0})$. Then $j(\Gamma) = \sup\{T(A_1^{n-1} \& A_2^{n-1} \& \dots \& A_s^{n-1} \rightarrow \bar{0}) \mid A_1, A_2, \dots, A_s \in \Gamma, s \in N\}$. $\square$

Theorem 3. Suppose that $\Gamma = \{A_1, A_2, \dots, A_m\}$, then,

- (i) in $\mathcal{L}_n$, $j(\Gamma) = T(A_1^{n-1} \& \dots \& A_m^{n-1} \rightarrow \bar{0}) = 1 - T(A_1^{n-1} \& \dots \& A_m^{n-1})$.
- (ii) In $\mathcal{L}_{uk}$, $j(\Gamma) = \sup\{T(A_1^{t_1} \& A_2^{t_2} \& \dots \& A_m^{t_m} \rightarrow \bar{0}) \mid (t_1, \dots, t_m) \in N^m\} = 1 - \inf\{T(A_1^{t_1} \& A_2^{t_2} \& \dots \& A_m^{t_m}) \mid (t_1, \dots, t_m) \in N^m\}$.

Proof. By Definition 9, Theorem 2, and Lemma 4, it is easy to check the theorem above. The proof is left to the reader. $\square$

Theorem 4. Suppose that Γ is a theory, and $\bar{0}$ is a contradiction. Then

$$j(\Gamma) = \sup\{T(A \rightarrow \bar{0}) \mid A \in D(\Gamma)\}. \quad (21)$$

Proof. For all $A_1, A_2, \dots, A_s \in \Gamma$, $(m_1, \dots, m_s) \in N^s$, since $A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \in D(\Gamma)$, thus $\{A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0} \mid A_1, A_2, \dots, A_s \in \Gamma, s \in N, (m_1, \dots, m_s) \in N^s\} \subseteq \{A \rightarrow \bar{0} \mid A \in D(\Gamma)\}$; then $j(\Gamma) \leq \sup\{T(A \rightarrow \bar{0}) \mid A \in D(\Gamma)\}$. Conversely, for all $A \in D(\Gamma)$, it follows from Remark 2 that $\exists A_1, \dots, A_t \in \Gamma$ and $(m_1, \dots, m_t) \in N^t$ such that $\vdash A_1^{m_1} \& A_2^{m_2} \& \dots \& A_t^{m_t} \rightarrow A$, $\vdash (A_1^{m_1} \& A_2^{m_2} \& \dots \& A_t^{m_t} \rightarrow A) \rightarrow ((A \rightarrow \bar{0}) \rightarrow (A_1^{m_1} \& A_2^{m_2} \& \dots \& A_t^{m_t} \rightarrow \bar{0}))$ by axioms; thus $\vdash (A \rightarrow \bar{0}) \rightarrow (A_1^{m_1} \& A_2^{m_2} \& \dots \& A_t^{m_t} \rightarrow \bar{0})$ by MP. It follows from Lemma 3 that $T(A \rightarrow \bar{0}) \leq T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_t^{m_t} \rightarrow \bar{0})$; then $j(\Gamma) \geq \sup\{T(A \rightarrow \bar{0}) \mid A \in D(\Gamma)\}$. This completes the proof. $\square$

Theorem 5. Suppose that Γ is a theory. Then the following are given.

- (i) Γ is consistent in $\mathcal{L}_{uk}$ and $\mathcal{L}_n$ if and only if $k(\Gamma) = 0$ in $\mathcal{L}_{uk}$ and $\mathcal{L}_n$, respectively.
- (ii) Γ is inconsistent in $\mathcal{L}_{uk}$ and $\mathcal{L}_n$ if and only if $k(\Gamma) = 1$ in $\mathcal{L}_{uk}$ and $\mathcal{L}_n$, respectively.

Proof. Since Γ is consistent or inconsistent, and $k(\Gamma) \in \{0, 1\}$, it suffices to prove (i).

