

Research Article

Valuing Time-Dependent CEV Barrier Options

C. F. Lo,¹ H. M. Tang,¹ K. C. Ku,¹ and C. H. Hui²

¹ *Institute of Theoretical Physics and Department of Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong*

² *Research Department, Hong Kong Monetary Authority, 55th Floor, Two International Finance Centre, 8 Finance Street, Hong Kong*

Correspondence should be addressed to C. F. Lo, cfo@phy.cuhk.edu.hk

Received 15 January 2009; Accepted 29 May 2009

Recommended by Henry Schellhorn

We have derived the analytical kernels of the pricing formulae of the CEV knockout options with time-dependent parameters for a parametric class of moving barriers. By a series of similarity transformations and changing variables, we are able to reduce the pricing equation to one which is reducible to the Bessel equation with constant parameters. These results enable us to develop a simple and efficient method for computing accurate estimates of the CEV single-barrier option prices as well as their upper and lower bounds when the model parameters are time-dependent. By means of the multistage approximation scheme, the upper and lower bounds for the exact barrier option prices can be efficiently improved in a systematic manner. It is also natural that this new approach can be easily applied to capture the valuation of other standard CEV options with specified moving knockout barriers. In view of the CEV model being empirically considered to be a better candidate in equity option pricing than the traditional Black-Scholes model, more comparative pricing and precise risk management in equity options can be achieved by incorporating term structures of interest rates, volatility, and dividend into the CEV option valuation model.

Copyright © 2009 C. F. Lo et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

In recent years European barrier options have become extremely popular in world markets. Unlike standard options, a barrier option is a path dependent option in which the existence of the option depends upon whether the underlying asset price has touched a critical value, called the *barrier*, during the option's lifetime. Should the price of the underlying asset breach this barrier before option expiration, the option will be extinguished immediately. An advantage of trading barrier options is that they provide more flexibility in tailoring the portfolio returns while lowering the cost of option premiums. The pricing of barrier options has been studied in many literatures assuming the underlying asset price to follow

a lognormal diffusion process, that is, the Black-Scholes environment [1]. Merton [2] was the first to derive a closed-form solution for a down-and-out European call option. Other closed-form pricing formulae of single-barrier options were published in [3–8]. The analytical valuation of double-barrier options was discussed in [9–14].

The Black-Scholes option pricing model is a member of the class of constant elasticity of variance (CEV) option pricing models. The diffusion process of stock price S in a CEV model can be expressed as

$$dS = \mu S dt + \sigma S^{\beta/2} dZ, \quad 0 \leq \beta < 2 \quad (1.1)$$

where μ is the instantaneous mean, $\sigma S^{\beta/2}$ is the instantaneous variance of the stock price, dZ is a Weiner process, and β is the elasticity factor. The equation shows that the instantaneous variance of the percentage price change is equal to $\sigma^2 / S^{2-\beta}$ and is a direct inverse function of the stock price. In the limiting case $\beta = 2$, the CEV model returns to the conventional Black-Scholes model in which the variance rate is independent of the stock price. In another case $\beta = 0$, it is the Ornstein-Uhlenbeck model. Several theoretical arguments imply an association between stock price and volatility. Black [15] and Christie [16] consider the effects of financial leverage on the variance of the stock. A fall in the stock price increases the debt-equity ratio of the firm; therefore, both the risk and the variance of the stock increase. Black also proposes that a downturn in the business cycle might lead to an increase in the stock price volatility and hence to a fall in the stock prices.

Empirical evidence has shown that the CEV process may be a better description of stock behavior than the more commonly used lognormal model because the CEV process allows for a nonzero elasticity of return variance with respect to prices. Schmalensee and Trippi [17] find a strong negative relationship between stock price changes and changes in implied volatility after examining over a year of weekly data on six stocks. By applying the trading profits approach on 19000 daily warrant price observations, Hauser and Lauterbach [18] find that the CEV model roughly doubles the trading excess returns of the Black-Scholes model. The superiority of the CEV model is strongest in out-of-the-money and longer time to expiration warrants. The results are consistent with the findings in Lauterbach and Schultz [19]. If the relationship between the variance and the stock price is deduced from the empirical data, an option pricing formula based on the CEV model could fit the actual market option prices better than the Black-Scholes model. Beckers [20] finds thirty-seven out of forty-seven stocks in a year daily data set to have estimated β to be less than two and concludes that the CEV diffusion process could be a better candidate of describing the actual stock price behavior than the Black-Scholes model.

In addition to providing a better description of stock behavior, the CEV process can be employed in the contingent-claims approach to valuing defaultable bonds. For example, in a valuation model of defaultable bonds proposed by Cathcart and El-Jahel [21] recently, default occurs when some signaling process hits some constant default barrier (i.e., the option to default can be considered as a barrier option). The model assumes the signaling process for each firm that determines the occurrence of default rather than the value of the assets of the firm. The signaling process can capture factors that can affect the probability of default. The use of the signaling process is also appropriate for entities such as sovereign issuers that issue defaultable debts but do not have an identifiable collection of assets. The signaling process could follow diffusion processes such as lognormal, Ornstein-Uhlenbeck, or CEV processes.

The derivation of the CEV option pricing formula with $\beta = 1$ (commonly known as the “square-root process”) was first presented by Cox and Ross [22] as an alternative diffusion process for valuation of options. Cox [23] also derived the option pricing formula for $\beta < 2$. All these derivations assume the model parameters such as volatility, interest rate, and dividend yield are constant. However, the model parameters are actually time-dependent in market. The time-dependent term structures of interest rates and volatility which can be implied from the money market and the option market, respectively, are expressed as time-dependent stepwise functions. The term structures can also be expressed as analytical functions to reflect expectation and dynamics of market factors. Only recently Lo et al. [24] succeeded in introducing time-dependent parameters into the CEV process, and obtaining the closed-form option pricing formula explicitly.

