Research Article

Dynamics and Matter-Wave Solitons in Bose-Einstein Condensates with Two- and Three-Body Interactions

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By means of similarity transformation, this paper proposes the matter-wave soliton solutions and dynamics of the variable coefficient cubic-quintic nonlinear Schrödinger equation arising from Bose-Einstein condensates with time-dependent two- and three-body interactions. It is found that, under the effect of time-dependent two- and three-body interaction and harmonic potential with time-dependent frequency, the density of atom condensates will gradually diminish and finally collapse.

1. Introduction

Bose-Einstein condensation was first predicted by Einstein and Indian physicist Bose in 1924-1925. It is an exotic quantum phenomenon that was observed in dilute atomic gases for the first time in 1995 [1–3]. The “condensate” here is a state of matter of a dilute gas of bosons at temperatures close to 0 kelvins, which is different from the “condensate” in day life. One of the interesting dynamical features in the context of Bose-Einstein condensate (BEC) is the formation of matter wave solitons such as bright solitons [4, 5], dark solitons [5], vortex solitons [6], and gap solitons [7], which have been experimentally achieved before. Recent experimental techniques for managing nonlinearity have attracted considerable attention. For example, nonlinearity management arises in atomic physics for the Feshbach resonance [4, 8] of the scattering length of interatomic interactions in BECs, where the interaction strength can be characterized by a single parameter, the s-wave scattering length a_s . Across a Feshbach resonance the length a_s can in principle be varied from $-\infty$ to $+\infty$, where $a_s < 0$ ($a_s > 0$) corresponds to effectively attractive (repulsive) interactions. Thus in this situation, one can deal with the governing equations with the nonlinearity coefficients being functions of time or space [4, 9–12].

In the mean-field theory, a BEC system can be well described by the Gross-Pitaevskii (GP) equation [13–15], whose coefficient in front of the cubic term comes from

the interatomic interaction. Under certain condition, the GP equation can be converted into the classical nonlinear Schrödinger equation (NLS). It is known that at low densities the three-body interactions can be neglected and the s-wave two-body interactions achieve a dominant position. However, the three-body interactions play a key role in BEC at high densities. Similarly, a BEC with two- and three-body interactions can be described by the GP equation with cubic-quintic nonlinearity, also called variable coefficient cubic-quintic nonlinear Schrödinger (CQNLS) equation [16–18]:

$$i \frac{\partial \psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + V(x, t) \psi + g(t) |\psi|^2 \psi + G(t) |\psi|^4 \psi = 0, \quad (1)$$

where ψ is matter-wave function, $V(x, t)$ is external potential, $g(t)$ is two-body interaction coefficient, and $G(t)$ is the three-body interaction coefficient. Function $g(t)$ is positive (negative) for attractive (repulsive) condensates and the same as function $G(t)$. Here functions $g(t)$, $G(t)$, and $V(x, t)$ are experimentally controlled.

In this paper, we investigate the matter-wave soliton solutions and dynamics in BEC with two- and three-body interactions trapped by harmonic potential. The paper is organized as follows. In Section 2, the exact matter-wave

soliton solutions of the variable coefficient cubic-quintic nonlinear Schrödinger equation are obtained by using similarity transformation. In Section 3, the density distributions and dynamics of the matter-wave solitons are investigated by analyzing their figures. We summarize our results in the conclusions.

2. Exact Matter-Wave Soliton Solutions of CQNLS Equation (1)

2.1. Similarity Transformation. In what follows, we solve the variable coefficient cubic-quintic nonlinear Schrödinger equation (1) by means of similarity transformation. Through analyzing the properties of the exact solutions, new density distributions and dynamics of BEC with two- and three-body interactions are found.

Assume the variable coefficient CQNLS equation (1) can be transformed into the following CQNLS equation:

$$i\frac{\partial U}{\partial T} + \frac{\partial^2 U}{\partial X^2} + \sigma_2|U|^2U + \sigma_3|U|^4U = 0, \quad (2)$$

where σ_2, σ_3 are constants.