Assume that Γ is consistent, that is, $\bar{0} \notin D(\Gamma)$. For any given $A_1, A_2, \dots, A_s \in \Gamma$ and $m_1, m_2, \dots, m_s \in N$, then it follows from generalized deduction theorems that $(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s}) \rightarrow \bar{0}$ is not a theorem; then, by Lemma 2 and Remark 1, $T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}) < 1$; thus $k(\Gamma) = 0$. Conversely, if $k(\Gamma) = 0$, then, for any $A_1, A_2, \dots, A_s \in \Gamma$ and $m_1, m_2, \dots, m_s \in N$, it follows from Definition 9 that $T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}) < 1$; hence $A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}$ is not a theorem by Lemma 2

and Remark 1. Thus $\bar{0} \notin D(\Gamma)$ by generalized deduction theorems; then, Γ is consistent. $\square$

Theorem 6. Suppose that Γ is a theory, then, in $\mathcal{L}_n$ and $\mathcal{L}_{uk}$,

- (i) Γ is consistent if and only if $1/2 \leq \text{Consist}(\Gamma) \leq 1$,
- (ii) Γ is inconsistent if and only if $\text{Consist}(\Gamma) = 0$,
- (iii) Γ is consistent and full divergent if and only if $\text{Consist}(\Gamma) = 1/2$.

Proof. (a) In $\mathcal{L}_n$, (i) by Theorem 5, Γ is consistent if and only if $k(\Gamma) = 0$; hence (i) holds.

(ii) If Γ is inconsistent, then $k(\Gamma) = 1$ by Theorem 5, thus $j(\Gamma) = 1$ and $\text{Consist}(\Gamma) = 0$. Conversely, if $\text{Consist}(\Gamma) = 0$, then $k(\Gamma) + j(\Gamma) = 2$; hence $k(\Gamma) = 1$ and Γ is inconsistent by Theorem 5.

(iii) If Γ is consistent and full divergent, then $\text{div}(\Gamma) = 1$ and $k(\Gamma) = 0$ by Theorem 5. For all $\rho(A, B) \in \{\rho(A, B) \mid A, B \in D(\Gamma)\}$, since $\rho(A, B) = 1 - T((A \rightarrow B) \wedge (B \rightarrow A))$ and $A \rightarrow B, B \rightarrow A \in D(\Gamma)$, it follows from generalized deduction theorems that there exist $A_1, \dots, A_s \in \Gamma$ and $B_1, \dots, B_t \in \Gamma$, such that $\vdash A_1^{n-1} \& \dots \& A_s^{n-1} \rightarrow (A \rightarrow B)$ and $\vdash B_1^{n-1} \& \dots \& B_t^{n-1} \rightarrow (B \rightarrow A)$. Since $\vdash A_1^{n-1} \& \dots \& A_s^{n-1} \& B_1^{n-1} \& \dots \& B_t^{n-1} \rightarrow (A \rightarrow B) \& (B \rightarrow A)$ and $\vdash (A \rightarrow B) \& (B \rightarrow A) \rightarrow (A \rightarrow B) \wedge (B \rightarrow A)$ holds, hence $\vdash A_1^{n-1} \& \dots \& A_s^{n-1} \& B_1^{n-1} \& \dots \& B_t^{n-1} \rightarrow (A \rightarrow B) \wedge (B \rightarrow A)$ by hypothetical syllogism. By axioms, $\vdash ((A \rightarrow B) \wedge (B \rightarrow A) \rightarrow \bar{0}) \rightarrow (A_1^{n-1} \& \dots \& A_s^{n-1} \& B_1^{n-1} \& \dots \& B_t^{n-1} \rightarrow \bar{0})$, so by Lemma 3 we have $\rho(A, B) = T((A \rightarrow B) \wedge (B \rightarrow A) \rightarrow \bar{0}) \leq T(A_1^{n-1} \& \dots \& A_s^{n-1} \& B_1^{n-1} \& \dots \& B_t^{n-1} \rightarrow \bar{0})$; thus $\text{div}(\Gamma) \leq j(\Gamma)$ and $j(\Gamma) = 1$; then $\text{Consist}(\Gamma) = 1/2$. Conversely, if $\text{Consist}(\Gamma) = 1/2$, then $k(\Gamma) + j(\Gamma) = 1$; thus $k(\Gamma) = 0$ and $j(\Gamma) = 1$. For any $A_1, \dots, A_t \in \Gamma$, $A_1^{n-1} \& \dots \& A_t^{n-1} \in D(\Gamma)$ by axioms. If A is a theorem, since $\rho(A, B) = 1 - T((A \rightarrow B) \wedge (B \rightarrow A)) = 1 - T(A \rightarrow B) = 1 - T(B) = 1 - T(A_1^{n-1} \& \dots \& A_t^{n-1}) = T(A_1^{n-1} \& \dots \& A_t^{n-1} \rightarrow \bar{0})$, where $B = A_1^{n-1} \& \dots \& A_t^{n-1}$, thus $j(\Gamma) \leq \text{div}(\Gamma)$ by Definitions 9 and 8, then $\text{div}(\Gamma) = 1$, so Γ is consistent and full divergent.