The valuation of European CEV barrier options with time-dependent model parameters is not a trivial extension. So far as we know, no simple and accurate approximation scheme is available yet. In a recent paper Lo and Hui [25] generalize the Lie-algebraic technique of Lo et al. [24] to derive the analytical kernels of the pricing formulae of the CEV knockout options with time-dependent parameters for a parametric class of moving barriers. By best fitting the fixed barrier by these parametric moving barriers, they also provide a simple method for computing tight upper and lower bounds of the prices of the single-barrier options (both call and put options). In this paper we try to provide an alternative derivation of the analytical pricing kernels via a more systematic approach, which involves a series of similarity transformations and changing variables so as to convert the pricing equation into one reducible to the Bessel equation with constant parameters. The solution of the resultant equation subject to absorbing boundary conditions is well known and discussed in most standard textbooks on partial differential equations. Then very tight upper and lower bounds of the exact barrier option prices can be calculated efficiently using the multistage approximation, and these bounds can also be systematically improved in a straightforward manner.

The remainder of this paper is structured as follows. In the next section we present the derivation of the analytical kernels of the pricing formulae of the CEV knockout options with time-dependent parameters for a parametric class of moving barriers, and describe our formulation for evaluating accurate approximation of the value of a single-barrier European CEV option with time-dependent parameters. Section 3 presents some illustrative examples and examines the accuracy and efficiency of our approximate approach. Tight upper and lower bounds of the exact barrier option prices are also calculated. In Section 4 the multistage approximation scheme is proposed to systematically tighten the upper and lower bounds. It is found that even a rather low-order approximation can yield very tight bounds of the exact barrier option prices. Numerical results are discussed in detail. In Section 5, we apply the multistage approximation scheme to value barrier options with time-dependent volatilities. In Section 6 we summarize our investigation and provide suggestions for future research.

2. CEV Single-Barrier Options

The CEV model with time-dependent model parameters for a standard European option is described by the partial differential equation [23]

$$\frac{\partial P(S, \tau)}{\partial \tau} = \frac{1}{2} \sigma(\tau)^2 S^\beta \frac{\partial^2 P(S, \tau)}{\partial S^2} + [r(\tau) - d(\tau)] S \frac{\partial P(S, \tau)}{\partial S} - r(\tau) P(S, \tau) \quad (2.1)$$

for $0 \leq \beta < 2$. Here P is the option value, S is the underlying asset price, τ is the time to maturity, σ is the volatility, r is the risk-free interest rate, and d is the dividend. Introducing a simple change of variables: $x = \sqrt{S^{(2-\beta)}}$, (2.1) can be recast in the following form:

$$\begin{aligned} \frac{\partial u(x, \tau)}{\partial \tau} &= \frac{1}{8} \tilde{\sigma}(\tau)^2 \frac{\partial^2 u(x, \tau)}{\partial x^2} + \frac{1}{2} \left[\tilde{\mu}(\tau)x - \frac{(4-\beta)\tilde{\sigma}(\tau)^2}{4(2-\beta)x} \right] \frac{\partial u(x, \tau)}{\partial x} \\ &\quad + \left[\frac{(4-\beta)\tilde{\sigma}(\tau)^2}{8(2-\beta)x^2} - r(\tau) - \frac{\tilde{\mu}(\tau)}{2} \right] u(x, \tau) \\ &\equiv H(\tau)u(x, \tau), \end{aligned} \quad (2.2)$$

where $\tilde{\sigma}(\tau) = (2-\beta)\sigma(\tau)$, $\tilde{\mu}(\tau) = (2-\beta)[r(\tau) - d(\tau)]$ and $u(x, \tau) = xP(S, \tau)$. This equation represents a generalization of the Fokker-Planck equation associated with the well-known Rayleigh process [26]. It is not difficult to show that the operator $H(\tau)$ can be rewritten as follows:

$$H(\tau) = a_1(\tau)K_+ + a_2(\tau)K_0 + a_3(\tau)K_- + b(\tau), \quad (2.3)$$

where

$$\begin{aligned} K_- &= \frac{1}{2} \left[\frac{\partial^2}{\partial x^2} - \frac{4-\beta}{(2-\beta)x} \frac{\partial}{\partial x} + \frac{4-\beta}{(2-\beta)x^2} \right], \\ K_0 &= \frac{1}{2} \left(x \frac{\partial}{\partial x} - \frac{1}{2-\beta} \right), \quad K_+ = \frac{1}{2} x^2, \\ a_3(\tau) &= \frac{1}{4} \tilde{\sigma}(\tau)^2, \quad a_2(\tau) = \tilde{\mu}(\tau), \\ a_1(\tau) &= 0, \quad b(\tau) = -\frac{1-\beta}{2(2-\beta)} \tilde{\mu}(\tau) - r(\tau). \end{aligned} \quad (2.4)$$

The operators K_+ , K_0 , and K_- are the generators of the Lie algebra $\mathfrak{su}(1,1)$ [27]:

$$[K_+, K_-] = -2K_0, \quad [K_0, K_{\pm}] = \pm K_{\pm}. \quad (2.5)$$

Without loss of generality we first assume the solution of (2.2) takes the form

$$u(x, \tau) = \exp \left\{ \varphi(\tau) - \frac{1}{2} \Gamma(\tau) x^2 \right\} \tilde{u}(x, \tau), \quad \varphi(\tau) = \int_0^\tau d\tau' b(\tau'). \quad (2.6)$$

Substituting (2.6) into (2.2) yields

$$\begin{aligned} \frac{\partial \tilde{u}(x, \tau)}{\partial \tau} = & \left\{ \left[\frac{d\Gamma(\tau)}{d\tau} + a_3(\tau)\Gamma^2(\tau) - a_2(\tau)\Gamma(\tau) \right] K_+ \right. \\ & \left. + [a_2(\tau) - 2a_3(\tau)\Gamma(\tau)]K_0 + a_3(\tau)K_- \right\} \tilde{u}(x, \tau). \end{aligned} \quad (2.7)$$