By the principle of similarity transformation, suppose the variable coefficient CQNLS equation (1) has the following form of exact solution:

$$\psi = \rho U(T, X) e^{i\varphi}, \quad (3)$$

$$|\psi|^2 = \rho^2 |U(T, X)|^2, \quad (4)$$

where ρ is amplitude, φ is phase, and ρ and φ are functions of t and x ; function $U(T, X)$ solves the CQNLS equation (2), T is a function of time t , and X is a function of t and x .

Making use of the symbolic computation software Maple, substituting the similarity transformation (3) and (4) into the variable coefficient CQNLS equation (1), and letting $U(T, X)$ satisfy the CQNLS equation (2), we obtain a set of partial differential equations (PDEs) of functions $T, X, \rho, \varphi, V, g$, and G as

$$\begin{aligned} \frac{\partial X}{\partial x} - 2\frac{\partial T}{\partial t} &= 0, \\ \frac{\partial^2 X}{\partial x^2} \rho + 2\frac{\partial \rho}{\partial x} \frac{\partial X}{\partial x} &= 0, \\ 2\frac{\partial X}{\partial x} \frac{\partial \varphi}{\partial x} \rho + 2\frac{\partial X}{\partial t} \rho &= 0, \\ \rho^2 g - \sigma_2 \frac{\partial T}{\partial t} &= 0, \\ \rho^4 G - \sigma_3 \frac{\partial T}{\partial t} &= 0, \\ 2V\rho - \left(\frac{\partial \varphi}{\partial x}\right)^2 \rho - 2\frac{\partial \varphi}{\partial t} \rho + \frac{\partial^2 \rho}{\partial x^2} &= 0, \\ 2\frac{\partial \rho}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} \rho + 2\frac{\partial \rho}{\partial x} \frac{\partial \varphi}{\partial x} &= 0. \end{aligned} \quad (5)$$

From the first equation in (5), that is, $(\partial X/\partial x) - 2(\partial T/\partial t) = 0$, it is found that function X is linear in variable x . Furthermore, from $(\partial^2 X/\partial x^2)\rho + 2(\partial \rho/\partial x)(\partial X/\partial x) = 0$ we have $(\partial \rho/\partial x)(\partial X/\partial x) = 0$. Because $\partial X/\partial x \neq 0$, we have $\partial \rho/\partial x = 0$, and then ρ is only a function of time t . So we can write functions X and ρ as

$$X = \tau_1 x + \tau_2, \quad (6)$$

$$\rho = \rho(t), \quad (7)$$

where τ_1 and τ_2 are functions of time t .

Inserting (6) and (7) into the fourth and fifth equations in (5), we have the expressions of g and G as

$$g = \frac{\sigma_2(\partial T/\partial t)}{\rho^2}, \quad G = \frac{\sigma_3(\partial T/\partial t)}{\rho^4}. \quad (8)$$

Because ρ is only a function of time t ; from the last equation in (5) we have $2(\partial \rho/\partial t) + (\partial^2 \varphi/\partial x^2)\rho = 0$, so function φ is a quadratic function of variable x . Thus we can assume function φ as

$$\varphi = \eta_1 x^2 + \eta_2 x + \eta_3, \quad (9)$$

where η_1, η_2 , and η_3 are all functions of time t .

Finally, the external potential $V(x, t)$ is usually harmonic potential in real experiments, so we let

$$V(x, t) = \frac{\omega^2 x^2}{2}, \quad (10)$$

where ω is the frequency of harmonic potential, which is a function of time t .