(b) The proof of $\mathcal{L}_{uk}$ is analogous to that of (a) and so it is omitted. $\square$

Definition 11. Suppose that Γ is a theory, and define $d(\bar{0}, \Gamma) = \inf\{\rho(\bar{0}, A) \mid A \in D(\Gamma)\}$. It can be verified that $\rho(\bar{0}, A) = 1 - T(A \rightarrow \bar{0})$ and $d(\bar{0}, \Gamma) = 1 - \sup\{T(A \rightarrow \bar{0}) \mid A \in D(\Gamma)\}$.

Theorem 7. In $\mathcal{L}_n$ and $\mathcal{L}_{uk}$, if Γ_1 is consistent, or Γ_1 is inconsistent but $\text{consist}(\Gamma_2) \neq 1/2$, then $\text{consist}(\Gamma_1) < \text{consist}(\Gamma_2)$ if and only if $d(\bar{0}, \Gamma_1) < d(\bar{0}, \Gamma_2)$.

Proof. Suppose that $\text{consist}(\Gamma_1) < \text{consist}(\Gamma_2)$. First of all, we prove that $j(\Gamma_1) > j(\Gamma_2)$, if Γ_1 is consistent, and $k(\Gamma_1) = 0$ by Theorem 5; hence $1 - (1/2)j(\Gamma_1) < 1 - (1/2)(j(\Gamma_2) + k(\Gamma_2))$ and $j(\Gamma_1) > (j(\Gamma_2) + k(\Gamma_2))$, so $k(\Gamma_2) = 0$, and then $j(\Gamma_1) > j(\Gamma_2)$. If Γ_1 is inconsistent but $\text{consist}(\Gamma_2) \neq 1/2$, it follows from Theorems 5 and 6 that $k(\Gamma_1) = 1$ and $\text{consist}(\Gamma_1) = 0$, hence $\text{consist}(\Gamma_2) > 0$ and $j(\Gamma_1) = 1$; it follows from Theorem 6 that Γ_2 is consistent and $\text{consist}(\Gamma_2) > 1/2$; thus we have $j(\Gamma_2) < 1$, and then $j(\Gamma_1) > j(\Gamma_2)$. Secondly, we prove that $d(\bar{0}, \Gamma_1) < d(\bar{0}, \Gamma_2)$.

Hence there exists $r \in (1, 0)$ such that $j(\Gamma_1) > r > j(\Gamma_2)$. By definition, there exist $A_1, A_2, \dots, A_s \in \Gamma_1$ and $m_1, m_2, \dots, m_s \in N$ such that $T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}) > r$ and, for any $B_1, B_2, \dots, B_t \in \Gamma_2$ and $m_1, m_2, \dots, m_t \in N$, $T(B_1^{m_1} \& B_2^{m_2} \& \dots \& B_t^{m_t} \rightarrow \bar{0}) < r$; thus $T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}) > T(B_1^{m_1} \& B_2^{m_2} \& \dots \& B_t^{m_t} \rightarrow \bar{0})$, where $A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \in D(\Gamma_1)$ and $B_1^{m_1} \& B_2^{m_2} \& \dots \& B_t^{m_t} \in D(\Gamma_2)$. Thus $\rho(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s}, \bar{0}) = 1 - T(A_1^{m_1} \& A_2^{m_2} \& \dots \& A_s^{m_s} \rightarrow \bar{0}) < 1 - r < 1 - T(B_1^{m_1} \& B_2^{m_2} \& \dots \& B_t^{m_t} \rightarrow \bar{0}) = \rho(B_1^{m_1} \& B_2^{m_2} \& \dots \& B_t^{m_t}, \bar{0})$; then $d(\bar{0}, \Gamma_1) < d(\bar{0}, \Gamma_2)$ holds.