Then we set the coefficient associated with K_+ in (2.7) equal to zero and obtain

$$\begin{aligned} \frac{d\Gamma(\tau)}{d\tau} &= -a_3(\tau)\Gamma^2(\tau) + a_2(\tau)\Gamma(\tau) \\ \Rightarrow \frac{d}{d\tau} \{ \Gamma(\tau) \exp[-c_2(\tau)] \} &= -\frac{dc_3(\tau)}{d\tau} \{ \Gamma(\tau) \exp[-c_2(\tau)] \}^2 \\ \Rightarrow \Gamma(\tau) &= \frac{\gamma \exp[c_2(\tau)]}{1 + \gamma c_3(\tau)}, \end{aligned} \quad (2.8)$$

where γ is an arbitrary real constant and the $c_i(\tau)$ are defined by

$$\begin{aligned} c_2(\tau) &= \int_0^\tau a_2(\tau') d\tau', \\ c_3(\tau) &= \int_0^\tau a_3(\tau') \exp[c_2(\tau')] d\tau'. \end{aligned} \quad (2.9)$$

Next, we perform the time-dependent similarity transformation

$$S_1 = \exp \{ [c_2(\tau) - 2 \ln |1 + \gamma c_3(\tau)|] K_0 \} \quad (2.10)$$

to (2.7) so that it becomes

$$\frac{\partial \bar{u}(x, \tau)}{\partial \tau} = \frac{a_3(\tau) \exp[c_2(\tau)]}{[1 + \gamma c_3(\tau)]^2} K_- \bar{u}(x, \tau) \Rightarrow \frac{\partial \bar{u}(x, \Omega)}{\partial \Omega} = K_- \bar{u}(x, \Omega), \quad (2.11)$$

where $\bar{u}(x, \tau) = S_1^{-1} \tilde{u}(x, \tau)$ and

$$\Omega(\tau) = \int_0^\tau \frac{a_3(\tau') \exp[c_2(\tau')]}{[1 + \gamma c_3(\tau')]^2} d\tau' = \frac{c_3(\tau)}{1 + \gamma c_3(\tau)}. \quad (2.12)$$

Assuming that $\bar{u}(x, \Omega) = x^{(\alpha+1)/2} v(x, \Omega)$, where $\alpha = (4 - \beta)/(2 - \beta)$, (2.11) is reduced to

$$\frac{\partial v(x, \Omega)}{\partial \Omega} = \frac{1}{2} \left[\frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} - \frac{(\alpha - 1)^2}{4x^2} \right] v(x, \Omega), \quad (2.13)$$

whose formal solution is given by

$$v(x, \Omega) = \exp \left\{ \frac{\Omega}{2} \left[\frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} - \frac{(\alpha - 1)^2}{4x^2} \right] \right\} v(x, 0). \quad (2.14)$$

It should be noted that (2.13) is reducible to the Bessel equation of order $(\alpha - 1)/2$ by the separation of variables, and detailed analysis of its solutions for different boundary conditions are well documented in most standard textbooks on partial differential equations [28–30].

2.1. Up-and-Out Moving Barrier Options

Now we try to solve (2.13) for $0 \leq x \leq L$ and $\tau \geq 0$ with absorbing boundary conditions. Without loss of generality, we assume that $v(x, 0)$ is defined in terms of the Fourier-Bessel integral [28]:

$$v(x, 0) = \sum_{n=1}^{\infty} \frac{2J_{\omega}(x_{\omega n}(x/L))}{L^2 J_{\omega+1}^2(x_{\omega n})} \int_0^L dy \, y \, J_{\omega}\left(x_{\omega n} \frac{y}{L}\right) v(y, 0), \quad (2.15)$$

for $\omega = (\alpha - 1)/2 > -1$. Here $x_{\omega n}$ denotes the n th zero of the Bessel function J_{ω} of the first kind of order ω . Then, making use of the fact that

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{1}{x} \frac{\partial}{\partial x} - \frac{\omega^2}{x^2} \right\} J_{\omega}\left(x_{\omega n} \frac{x}{L}\right) = -\left(\frac{x_{\omega n}}{L}\right)^2 J_{\omega}\left(x_{\omega n} \frac{x}{L}\right), \quad (2.16)$$

it is not difficult to show that

$$v(x, \Omega) = \sum_{n=1}^{\infty} \frac{2J_{\omega}(x_{\omega n}x/L)}{L^2 J_{\omega+1}^2(x_{\omega n})} \exp\left(-\frac{\Omega x_{\omega n}^2}{2L^2}\right) \int_0^L dy \, y \, J_{\omega}\left(x_{\omega n} \frac{y}{L}\right) v(y, 0). \quad (2.17)$$

As a result, the price of the corresponding up-and-out option is given by

$$P_{\text{up-and-out}}(S, \tau) = \frac{u(x, \tau)}{x} = \frac{1}{x} \int_0^L dy \, K(x, \tau; y, 0) \, u(y, 0), \quad (2.18)$$

where

$$\begin{aligned}
 K(x, \tau; y, 0) = & \sum_{n=1}^{\infty} \frac{2y}{L^2 J_{\omega+1}^2(x_{\omega n})} \left(\frac{x}{y}\right)^{\omega+1} \frac{\exp[c_2(\tau)/2 + \int_0^{\tau} d\tau' b(\tau')]}{|1 + \gamma c_3(\tau)|} \\
 & \times \exp\left\{-\frac{\gamma \exp[c_2(\tau)]}{2[1 + \gamma c_3(\tau)]} x^2\right\} \exp\left\{-\frac{c_3(\tau)}{2[1 + \gamma c_3(\tau)] L^2} x_{\omega n}^2\right\} \\
 & \times J_{\omega}\left(x_{\omega n} \frac{\exp[c_2(\tau)/2]}{|1 + \gamma c_3(\tau)|} \frac{x}{L}\right) J_{\omega}\left(x_{\omega n} \frac{y}{L}\right) \exp\left[\frac{1}{2} \gamma y^2\right].
 \end{aligned} \quad (2.19)$$

