Research Article

Stability and Dissipation of Many Components' LC Systems and Application to Metamaterials

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Dykhne's method, based on rotational symmetry of two-dimensional equations of constant direct current, was applied to study the properties of the medium, consisting of reactances—of inductors and capacitors (nondissipative elements). The exact solution for the 4-component system consisting of the two different types of 2 inductors and 2 capacitors, randomly placed and connected, was obtained. The obtained solution for investigated system has the two features: (1) the finite dissipation in the nondissipative system of inductors and capacitors; (2) appearance of plateau—the constant value of effective conductivity in a wide range of concentrations of components. The obtained results were useful for description of composite metamaterials.

1. Introduction: Dual Symmetry of Two-Dimensional Systems

As it was well known, the conductivity problem of the randomly composite systems due to computational difficulties can not be solved exactly in general case; usually the effective medium approximation [1] and numerical methods have been used. However, in two-dimensional case due to internal dual symmetry of DC equations the exact results for the effective conductivity of two-phase inhomogeneous medium have been obtained. These results include Keller's theorem [2] and a general approach, developed in the works of Dykhne [3, 4]. The reciprocity (duality) relation for the effective conductivity at arbitrary concentrations of the phases has been established:

$$\sigma_e(\varepsilon) \sigma_e(-\varepsilon) = \sigma_1 \sigma_2. \quad (1)$$

Here σ_e is the effective conductivity of the system, ε is a deviation of phase concentration from the percolation threshold, and σ_1 and σ_2 are phase conductivities. (Dual system is a system that differs from the initial by the interchange of phases with the same geometric arrangement of the phases;

see Figure 1) Therefore the effective conductivity of two-phase medium at the percolation threshold (at equal phase concentrations) is equal to

$$\sigma_e(0) = \sqrt{\sigma_1 \sigma_2}. \quad (2)$$

Let us recall briefly the using of dual symmetry and the results previously established, resulting from this dual symmetry. Two-dimensional conductive medium was described by DC equations and Ohm's law.

$$\begin{aligned} \operatorname{div} \vec{j} &= 0, \\ \operatorname{curl} \vec{e} &= 0. \end{aligned} \quad (3)$$

As firstly established in the works of Keller [2] and Dykhne [3, 4], these equations in the two-dimensional case have internal dual symmetry—the invariance relatively on linear transformations of rotation:

$$\begin{aligned} \vec{j} &= b [\vec{n}, \vec{e}'], \\ \vec{e} &= d [\vec{n}, \vec{j}']. \end{aligned} \quad (4)$$

Here b, d are constant coefficients of rotational transformations; vector $\vec{n}$ is the normal vector to the plane.

FIGURE 1: (a) Model two-phase medium. (b) The model of the medium, which is dual with respect to the initial original one.

Due to the linearity of these transformations Ohm's law in a new primed system was maintained:

$$\vec{j}' = \sigma' \vec{e}'. \quad (5)$$

And the conductivity of the primed system is

$$\sigma' = \frac{b}{d\sigma}. \quad (6)$$

The same arguments were held for the averaged values and it is easy to obtain relation (6), from which we obtain the known result for the effective conductivities of two-phase medium:

$$\sigma_e(\varepsilon) \sigma_e'(\varepsilon) = \sigma_1 \sigma_2. \quad (7)$$

In the general case of an anisotropic medium, for example, in a magnetic field, Ohm's law has a tensor form:

$$\vec{j} = \hat{\sigma} \vec{e}. \quad (8)$$

Here $\hat{\sigma}$ is a two-dimensional conductivity tensor with components σ_{xx} and σ_{xy} . Therefore, in this case, to describe the dual symmetry of two-dimensional anisotropic media generalized linear transformations of rotation were used [5–8]:

$$\begin{aligned} \vec{j} &= a \vec{j}' + b [\vec{n}, \vec{e}'], \\ \vec{e} &= c \vec{e}' + d [\vec{n}, \vec{j}']. \end{aligned} \quad (9)$$

In the transformed primed system Ohm's law also has a tensor form. The components of the tensor of the new primed system related to the components of the tensor of the original system through the following relationships:

$$\begin{aligned} \sigma'_{xx} &= \frac{\sigma_{xx}(ac + bd)}{(\sigma_{xx}d)^2 + (\sigma_{xy}d + a)^2} \\ \sigma'_{xy} &= \frac{\sigma_{xx}^2 cd + (c\sigma_{xy} - b)(d\sigma_{xy} - a)}{(\sigma_{xx}d)^2 + (\sigma_{xy}d + a)^2}. \end{aligned} \quad (10)$$

In this paper we applied general Dykhne's approach to study the conductivity of the new medium, consisting of 4 components such as reactive impedances—the two different inductors and the two capacitors randomly have been placed and connected.

The obtained results may to apply to metamaterials with negative refractive index [9–12], because the metamaterials usually have been constructed from capacitors and inductors. So the investigated model illustrated some features of response for complex composite metamaterials and stability of its response signal in the wide region of component concentrations.