Substituting (6)–(10) into (5) and simplifying, we have

$$\tau_1^2 - 2\frac{\partial T}{\partial t} = 0, \quad (11)$$

$$2\tau_1\eta_1 x + \tau_1\eta_2 + \frac{\partial \tau_1}{\partial t} x + \frac{\partial \tau_2}{\partial t} = 0, \quad (12)$$

$$\begin{aligned} 4\eta_1^2 x^2 + 4\eta_1\eta_2 x + \eta_2^2 \\ + 2\frac{\partial \eta_1}{\partial t} x^2 + 2\frac{\partial \eta_2}{\partial t} x + 2\frac{\partial \eta_3}{\partial t} - \omega^2 x^2 &= 0, \end{aligned} \quad (13)$$

$$\frac{\partial \rho}{\partial t} + \eta_1 \rho = 0, \quad (14)$$

which become the following ordinary differential equations (ODEs) of functions $\tau_1, \tau_2, \eta_1, \eta_2, \eta_3, T$, and ρ by further calculating

$$\tau_1^2 - 2\frac{\partial T}{\partial t} = 0, \quad (15)$$

$$\tau_1\eta_2 + \frac{\partial \tau_2}{\partial t} = 0, \quad 2\tau_1\eta_1 + \frac{\partial \tau_1}{\partial t} = 0, \quad (16)$$

$$2\frac{\partial \eta_2}{\partial t} + 4\eta_1\eta_2 = 0, \quad \omega^2 - 2\frac{\partial \eta_1}{\partial t} - 4\eta_1^2 = 0, \quad (17)$$

$$2\frac{\partial \rho}{\partial t} + 2\eta_1 \rho = 0, \quad -2\frac{\partial \eta_3}{\partial t} - \eta_2^2 = 0. \quad (18)$$

The exact solutions of the ODEs in (15)–(18) are

$$\eta_1 = \eta_1, \quad (19)$$

$$\tau_1 = C_6 e^{-2 \int \eta_1 dt}, \quad (20)$$

$$\eta_2 = C_5 e^{-2 \int \eta_1 dt}, \quad (21)$$

$$T = \frac{1}{2} \int \tau_1^2 dt + C_4, \quad (22)$$

$$\eta_3 = -\frac{1}{2} \int \eta_2^2 dt + C_3, \quad (23)$$

$$\rho = C_2 e^{-\int \eta_1 dt}, \quad (24)$$

$$\omega = \pm \sqrt{2 \left(\frac{\partial \eta_1}{\partial t} + 2\eta_1^2 \right)}, \quad (25)$$

$$\tau_2 = - \int \tau_1 \eta_2 dt + C_1, \quad (26)$$

where we assume the frequency of the harmonic potential to be positive; that is, $\omega = \sqrt{2((\partial \eta_1 / \partial t) + 2\eta_1^2)}$, and $C_1, \dots, C_6$ are constants.

Now we discuss the frequency of the harmonic potential ω by six cases.

Case 1. Let $\omega = \Omega_0$ (constant); that is, the frequency of the harmonic potential is invariant. Then we can get function η_1 from $\omega = \sqrt{2((\partial \eta_1 / \partial t) + 2\eta_1^2)}$. So other variables can be determined by (19)–(26). In current case, the function η_1 is

$$\eta_1 = -\frac{\Omega_0}{2} \frac{e^{2\Omega_0 t} + e^{2C_0 \Omega_0}}{e^{2C_0 \Omega_0} - e^{2\Omega_0 t}}, \quad (27)$$

where Ω_0 is a constant.

Case 2. Let function η_1 be linear in time t ; that is, $\eta_1 = \Omega_1 t$ with Ω_1 a constant, and then we have

$$\omega = \sqrt{2\Omega_1 (1 + 2\Omega_1 t^2)}. \quad (28)$$

Case 3. Let function η_1 be an exponential function of time t ; that is, $\eta_1 = e^{\Omega_2 t}$, and then we have

$$\omega = \sqrt{2e^{t\Omega_2} (2e^{t\Omega_2} + \Omega_2)}. \quad (29)$$

Case 4. Let function η_1 be a hyperbolic function of time t ; that is, $\eta_1 = \cosh(\Omega_3 t)$, and then we have

$$\omega = \sqrt{2 \sinh(\Omega_3 t) + 4 \cosh(\Omega_3 t)^2}, \quad (30)$$

where $\sinh$ denotes hyperbolic sine function and $\cosh$ denotes hyperbolic cosine function.