In the above derivation we have made use of the well-known relation

$$\exp\left(\eta x \frac{\partial}{\partial x}\right) f(x) = f(x \exp(\eta)). \quad (2.20)$$

It can be easily seen that at time $\tau \geq 0$ the kernel $K(x, \tau; y, 0)$ vanishes at $x = x^*(\tau) = L|1 + \gamma c_3(\tau)| \exp[-c_2(\tau)/2]$. That is, we have derived the kernel of the pricing equation in (2.2) with an up-and-out barrier belonging to the class of trajectories:

$$S^*(\tau) = [x^*(\tau)]^{2/(2-\beta)} = \left\{ L|1 + \gamma c_3(\tau)| \exp\left[-\frac{1}{2} c_2(\tau)\right] \right\}^{2/(2-\beta)} \quad (2.21)$$

parametrized by the real adjustable parameter γ .

2.2. Down-and-Out Moving Barrier Options

On the other hand, for a down-and-out option with the barrier following the trajectory defined by (2.21), we would suppose that $v(x, 0)$ is defined in terms of the Weber transform [29]:

$$\begin{aligned}
 v(x, 0) = & \int_0^{\infty} d\xi \frac{J_{\omega}(x\xi) Y_{\omega}(\xi L) - Y_{\omega}(x\xi) J_{\omega}(\xi L)}{J_{\omega}^2(\xi L) + Y_{\omega}^2(\xi L)} \xi \\
 & \times \int_L^{\infty} dy [J_{\omega}(y\xi) Y_{\omega}(\xi L) - Y_{\omega}(y\xi) J_{\omega}(\xi L)] y v(y, 0)
 \end{aligned} \quad (2.22)$$

for $L \leq x < \infty$. Then, it is straightforward to show that $u(x, \tau)$ is simply given by

$$u(x, \tau) = \int_L^{\infty} dy G(x, \tau; y, 0) u(y, 0), \quad (2.23)$$

where

$$\begin{aligned}
 G(x, \tau; y, 0) = & \int_0^\infty d\xi \, y \xi \left(\frac{x}{y} \right)^{\omega+1} \frac{\exp[c_2(\tau)/2 + \int_0^\tau d\tau' b(\tau')]}{|1 + \gamma c_3(\tau)|} \exp\left[\frac{1}{2}\gamma y^2\right] \\
 & \times \exp\left\{-\frac{\gamma \exp[c_2(\tau)]}{2[1 + \gamma c_3(\tau)]} x^2\right\} \exp\left\{-\frac{c_3(\tau)}{2[1 + \gamma c_3(\tau)]} \xi^2\right\} \\
 & \times \left[J_\omega\left(\frac{x\xi \exp[c_2(\tau)/2]}{|1 + \gamma c_3(\tau)|}\right) Y_\omega(\xi L) - Y_\omega\left(\frac{x\xi \exp[c_2(\tau)/2]}{|1 + \gamma c_3(\tau)|}\right) J_\omega(\xi L) \right] \\
 & \times \frac{J_\omega(y\xi) Y_\omega(\xi L) - Y_\omega(y\xi) J_\omega(\xi L)}{J_\omega^2(\xi L) + Y_\omega^2(\xi L)}
 \end{aligned} \tag{2.24}$$

is the kernel of the pricing equation in (2.2) associated with a down-and-out barrier moving along the trajectory given in (2.21). Here Y_ω denotes the Bessel function of the second kind of order ω . It should also be noted that the Gaussian decaying factor of the integrand ensures the rapid convergence of the integration over ξ . Accordingly, the price of the corresponding down-and-out moving barrier option is found to be

$$P_{\text{down-and-out}}(S, \tau) = \frac{u(x, \tau)}{x} = \frac{1}{x} \int_L^\infty dy \, G(x, \tau; y, 0) u(y, 0). \tag{2.25}$$

Furthermore, it is not difficult to see that in the special case of $L = 0$, that is, *no barrier*, the kernel in (2.24) is reduced to the one obtained by Lo et al. [24] which has a Gaussian decaying factor in the variable y . By the *maximum principle* for the parabolic partial differential equation [30], we can thus conclude that the kernel in (2.24) must have a decaying factor in the variable y , which decays at least as fast as the Gaussian decaying factor in the special case of $L = 0$.

2.3. Illustrative Applications

If we take a closer look at the trajectory of the moving barrier defined in (2.21), we would immediately realize that the special case of a *fixed barrier* does not belong to the class of parametric barriers. In order to simulate a fixed barrier, we will thus choose an optimal value of the adjustable parameter γ in such a way that the integral

$$\int_0^T [x^*(\tau) - L]^2 d\tau \tag{2.26}$$

is minimum. In other words, we try to minimize the deviation from the fixed barrier by varying the parameter γ . Here T denotes the time at which the option price is evaluated. An illustrative example of such an optimal fluctuating barrier (represented by the dashed line) is shown in Figure 1. It is clear that the approximation is indeed very good. Within the framework of the new approach, we can also determine the upper and lower bounds for the exact barrier option prices. It is not difficult to show that for an *up-and-out* option the upper bound can be provided by the option price associated with a moving barrier whose

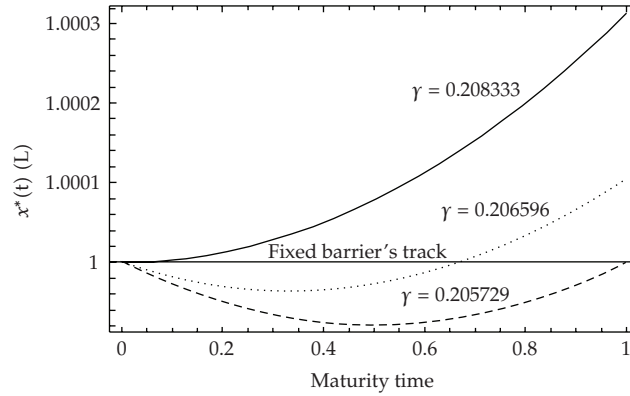


Figure 1: Barrier tracks for the bounds and barrier option price estimate within the single-stage approximation scheme. Upper bound's track (solid line), lower bound's track (long dashed line) and barrier option price estimate's track (dashed line). Time to maturity is equal to one. Other input parameters are $\beta = 1.0$, $S_0 = 26$, $X = 20$, $\sigma_{BS}^2 = 0.02$, $d = 0$ and $r = 0.05$. The values of γ for the barrier tracks are shown along the tracks.