2. Effective Conductivity of System, Consisting of Randomly Connected L and C

Let us apply Dykhne's approach, based on rotational transformations, to calculate the conductivity of randomly connected discrete LC system. Note that in the case of media with metallic conductivity this method of linear transformations has applied only for two-component and very special case of three-component media. But below by using of this method we calculate expressions for the effective conductivity of the medium, consisting of randomly connected inductances and capacitances, taking four different values. Let us consider a system, consisting of inductors and capacitors, and present the capacitive (inductive) resistance in the mathematical form of the imaginary variable: $Y_C = 1/i\omega C$ or $Y_L = i\omega L$.

Ohm's law for medium with nondissipative elements keeps their form:

$$j = Y \quad E = i|Y|E. \quad (11)$$

Here $Y = i|Y|$ is the impedance of system, imaginary part of usual conductivity, presented in the complex variable form. We want to stress that the main feature of formula (11) consists of that electric field E perpendicular to the direction of electric current j ; this fact has been emphasized by imaginary unit in (11). Due to this feature the considered system has a nondissipative character and the possibility of exact solution for the 4-component system arose.

Let us study the medium, consisting of capacitances and inductances of the two types and having different values. Let us assume also that the concentrations p_i of components (reactances) with even and odd numeration are equal to each other:

$$\begin{aligned} p_1 &= p_3, \\ p_2 &= p_4, \end{aligned} \quad (12)$$

and they are arbitrary in magnitude:

$$2(p_1 + p_2) = 1. \quad (13)$$

For convenience, we write the concentration of reactive resistance of the first and the second type as follows:

$$\begin{aligned} p_1 &= \frac{1}{4} + \delta, \\ p_2 &= \frac{1}{4} - \delta. \end{aligned} \quad (14)$$

Following the general method of Dykhne, we established the possible permutations of the symmetry between the components of different types and the corresponding relations of duality for the effective conductivity.

(1) First permutation symmetry of the system: interchange of reactance components with odd and even numeration by positions:

$$\begin{aligned} Y_1'' &= Y_2, \\ Y_2'' &= Y_1, \\ Y_3'' &= Y_4, \\ Y_4'' &= Y_3, \end{aligned} \quad (15)$$

$$(1 \longleftrightarrow 2, 3 \longleftrightarrow 4).$$

According to terms (15) the formula for the conductivity of the transformed system has been given by the transformation coefficients

$$\begin{aligned} a_1 &= -c_1; \\ b_1 &= \frac{Y_1 Y_2 (Y_3 + Y_4) - Y_3 Y_4 (Y_1 + Y_2)}{Y_1 Y_2 - Y_3 Y_4}; \\ d_1 &= \frac{Y_1 + Y_2 - Y_3 - Y_4}{Y_1 Y_2 - Y_3 Y_4}. \end{aligned} \quad (16)$$

We emphasize that such a permutation is possible for arbitrary values of the conductivities only for the nondissipation system. In the case of usual metallic conductivity and in the three-phase case, there is an additional condition on the conductivity of the third phase: $\sigma_3 = \sqrt{\sigma_1 \sigma_2}$.

In a four-phase case such a transformation is absent. For arbitrary concentrations of phases the primed system is dual with respect to the original: $Y_e'(\delta) = Y_e(-\delta)$. Consequently, we obtain the following duality (reciprocity) relation for the effective conductivity of four-component nondissipation system:

$$d_1 Y_e(-\delta) Y_e(\delta) + b_1 = a_1 (Y_e(-\delta) + Y_e(\delta)). \quad (17)$$

(2) The second symmetry of the system: interchange of reactances with odd and even numeration in its positions:

$$\begin{aligned} Y_1'' &= Y_4, \\ Y_2'' &= Y_3, \\ Y_3'' &= Y_2, \\ Y_4'' &= Y_1, \end{aligned} \quad (18)$$

$$(1 \longleftrightarrow 4, 2 \longleftrightarrow 3).$$

In this case, the coefficients a, b, d have the meanings

$$\begin{aligned} a_2 &= -c_2, \\ b_2 &= \frac{Y_1 Y_4 (Y_2 + Y_3) - Y_2 Y_3 (Y_1 + Y_4)}{Y_1 Y_4 - Y_2 Y_3}; \\ d_2 &= \frac{Y_1 + Y_4 - Y_2 - Y_3}{Y_1 Y_4 - Y_2 Y_3} \end{aligned} \quad (19)$$

and get another duality (reciprocity) relation similar to (17), but with its coefficients a_2, b_2, d_2 :

$$d_2 Y_e(-\delta) Y_e(\delta) + b_2 = a_2 (Y_e(\delta) + Y_e(-\delta)). \quad (20)$$

Then we introduce the even part of the effective conductivity $\psi(\delta) = (Y_e(\delta) + Y_e(-\delta))/2$ and the odd part of conductivity $\chi(\delta) = (Y_e(\delta) - Y_e(-\delta))/2$. As we write above δ is the phase concentration with respect to value of $1/4$. Then from (20) we obtain the following relation, which determines the structure of the effective conductivity:

$$\psi^2(\delta) - 2\psi(\delta) \frac{a}{d} + \chi^2(\delta) + \frac{b}{d} = 0. \quad (21)$$

And as a result

$$\psi(\delta) = \frac{a}{d} \pm \sqrt{\left(\frac{a}{d}\right)^2 - \frac{b}{d} - \chi^2(\delta)}. \quad (22)$$

It is interesting to note that, in special cases, when the odd part vanishes $\chi(\delta) = 0$, expression (26) gives a complete solution to the problem.