Case 5. Let function η_1 be another exponential function of time t ; that is, $\eta_1 = e^{\Omega_4 t^2}$, and then we have

$$\omega = 2\sqrt{e^{\Omega_4 t^2} (e^{\Omega_4 t^2} + \Omega_4 t)}. \quad (31)$$

Case 6. Finally, let function η_1 be $\eta_1 = e^{\sin(\Omega_5 t)}$, and then we have

$$\omega = \sqrt{2e^{\sin(\Omega_5 t)} [2e^{\sin(\Omega_5 t)} + \Omega_5 \cos(\Omega_5 t)]}, \quad (32)$$

and here and above $\Omega_1, \Omega_2, \dots, \Omega_5$ are nonzero constants.

2.2. Exact Solutions of the CQNLS Equation (2). Up to now, we have derived the coefficients in the similarity transformation. The next thing is to find the exact solutions of the CQNLS equation (2) to formulate the wave function $U(T, X)$. In what follows, we list three types of exact solutions of the CQNLS equation (2).

Type 1. Consider

$$U_1(T, X) = \frac{cn(l_1 X + l_2 T, k) e^{i(l_0 X + d_0 T)}}{\sqrt{d_1 + d_2 [cn(l_1 X + l_2 T, k)]^2}}, \quad (33)$$

where cn is Jacobi elliptic cosine function [19, 20] with module k ($0 \leq k \leq 1$), and l_0 and d_2 are nonzero constants and

$$\begin{aligned} d_1 &= -\frac{(4K^2 k^2 + \sigma_2 d_2 + 2\sigma_3 - 2K^2) d_2}{4K^2 k^2} \\ l_1 &= \frac{K}{d_2}, \quad l_2 = -2\frac{K l_0}{d_2}, \\ d_0 &= \frac{-4l_0^2 d_2^2 + 3\sigma_2 d_2 - 4K^2 k^2 + 2\sigma_3 + 2K^2}{4d_2^2}, \end{aligned} \quad (34)$$

where the constant K satisfies

$$\begin{aligned} K &= \frac{1}{6} \left(-24\sigma_3 k^2 + 12\sigma_3 \right. \\ &\quad \left. + 6\sqrt{9\sigma_2^2 d_2^2 + 24\sigma_2 \sigma_3 d_2 + (16k^4 - 16k^2 + 16)\sigma_3^2} \right)^{1/2}. \end{aligned} \quad (35)$$

Type 2. Consider

$$U_2(T, X) = \frac{\sqrt{2}\text{sech}\sqrt{6Q-12\sigma_3}(X-2l_0T)e^{i((12l_0d_2^2X+(8\sigma_3-12l_0^2d_2^2+9\sigma_2d-Q)T)/12d_2^2)}}{\sqrt{d_2[3\sigma_2d_2+4\sigma_3+Q+2\text{sech}^2((\sqrt{6Q-12\sigma_3}(X-2l_0T))/6d_2)(2\sigma_3-Q)](2\sigma_3-Q)^{-1}}}, \quad (36)$$

where l_0 and d_2 are nonzero constants and $Q = \sqrt{(4\sigma_3+3\sigma_2d_2)^2}$.
Type 3. Consider

$$U_3(T, X) = \frac{\sinh((\sqrt{-2\sigma_2d_2-4\sigma_3}/2d_2)(X-2l_0T))e^{i(l_0Xd_2^2+(\sigma_2d_2+\sigma_3+l_0^2d_2^2)T)/d_2^2}}{\sqrt{d_2[3\sigma_2d_2+6\sigma_3+\sinh^2((\sqrt{-2\sigma_2d_2-4\sigma_3}(X-2l_0T))/2d_2)(4\sigma_3+3\sigma_2d_2)](4\sigma_3+3\sigma_2d_2)^{-1}}}, \quad (37)$$

where l_0 and d_2 are nonzero constants.