$x^*(\tau)$ is greater than or equal to L for the duration of interest. (The proof is based upon the maximum principle for the parabolic partial differential equation [30].) Similarly, the option price associated with a moving barrier whose $x^*(\tau)$ is less than or equal to L for the duration of interest can serve as the desired lower bound. In this example, the best lower bound can be obtained by choosing an appropriate value of γ such that $x^*(\tau = 0) = x^*(\tau = T) = L$. That is, at time $\tau = T$ the moving barrier will return to its initial position and merge with the fixed barrier. In Figure 1 an example of such a barrier movement is denoted by the long-dashed line. On the other hand, the best upper bound can be obtained by choosing a γ value which satisfies the requirement that $dx^*(\tau)/d\tau = 0$ at $\tau = 0$. That is, the instantaneous rate of change of $x^*(\tau)$ is required to be zero at time $\tau = 0$. An example of such a barrier movement is represented by the solid line in Figure 1. On the contrary, for a *down-and-out* option we can simply switch the above two choices of barrier movement in order to determine the upper and lower bounds of the option price.

3. Illustrative Examples

For illustration, we apply the approximate method to a " $\beta = 1$ "-CEV *up-and-out* call option with constant model parameters: $\sigma_{BS}^2 = 0.02$, $r = 0.05$, $d = 0$. (Note that the value of σ to be used for the CEV model is adjusted to be $\sigma = \sigma_{BS}S^{(2-\beta)/2}$.) The strike price X and the knockout barrier S_0 are set equal to 20 and 26, respectively. We now try to evaluate the barrier option price $P(S, \tau)$ associated with the current underlying asset price $S = 24$ at time $\tau = 1$.

First of all, we determine the optimal value of the adjustable parameter γ :

$$\gamma_{\text{opt}} = 0.206596. \quad (3.1)$$

Then an estimate of the exact up-and-out barrier option price can be evaluated by numerically computing the integral in (2.18) (with, e.g., *Mathematica*):

$$P(S = 24, \tau = 1) = 0.71396. \quad (3.2)$$

Since the exact value of the barrier option price is found to be [8]

$$P_{\text{exact}}(S = 24, \tau = 1) = 0.71401, \quad (3.3)$$

the approximate estimate is indeed very close to the exact result with a percentage error of -0.00658% only. The numerical results for the corresponding upper and lower bounds are determined as follows:

$$\begin{aligned} \text{Upper bound} &= 0.71641 && (\text{percentage error} = 0.33613\%), \\ \text{Lower bound} &= 0.71274 && (\text{percentage error} = -0.17843\%). \end{aligned} \quad (3.4)$$

The barrier tracks for the estimate and bounds of the option price are shown in Figure 1. Clearly, the new approach is able to yield very tight upper and lower bounds for the exact barrier option price. To further illustrate the accuracy of the new approach, we also calculate the estimates and bounds of the option prices corresponding to some other underlying asset prices, namely, $S = 16, 18, 20$, and 22 . The numerical results are listed in Tables 1(a) and 1(b). It is clear that the estimates and bounds are remarkably good; all of them have a percentage error of less than 0.1 percent above or below the exact option value.

In order to assess the efficiency of the new approach, we also perform Monte Carlo simulations to evaluate the option prices. As shown in Tables 1(a) and 1(b), using a time-step of 10^{-5} and a sample of 10^5 random paths of the underlying asset price, the Monte Carlo method gives much poorer estimates in comparison with the new approach. Furthermore, to examine the robustness of the new approach, we carry out the same kind of investigations for *up-and-out* call options in different CEV environments too. The numerical results corresponding to different β values, namely, $\beta = 0.5$ and 1.5 , are tabulated in Tables 2 and 3, respectively. Beyond question, the advantages of the new approach are clearly demonstrated by these data.

4. Systematic Multistage Approximation

As the time to maturity increases beyond one year, that is, $\tau > 1$, or the model parameters have more dramatic term structures, the accuracy of the estimates and bounds of the approximate method decreases. To obtain the same accuracy as before, we can approximate the fixed barrier by a continuous and piecewise smooth barrier, leading to the multistage approximation scheme. For simplicity, we will concentrate on the *up-and-out* options to demonstrate the multistage approximation in the following. Generalization to the *down-and-out* options should be very straightforward. First of all, we consider the estimate of the lower bound and perform the evaluation in two stages.

Stage 1 (the time interval $[0, T/2]$). Following the same procedure as that discussed in Section 3, we choose an appropriate value of the parameter γ , denoted by γ_{L1} , such that $x^*(\tau = 0) = x^*(\tau = T/2) = 0$. This determines the movement of the barrier within the time interval $[0, T/2]$. The corresponding price function is given by the integral in (2.18) with the kernel K associated with γ_{L1} . One can efficiently calculate the integral numerically using either the *Mathematica* or the Gauss quadrature method.