In other words, duality relation (20) in these special cases gives the full solution of the problem.

(3) Indeed, let us consider a third symmetry of the system: an interchange of the components (resistance) with even and odd numeration between them:

$$\begin{aligned} Y_1'' &= Y_3, \\ Y_2'' &= Y_4, \\ Y_3'' &= Y_1, \\ Y_4'' &= Y_2, \end{aligned} \quad (23)$$

$$(1 \longleftrightarrow 3, 2 \longleftrightarrow 4).$$

In this case, the primed system is macroscopically equivalent to the original: the effective conductivity has been determined by the equation

$$d_3 Y_e(\delta)^2 + b_3 - 2a_3 Y_e(\delta) = 0. \quad (24)$$

Here, the coefficients a_3, b_3, d_3 are determined by condition (23). Thus, the effective conductivity of the four-phase medium consisting of nondissipative resistance-capacitance and inductance is equal to

$$\begin{aligned} Y^e(\pm\delta) &= \frac{(Y_1 Y_3 - Y_2 Y_4)}{(Y_1 + Y_3 - Y_2 - Y_4)} \\ &\pm \sqrt{\left(\frac{Y_1 Y_3 - Y_2 Y_4}{Y_1 + Y_3 - Y_2 - Y_4}\right)^2 - \frac{Y_2 Y_4 (Y_1 + Y_3) - Y_1 Y_3 (Y_2 + Y_4)}{Y_1 + Y_3 - Y_2 - Y_4}}, \end{aligned} \quad (25)$$

where Y_i is the i -conductivity component; $Y^e(+\delta)$ is the effective conductivity of the percolation cluster formed mainly by the resistors Y_1 and Y_3 and a small amount of resistance Y_2 и Y_4 ; $Y^e(-\delta)$ is the effective conductivity of the percolation cluster formed mainly by the resistors Y_2 and Y_4 and a small amount of odd Y_1 and Y_3 resistance.

But this solution has an additional feature: in the case of equal values of sum of even and sum of odd conductivities as $Y_1 + Y_3 = Y_2 + Y_4$ the uncertainty of value of effective conductivity has appeared in (25). To solve this uncertainty it is necessary to return to initial formulas as (10) and condition (23). From this condition (23) we obtain that the coefficient $d_3 = (Y_1 + Y_3 - Y_2 - Y_4)/(Y_1 Y_3 - Y_2 Y_4) = 0$ and consequently from (24) the following value for effective conductivity has followed along the lines $Y_1 + Y_3 = Y_2 + Y_4$:

$$Y^e(\delta) = \frac{(Y_1 + Y_3)}{2}. \quad (26)$$

The transition to homogeneous medium is obvious.

3. Conclusion

Let us analyze expression (25). The transition to medium with few elements has been carried out in an obvious way: three-component system is equal to $Y_2 = Y_4$ and a two-component system corresponds to the limit $Y_1 = Y_3$, $Y_2 = Y_4$. We emphasize again that the calculated effective conductivity of nondissipative four-element system is constant over a wide range of concentrations; that is, it takes the form of a plateau.

Another interesting fact is that the height of the plateau depends on the conductivity of all four components up to arbitrarily low concentrations of two components. This reflects the fact that it is impossible to construct percolation cluster, a set of conductive paths of two phases only in a two-dimensional case, and using another two phases is required, and therefore effective conductivity of four-component system depends on the conductivities of all phases included.

Firstly the percolation method has been used for study of nonstationary conductivity in [13].

Near to the percolation threshold the conductive cluster was built from a combination of two dominant phases with the remaining two phases; after crossing the threshold this cluster was also constructed from the other two dominant phases with the remaining two phases. The presence of a plateau in the solution could not be predicted in advance; it reflects the peculiarities of the current flow in the LC system.

And additional unexpected feature of the obtained solution (25) is appearance of the finite dissipation in system, consisting only of the nondissipation components. For the first time it seems that obtained solution is not correct. But we explain this so that inductors L and capacitors C represent by themselves oscillatory circuits and during the flow of electric current the local oscillations are generated and as a result after averaging over medium we obtain the finite dissipation. Firstly this phenomena has been discussed in [3] and after that in [14].

It is necessary that obtained results may be used for time-delayed systems and experiments [15, 16] and for analysis of stability of signal of composite many components' metamaterials.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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