2.3. *Exact Matter-Wave Soliton Solutions of (1).* Three types of exact solutions $U_1(T, X)$, $U_2(T, X)$, $U_3(T, X)$ of the CQNLS equation (2) have been obtained above. In order to achieve the exact matter-wave soliton solutions of the variable coefficient CQNLS equation (1), we only need to combine the solutions $U_1(T, X)$, $U_2(T, X)$, $U_3(T, X)$ with the similarity transformation (3). Thus the exact matter-wave soliton solution of the variable coefficient CQNLS equation is

$$\psi(x, t) = \rho U(T, X) e^{i\varphi}, \quad (38)$$

where X , φ , and ρ are given by (6), (9), and (24) and functions τ_1 , τ_2 , η_1 , η_2 , η_3 , and T are given by (19)–(26). Here the value of function η_1 is the key to the solution. We have listed six cases of choices of function η_1 above. But we only choose $\eta_1 = \Omega_1 t$ in the following calculation. So from (19)–(26), we have

$$\begin{aligned} \omega &= \sqrt{2\Omega_1(1+2\Omega_1t^2)}, \\ \eta_2 &= C_5 e^{-\Omega_1^2 t^2}, \\ \eta_3 &= -\frac{C_5 C_6}{2} \frac{\sqrt{2\pi} \operatorname{erf}(\sqrt{2}\Omega_1 t)}{4\Omega_1}, \\ T &= \frac{C_6^2}{2} \frac{\sqrt{2\pi} \operatorname{erf}(\sqrt{2}\Omega_1 t)}{4\Omega_1} + C_4, \end{aligned}$$

$$\rho = C_2 e^{-\Omega_1 t^2/2},$$

$$X = C_6 e^{-\Omega_1^2 t^2} x - \frac{C_5 C_6 \sqrt{2\pi} \operatorname{erf}(\sqrt{2}\Omega_1 t)}{4\Omega_1} + C_1, \quad (39)$$

where erf is error function (also named Gaussian error function).

To sum up, when the external potential is time-dependent harmonic potential $V(x, t) = 2\Omega_1(1+2\Omega_1t^2)x^2$, the coefficients of two- and three-body interactions are $g(t) = (\sigma_2 C_6^2/2\rho^2)e^{-2\Omega_1^2 t^2}$, $G(t) = (\sigma_3 C_6^2/2\rho^4)e^{-2\Omega_1^2 t^2}$, and we derive three families of exact matter-wave soliton solutions of the variable coefficient CQNLS equation (1) as follows.

Family 1. Consider

$$\begin{aligned} \psi_1(t, x) &= C_2 e^{-\Omega_1 t^2/2} \\ &\cdot \frac{cn(l_1 X + l_2 T, k) e^{i(l_0 X + d_0 T)}}{\sqrt{d_1 + d_2 [cn(l_1 X + l_2 T, k)]^2}} \\ &\cdot e^{i[\Omega_1 t x^2 + C_5 e^{-\Omega_1^2 t^2} x - (C_5 C_6/2)(\sqrt{2\pi} \operatorname{erf}(\sqrt{2}\Omega_1 t)/4\Omega_1)]}. \end{aligned} \quad (40)$$

Family 2. Consider

$$\begin{aligned} \psi_2(t, x) &= C_2 e^{-\Omega_1 t^2/2} \\ &\cdot \frac{\sqrt{2}\text{sech}\sqrt{6Q-12\sigma_3}(X-2l_0T)e^{i((12l_0Xd_2^2+(8\sigma_3-12l_0^2d_2^2+9\sigma_2d_2-Q)T)/12d_2^2)}}{\sqrt{d_2[3\sigma_2d_2+4\sigma_3+Q+2\text{sech}^2((\sqrt{6Q-12\sigma_3}(X-2l_0T))/6d_2)(2\sigma_3-Q)](2\sigma_3-Q)^{-1}}} \\ &\cdot e^{i[\Omega_1 t x^2 + C_5 e^{-\Omega_1^2 t^2} x - (C_5 C_6/2)(\sqrt{2\pi} \operatorname{erf}(\sqrt{2}\Omega_1 t)/4\Omega_1)]}. \end{aligned} \quad (41)$$