Table 1: Comparison of estimates and bounds of option prices with the analytical results from Lo et al., "Pricing barrier options with square root process," *International Journal of Theoretical and Applied Finance*, 4(5):805-818 (2001). Percentage error is defined as (estimate - analytical result)/analytical result $\times 100\%$. Other input parameters are $\beta = 1.0$, $S_0 = 26$, $X = 20$, $\sigma_{BS}^2 = 0.02$, $d = 0$, and $r = 0.05$. In the Monte Carlo simulation $\Delta t = 0.00001$ and number of ensembles = 100000.

(a) CEV up-and-out call option with $\beta = 1.0$.

S	P 's estimate	γ_{opt}	% error of estimate	Analytical result of P	Monte-Carlo method	% error of MC	Lower bound of P	γ_L	% error of lower bound
16	0.10494	0.30900	0.00000	0.10494	0.10573	0.75281	0.10494	0.30859	0.00000
18	0.51902	0.27546	-0.00578	0.51905	0.51713	-0.36991	0.51900	0.27431	-0.00963
20	1.11794	0.24792	-0.01699	1.11813	1.12344	0.47490	1.11777	0.24688	-0.03220
22	1.25156	0.22538	-0.03435	1.25199	1.27467	1.81152	1.25096	0.22443	-0.08227
24	0.71396	0.20660	-0.00658	0.71401	0.73651	3.15122	0.71274	0.20573	-0.17843

(b) CEV up-and-out call option with $\beta = 1.0$.

Upper bound of P	γ_U	% error of upper bound	Improved lower bound of P by a two-stage scheme	% error of improved lower bound of P by a two-stage scheme	Improved upper bound of P by a two-stage scheme	% error of improved upper bound of P by a two-stage scheme
0.10494	0.31250	0.00000	0.10494	0.00000	0.10494	0.00000
0.51906	0.27778	0.00212	0.51903	-0.00385	0.51907	0.00385
1.11827	0.25000	0.01252	1.11802	-0.00984	1.11824	0.00984
1.25277	0.22727	0.06230	1.25172	-0.02157	1.25243	0.03514
0.71641	0.20833	0.33613	0.71368	-0.04622	0.71458	0.07983

Table 2: CEV up-and-out call option with $\beta = 0.5$. Comparison of estimates and bounds of option prices with numerical results by Monte-Carlo (MC) method. Other input parameters are $\beta = 0.5$, $S_0 = 26$, $X = 20$, $\sigma_{BS}^2 = 0.02$, $d = 0$, and $r = 0.05$. In the Monte Carlo simulation $\Delta t = 0.00001$ and number of ensembles = 100000.

S	P 's estimate	γ_{opt}	Monte-Carlo result	Lower bound of P	γ_L	Upper bound of P	γ_U	Improved lower bound of p by a two-stage scheme	Improved upper bound of p by a two-stage scheme
16	0.09542	0.05143	0.09744	0.09542	0.05111	0.09542	0.05208	0.09542	0.09542
18	0.52449	0.04310	0.52401	0.52447	0.04283	0.52454	0.04365	0.52450	0.52454
20	1.16472	0.03680	1.17240	1.16448	0.03657	1.16521	0.03727	1.16484	1.16519
22	1.28880	0.31899	1.30895	1.28788	0.03170	1.29065	0.03230	1.28904	1.29019
24	0.71159	0.27996	0.72594	0.70974	0.02782	0.71527	0.02835	0.71116	0.71254

Stage 2 (the time interval $[T/2, T]$). We repeat the procedure in Stage 1 such that $x^*(\tau = T/2) = x^*(\tau = T) = 0$. This will give us another value of γ , denoted by γ_{12} , and determine

Table 3: CEV up-and-out call option with $\beta = 1.5$. Comparison of estimates and bounds of option prices with numerical results by Monte Carlo (MC) method. Other input parameters are $\beta = 1.5$, $S_0 = 26$, $X = 20$, $\sigma_{BS}^2 = 0.02$, $d = 0$, and $r = 0.05$. In the Monte Carlo simulation, $\Delta t = 0.00001$ and number of ensembles = 100000.

S	P 's estimate	γ_{opt}	Monte Carlo result	Lower bound of P	γ_L	Upper bound of P	γ_U	Improved lower bound of P by a two-stage scheme	Improved upper bound of P by a two-stage scheme
16	0.11415	2.48958	0.11526	0.11415	2.48438	0.11415	2.5000	0.11415	0.11415
18	0.51031	2.34720	0.51011	0.51030	2.34229	0.51033	2.35702	0.51032	0.51033
20	1.07084	2.22675	1.07658	1.07075	2.22209	1.07101	2.23607	1.07088	1.07100
22	1.21585	2.12312	1.23098	1.21556	2.11868	1.21644	2.13201	1.21592	1.21628
24	0.71679	2.03274	0.73793	0.71618	2.02848	0.71801	2.04124	0.71664	0.71709

the moving barrier's trajectory for the time interval $[T/2, T]$. Then, the corresponding price function is given by

$$u\left(x, \frac{T}{2} \leq \tau \leq T\right) = \int_0^L dy \tilde{K}\left(x, \tau; y, \frac{T}{2}\right) u\left(y, \frac{T}{2}\right), \quad (4.1)$$

where

$$\begin{aligned} \tilde{K}\left(x, \tau; y, \frac{T}{2}\right) &= \sum_{n=1}^{\infty} \frac{2y}{L^2 J_{\omega+1}^2(x_{\omega n})} \left(\frac{x}{y}\right)^{\omega+1} \frac{\exp\left[\tilde{c}_2(\tau)/2 + \int_{T/2}^{\tau} d\tau' b(\tau')\right]}{|1 + \gamma_{L2} \tilde{c}_3(\tau)|} \\ &\quad \times \exp\left\{-\frac{\gamma_{L2} \exp[\tilde{c}_2(\tau)]}{2[1 + \gamma_{L2} \tilde{c}_3(\tau)]} x^2\right\} \exp\left\{-\frac{\tilde{c}_3(\tau)}{2[1 + \gamma_{L2} \tilde{c}_3(\tau)] L^2} x_{\omega n}^2\right\} \\ &\quad \times J_{\omega}\left(x_{\omega n} \frac{\exp[\tilde{c}_2(\tau)/2]}{|1 + \gamma_{L2} \tilde{c}_3(\tau)|} \frac{x}{L}\right) J_{\omega}\left(x_{\omega n} \frac{y}{L}\right) \exp\left[\frac{1}{2} \gamma_{L2} y^2\right], \quad (4.2) \\ \tilde{c}_2(\tau) &= \int_{T/2}^{\tau} \tilde{\mu}(\tau') d\tau', \\ \tilde{c}_3(\tau) &= \frac{1}{4} \int_{T/2}^{\tau} \tilde{\sigma}(\tau')^2 \exp[\tilde{c}_2(\tau')] d\tau'. \end{aligned}$$