Conversely, suppose that $d(\bar{0}, \Gamma_1) < d(\bar{0}, \Gamma_2)$; since there exists $r \in (0, 1)$ such that $d(\bar{0}, \Gamma_1) < r < d(\bar{0}, \Gamma_2)$, it follows from Definition 11 that $\exists B \in D(\Gamma_1)$, and for all $A \in D(\Gamma_2)$ such that $1 - T(A \rightarrow \bar{0}) > r > 1 - T(B \rightarrow \bar{0})$, thus $T(B \rightarrow \bar{0}) > 1 - r > T(A \rightarrow \bar{0})$, and $j(\Gamma_1) > 1 - r \geq j(\Gamma_2)$. Then $\text{consist}(\Gamma_1) < \text{consist}(\Gamma_2)$. This completes the proof. $\square$

Remark 4. From Theorem 7, the definition of consistency degree is reasonable. If the consistency degree is higher, the distance between every conclusion of Γ and contradiction should be longer.

Example 2. In L_8 , calculate $j(\Gamma)$ and $\text{Consist}(\Gamma)$ for (i) $\Gamma = \{p\}$, and (ii) $\Gamma = S = \{p_1, p_2, \dots\}$.

Solution. (i) Since $[p^7]_{i/7} = \{\vec{v} \in L_8^\infty \mid \vec{v} = \varphi(v), v(p^7) = i/7, v \in \Omega\}$, and

$$v(p^7) = \begin{cases} 7v(p) - 6, & v(p) > \frac{6}{7}, \\ 0, & v(p) \leq \frac{6}{7}, \end{cases} \quad (22)$$

hence $\mu([p^7]_0) = 7/8, \mu([p^7]_{i/7}) = 0 (i = 1, 2, \dots, 6)$, $\mu([p^7]_1) = 1/8, T(p^7) = \sum_{i=0}^7 (i/7) \mu([p^7]_{i/7}) = 1/8$. By Theorem 3, $j(\Gamma) = 1 - T(p^7) = 7/8, k(\Gamma) = \max\{[T(p^m \rightarrow \bar{0})] \mid m \in N\} = 0$, and $\text{Consist}(\Gamma) = 1 - (1/2)(7/8) = 9/16$.

(ii) By [1], $\vdash p_1^7 \& p_2^7 \& \dots \& p_m^7 \rightarrow p_1 \wedge p_2 \wedge \dots \wedge p_m$; then it follows from Lemma 3 that $T(p_1^7 \& p_2^7 \& \dots \& p_m^7) \leq T(p_1 \wedge p_2 \wedge \dots \wedge p_m)$. Let $A = p_1 \wedge p_2 \wedge \dots \wedge p_m$. Then $A^*(x_1, x_2, \dots, x_m) = x_1 \wedge x_2 \wedge \dots \wedge x_m$. Hence $A^*(x_1, x_2, \dots, x_m) = i/7$ if and only if $\exists j \in \{1, 2, \dots, m\}, x_j = i/7$, and $x_k \geq i/7$ (for all $k \neq j$). Since there are a total of 8^m m -tuples $(x_1, x_2, \dots, x_m)$ and among them only $(8-i)^7 + (8-i-1)(8-i)^6 + (8-i-1)^2(8-i)^5 + \dots + (8-i-1)^5(8-i)^2 + (8-i-1)^6(8-i) + (8-i-1)^7$ imply that $A^*(x_1, x_2, \dots, x_m) = i/7$, hence $\mu([p_1 \wedge p_2 \wedge \dots \wedge p_m]_{i/7}) = (1/8^m)[(8-i)^7 + (8-i-1)(8-i)^6 + (8-i-1)^2(8-i)^5 + \dots + (8-i-1)^5(8-i)^2 + (8-i-1)^6(8-i) + (8-i-1)^7]$; thus $T(p_1 \wedge p_2 \wedge \dots \wedge p_m) = (1/8^m) \sum_{i=0}^7 (i/7) [(A^*)^{-1}(i/7)] \rightarrow 0, (m \rightarrow \infty)$. We have $\lim_{m \rightarrow \infty} (T(p_1^7 \& p_2^7 \& \dots \& p_m^7)) = 0$. It is easy to check that $k(\Gamma) = 0$; then $j(\Gamma) = 1 - \inf\{T(p_1^7 \& p_2^7 \& \dots \& p_m^7) \mid p_1, p_2, \dots, p_m \in S, m \in N\} = 1$ and $\text{Consist}(\Gamma) = 1 - 1/2 = 1/2$.

Example 3. In Luk, calculate $j(\Gamma)$ and $\text{Consist}(\Gamma)$ for (i) $\Gamma = \{p\}$, and (ii) $\Gamma = S = \{p_1, p_2, \dots\}$.