Family 3. Consider

$$\psi_3(t, x) = C_2 e^{-\Omega_1 t^2/2} \cdot \frac{\sinh\left(\left(\sqrt{-2\sigma_2 d_2 - 4\sigma_3/2d_2}\right)(X - 2l_0 T)\right) e^{i(l_0 X d_2^2 + (\sigma_2 d_2 + \sigma_3 l_0^2 d_2^2)T)/d_2^2}}{\sqrt{d_2 \left[3\sigma_2 d_2 + 6\sigma_3 + \sinh^2\left(\left(\sqrt{-2\sigma_2 d_2 - 4\sigma_3/2d_2}\right)(X - 2l_0 T)\right) (4\sigma_3 + 3\sigma_2 d_2)\right] (4\sigma_3 + 3\sigma_2 d_2)^{-1}}} \cdot e^{i[\Omega_1 t x^2 + C_5 e^{-\Omega_1 t^2} x - (C_5 C_6/2)(\sqrt{2\pi} \operatorname{erf}(\sqrt{2}\Omega_1 t)/4\Omega_1)]}, \quad (42)$$

where the functions T, X are given by (39) and $Q = \sqrt{(4\sigma_3 + 3\sigma_2 d_2)^2}$.

Remark 1. By considering the rest five cases of frequency ω and function η_1 , we can also obtain other exact matter-wave soliton solutions of the variable coefficient CQNLS equation (1) under different types of two- and three-body interactions.

3. Density Distributions and Dynamics of the Matter-Wave Solitons

In this Section, we investigate the density distributions and dynamics of the matter-wave soliton solutions (40)–(42) by analyzing their figures. It is noted that the frequency of the harmonic potential is $\omega = \sqrt{2\Omega_1(1 + 2\Omega_1 t^2)}$, and the coefficients of two- and three-body interactions are $g(t) = (\sigma_2 C_6^2/2\rho^2)e^{-2\Omega_1 t^2}$, $G(t) = (\sigma_3 C_6^2/2\rho^4)e^{-2\Omega_1 t^2}$, which are monotone nonincreasing function of time t .

3.1. Density Distribution and Dynamics of Solution $\psi_1(x, t)$. For the matter-wave soliton solution $\psi_1(x, t)$ in (40), we choose the parameters as follows: $C_1 = C_2 = \dots = C_6 = 1$, $\Omega_1 = 0.01$, $\sigma_2 = -0.5$, $\sigma_3 = -1$, $d_2 = 2$, $l_0 = 1$, and $k = 0.99$. Here the time range is $[0, 15]$, time size is 0.01, the space range is $[-20, 20]$, and space size is 0.05.

The density distributions of the matter-wave soliton solution $\psi_1(x, t)$ are shown in Figure 1. Here the coefficient of two-body interaction is $g(t) < 0$ and that of three-body interaction is $G(t) < 0$, which denotes that both two-body and three-body interactions are repulsive. According to the periodical property of the Jacobi elliptic functions, the density distribution $|\psi_1(x, t)|^2$ is periodic. We can choose elliptic modulus k from $0 < k < 1$, and we let $k = 0.99$ here. It is observed from Figure 1 that there are five peaks in the density distribution; that is, it has five periods in space. Under the effects of time-dependent harmonic potential and the repulsive two- and three-body interactions, the density distribution of wave function $\psi_1(x, t)$ decreases with time. We can also get various density distributions by choosing k from $0 < k < 1$.