Again, the integration in (4.1) can be efficiently evaluated using either the *Mathematica* or the Gauss quadrature method.

In Figure 2 the long-dashed line gives an illustrative example of the moving barrier's trajectory within the two-stage approximation scheme. It is clear that the deviation from the fixed barrier is much reduced in this two-stage approximation. The corresponding numerical results for different CEV up-and-out call options in Tables 1–3 also demonstrate that the lower bounds are dramatically improved. Apparently, one can further improve the lower bounds by splitting the evaluation process into more stages instead.

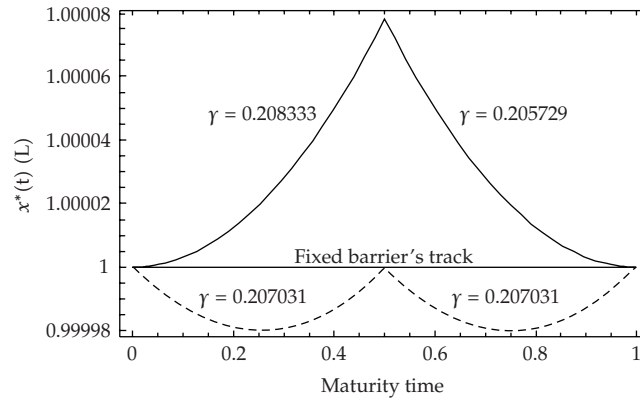


Figure 2: Barrier tracks for the bounds improved by a two-stage approximation scheme. Upper bound's track under a two-stage scheme (solid line) and lower bound's track under a two-stage scheme (long dashed line). Time to maturity is equal to one. Other input parameters are $\beta = 1.0$, $S_0 = 26$, $X = 20$, $\sigma_{BS}^2 = 0.02$, $d = 0$ and $r = 0.05$. The values of γ for the barrier tracks are shown along the tracks.

Next, we discuss how to implement the multistage approximation scheme to improve the upper bound. For the two-stage approximation, the γ_U in the single-stage approximation is used for the time interval $[0, T/2]$, that is, we set $\gamma_{U1} = \gamma_U$. At $\tau = T/2$, another value of γ , denoted by γ_{U2} , is selected so that the moving barrier will then start moving back to its initial position and merge with the fixed barrier at $\tau = T$. As a result, the corresponding price function is given by (i) the integral in (2.18) with the kernel K associated with the parameter γ_{U1} for $0 \leq \tau \leq T/2$, and (ii) the following expression:

$$u\left(x, \frac{T}{2} \leq \tau \leq T\right) = \int_0^L dy \bar{K}\left(\phi x, \tau; \theta y, \frac{T}{2}\right) u\left(\theta y, \frac{T}{2}\right), \quad (4.3)$$

where

$$\begin{aligned} \bar{K}\left(x, \tau; y, \frac{T}{2}\right) &= \sum_{n=1}^{\infty} \frac{2y}{L^2 J_{\omega+1}^2(x_{\omega n})} \left(\frac{x}{y}\right)^{\omega+1} \frac{\exp\left[\tilde{c}_2(\tau)/2 + \int_{T/2}^{\tau} d\tau' b(\tau')\right]}{|1 + \gamma_{U2} \tilde{c}_3(\tau)|} \\ &\quad \times \exp\left\{-\frac{\gamma_{U2} \exp[\tilde{c}_2(\tau)]}{2[1 + \gamma_{U2} \tilde{c}_3(\tau)]} x^2\right\} \exp\left\{-\frac{\tilde{c}_3(\tau)}{2[1 + \gamma_{U2} \tilde{c}_3(\tau)] L^2} x_{\omega n}^2\right\} \\ &\quad \times J_{\omega}\left(x_{\omega n} \frac{\exp[\tilde{c}_2(\tau)/2]}{|1 + \gamma_{U2} \tilde{c}_3(\tau)|} \frac{x}{L}\right) J_{\omega}\left(x_{\omega n} \frac{y}{L}\right) \exp\left[\frac{1}{2} \gamma_{U2} y^2\right], \\ \theta &= \left|1 + \gamma_{U1} c_3\left(\frac{T}{2}\right)\right| \cdot \exp\left[-\frac{1}{2} c_2\left(\frac{T}{2}\right)\right], \\ \phi &= |1 + \gamma_{U2} \tilde{c}_3(\tau)|^{-1} \cdot \exp\left[\frac{1}{2} \tilde{c}_2(\tau)\right] \end{aligned} \quad (4.4)$$

for $T/2 \leq \tau \leq T$. All the convolution integrals can be numerically evaluated using either the *Mathematica* or the Gauss quadrature method. Further improvement in the estimation