Solution. (i) Since

$$x^m = \begin{cases} mx - (m-1), & x > \frac{m-1}{m}, \\ 0, & x \leq \frac{m-1}{m}, \end{cases} \quad (23)$$

$T(p^m) = \int_0^1 x^m dx = \int_{(m-1)/m}^1 [mx - (m-1)] dx = 1/2m$, then $j(\Gamma) = \sup\{T(p^m \rightarrow \bar{0}) \mid m \in N\} = 1 - \inf\{T(p^m) \mid m \in N\} = 1, k(\Gamma) = \max\{[T(p^m \rightarrow \bar{0})] \mid m \in N\} = \max\{[1 - T(p^m)] \mid m \in N\} = 0$, and $\text{Consist}(\Gamma) = 1 - (1/2)(j(\Gamma) + k(\Gamma)) = 1/2$.

(ii) Since $\vdash p_1^{m_1} \& p_2^{m_2} \& \dots \& p_s^{m_s} \rightarrow p_1 \wedge p_2 \wedge \dots \wedge p_s$ by [1], then $T(p_1^{m_1} \& p_2^{m_2} \& \dots \& p_s^{m_s}) \leq T(p_1 \wedge p_2 \wedge \dots \wedge p_s)$. Let $\delta_s = \{(x_1, x_2, \dots, x_s) \in [0, 1]^s \mid x_i > (\ln s/s), i = 1, 2, \dots, s\}$ and $\Delta = [0, 1]^s$. Lebesgue measure $m(\delta_s) = (1 - (\ln s)/s)^s$. In $\Delta - \delta_s$, we have $x_1 \wedge x_2 \wedge \dots \wedge x_s \leq (\ln s)/s$. Thus

$$\begin{aligned} T(p_1 \wedge p_2 \wedge \dots \wedge p_s) &= \underbrace{\int_0^1 \dots \int_0^1}_{s} (x_1 \wedge x_2 \wedge \dots \wedge x_s) dx_1 dx_2 \dots dx_s \\ &= \int_{\Delta - \delta_s} (x_1 \wedge x_2 \wedge \dots \wedge x_s) dx_1 dx_2 \dots dx_s \\ &\quad + \int_{\delta_s} (x_1 \wedge x_2 \wedge \dots \wedge x_s) dx_1 dx_2 \dots dx_s \\ &\leq \frac{\ln s}{s} + \left(1 - \left(\frac{\ln s}{s}\right)\right)^s \rightarrow 0, \quad (s \rightarrow \infty). \end{aligned} \quad (24)$$

Thus $\lim_{s \rightarrow \infty} (T(p_1^{m_1} \& p_2^{m_2} \& \dots \& p_s^{m_s})) = 0$. It is easy to check that $k(\Gamma) = 0$; then $j(\Gamma) = 1 - \inf\{T(p_1^{m_1} \& p_2^{m_2} \& \dots \& p_s^{m_s}) \mid p_1, p_2, \dots, p_s \in S, s \in N\} = 1$ and $\text{Consist}(\Gamma) = 1/2$.

Remark 5. We compared the consistency degree concepts in [2, 3] with the definitions of this paper. From [2, 3], we see that the concept of consistency degree $\xi(\Gamma)$ and $\xi^*(\Gamma)$ of a theory Γ based on the concept of the divergence degree, moreover, though [2] has given the concept of consistency degree based on the condition of finite theories while the concept of consistency degree $\text{Consist}(\Gamma)$ of Γ is clearly based on the truth degrees of formulas and deduction theorems. From the logical point of view, the definition of $\text{Consist}(\Gamma)$ and $j(\Gamma)$ is more natural and reasonable. It is a more theoretical idea to understand while comparing to the facilitating calculation in this paper. Therefore, it is important and essential to give out the definition and characterization of the consistency degree in this paper.

4. Concluding Remarks

Based on deduction theorems, completeness theorems, and by means of the theory of truth degrees of formulas, the present paper defines a new polar index of a theory Γ and a new concept of consistency degrees of theories in Łukasiewicz fuzzy and n -valued propositional logic system, which is more natural and reasonable from the logical

point of view. Moreover, some important properties of truth degree of formula are proposed. Finally, sufficient and necessary conditions for a theory to be consistent, inconsistent, and fully divergent are obtained.

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