3.2. Density Distribution and Dynamics of Solution $\psi_2(x, t)$. We now study the matter-wave soliton solution $\psi_2(x, t)$ in (41). The parameters here are $C_1 = C_2 = \dots = C_6 = 1$,

$\Omega_1 = 0.01$, $\sigma_2 = 1/3$, $\sigma_3 = -0.5$, $d_2 = -1$, and $l_0 = 1$. The time range is $[0, 10]$, time size is 0.05, the space range is $[-200, 200]$, and space size is 0.1.

The density distributions of the matter-wave soliton solution $\psi_2(x, t)$ are shown in Figure 2. Here the coefficient of two-body interaction is $g(t) > 0$, and that of three-body interaction is $G(t) < 0$, which denotes that the two-body interaction is attractive and the three-body interaction is repulsive. This is a matter-wave bright soliton and is a localized nonlinear wave. It is seen from the color bar that as time goes on the density distribution of wave function, $\psi_2(x, t)$ diminishes. Thus we find that attractive two-body interaction and repulsive three-body interaction do not support stable matter-wave bright soliton, which is consistent with the real experiments.

3.3. Density Distribution and Dynamics of Solution $\psi_3(x, t)$. Finally, we analyze the matter-wave soliton solution $\psi_3(x, t)$ in (42). The parameters here are $C_1 = C_2 = \dots = C_6 = 1$, $\Omega_1 = 0.01$, $\sigma_2 = -2$, $\sigma_3 = -1$, $d_2 = 5$, and $l_0 = 1$. The time range is $[0, 10]$, time size is 0.02, the space range is $[-10, 10]$, and space size is 0.01.

The density distributions of the matter-wave soliton solution $\psi_3(x, t)$ are shown in Figure 3. Here the coefficient of two-body interaction is $g(t) < 0$, and that of three-body interaction is $G(t) < 0$, which denotes that both two-body interaction and three-body interactions are repulsive. This is a matter-wave dark soliton and is also a localized nonlinear wave. It is also observed that the density distribution of wave function $\psi_3(x, t)$ also diminishes with time.

4. Conclusions

In summary, we have studied the matter-wave solitons and dynamics of Bose-Einstein condensates with time-dependent two- and three-body interactions in time-dependent external potential. We find that when the nonlinear coefficients $g(t) = (\sigma_2 C_6^2/2\rho^2)e^{-2\Omega_1 t^2}$, $G(t) = (\sigma_3 C_6^2/2\rho^4)e^{-2\Omega_1 t^2}$, the cubic-quintic nonlinear Schrödinger equation supports three families of exact solutions. Moreover, six possible frequencies of harmonic potential are given. Finally, in the case of the harmonic potential $V(x, t) = 2\Omega_1(1 + 2\Omega_1 t^2)x^2$, we examine the density distributions and dynamics of the matter-wave soliton solutions by analyzing their plots. It is found that

FIGURE 1: Density distributions of the matter-wave soliton solution $\psi_1(x, t)$: (a) the three-dimensional density distribution and (b) the contour profile.

FIGURE 2: Density distributions of the matter-wave soliton solution $\psi_2(x, t)$: (a) the three-dimensional density distribution and (b) the contour profile.

FIGURE 3: Density distributions of the matter-wave soliton solution $\psi_3(x, t)$: (a) the three dimensional density distribution and (b) the contour profile.

under the effect of time-dependent two- and three-body interactions along with time-dependent harmonic potential, the density distributions of the matter-wave solitons diminish with time. This is consistent with the real Bose-Einstein condensate experiments. There are indeed many papers

[16–18, 21–28] studying the exact solutions of the nonlinear Schrödinger equations by similarity transformations, but they discuss either the matter-wave solitons of Bose-Einstein condensate with two-body interactions or the matter-wave solitons of Bose-Einstein condensate with spatially

inhomogeneous interactions. To our knowledge, the three families of exact solutions (40)–(42) for the cubic-quintic nonlinear Schrödinger are proposed for the first time.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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