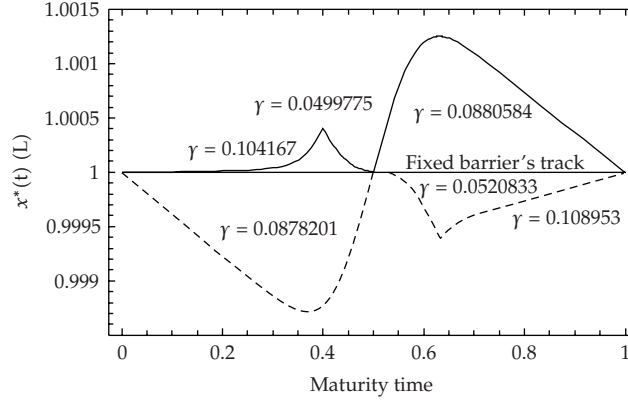


Figure 3: Barrier tracks for the bounds improved by a three-stage approximation scheme. Upper bound's track (solid line) and lower bound's track (long dashed line). Time to maturity is equal to one. Other input parameters are $\beta = 1.0$, $S_0 = 26$, $X = 20$, $\sigma_{BS}^2 = 0.02$, $d = 0$ and $r = 0.05$. The values of γ for the barrier tracks are shown along the tracks.

of the upper bound can be easily achieved by the approximation involving more stages. In Figure 2 the solid line gives an example of the moving barrier's trajectory for the two-stage approximation. The numerical results in Tables 1–3 show that the upper bounds are significantly improved.

As expected, the multistage approximation for both the upper and lower bounds becomes better and better as the number of stages increases; in fact, the gap between the bounds is asymptotically reduced to zero. In practice even a rather low-order approximation can yield very tight upper and lower bounds to the exact option price function. It should be pointed out that the above multistage approximation can be applied to an *up-and-out* put option in a similar manner as well.

5. Barrier Options with Time-Dependent Volatilities

Now, we apply the multistage approximation method to the case of a time-dependent volatility with the term structure.

$$\sigma_{BS}^2 = \sigma_0^2 \left\{ 1 + a_0 \exp \left[-\frac{(\tau - \tau_0)^2}{b_0} \right] \right\}, \quad (5.1)$$

where $\sigma_0 = 0.2$, $a_0 = 1$, $b_0 = 0.01$, and $\tau_0 = 0.5$. This term structure can be interpreted as a pulse of surge or drop (depending upon the sign of a_0) in market volatility. The centre of the pulse is at time τ_0 , and the width of the pulse is determined by b_0 . Other input parameters remain unchanged. In this example the three-stage approximation is used, and Figure 3 shows a couple of the typical barrier tracks. Because of the term structure of the volatility, the barrier movement is a little bit more complicated than those cases with constant volatilities. However, the spirit of the approximation scheme remains the same. Numerical results of the bounds of the barrier option prices for $\beta = 0.5, 1.0$, and 1.5 are shown in Table 4. For comparison, we also include the numerical data generated by Monte Carlo simulations using a time step of 10^{-5} and a sample of 10^5 random paths of the underlying asset price. It

Table 4: CEV up-and-out call option with time-dependent volatility. We extend our model to the time-dependent case with the volatility term structure expressed as: $\sigma_{BS}(\tau)^2 = \sigma_0^2 \{1 + a_0 \exp[-(\tau - \tau_0)^2/b_0]\}$ where $\sigma_0 = 0.2$, $a_0 = 1$, $b_0 = 0.01$, and $\tau_0 = 0.5$. Other input parameters are $S_0 = 26$, $X = 20$, $d = 0$, and $r = 0.05$. In the Monte Carlo simulation, $\Delta t = 0.00001$ and number of ensembles = 100000.

β	S	Lower bound of P	Upper bound of P	Monte-Carlo result
1.5	16	0.27326	0.27458	0.27240
	18	0.51040	0.51414	0.51468
	20	0.62600	0.63327	0.63608
	22	0.52915	0.53918	0.53809
	24	0.27824	0.28684	0.29050
1.0	16	0.28462	0.28602	0.28539
	18	0.55640	0.56046	0.56073
	20	0.67566	0.68348	0.68740
	22	0.55016	0.56068	0.56887
	24	0.27571	0.28432	0.28782
0.5	16	0.29076	0.29228	0.29136
	18	0.60248	0.60683	0.60574
	20	0.72862	0.73700	0.73813
	22	0.57272	0.58380	0.58778
	24	0.27361	0.28224	0.28672

Table 5: CEV down-and-out put option with time-dependent volatility. We extend our model to the time-dependent case with the volatility term structure expressed as $\sigma_{BS}(\tau)^2 = \sigma_0^2 \{1 + a_0 \exp[-(\tau - \tau_0)^2/b_0]\}$ where $\sigma_0 = 0.2$, $a_0 = 1$, $b_0 = 0.01$, and $\tau_0 = 0.5$. Other input parameters are $S_0 = 14$, $X = 20$, $d = 0$ and, $r = 0.05$. In the Monte Carlo simulation, $\Delta t = 0.00001$ and number of ensembles = 100000.

β	S	Lower bound of P	Upper bound of P	Monte-Carlo result
1.5	16	0.80290	0.81341	0.82993
	18	0.93028	0.93662	0.95328
	20	0.73071	0.73371	0.74182
	22	0.47491	0.47632	0.47829
	24	0.27759	0.27822	0.27923
1.0	16	0.79012	0.80025	0.81260
	18	0.87756	0.88452	0.89777
	20	0.67388	0.67754	0.68690
	22	0.44145	0.44324	0.44091
	24	0.26678	0.26764	0.26970
0.5	16	0.77761	0.78771	0.79748
	18	0.82329	0.83014	0.84160
	20	0.61817	0.62192	0.62888
	22	0.40516	0.40728	0.40804
	24	0.25092	0.25193	0.25603

can be seen that the bounds are indeed very tight whereas the Monte Carlo simulations give rather poor estimates. Furthermore, even though the approximation involves three stages, the convergence of the calculations is still very rapid for it takes less than two minutes to give one estimate of the barrier option value.

Finally, we generalize the multistage approximation scheme to the CEV *down-and-out* options. Table 5 shows the results of a *down-and-out* put option with time-dependent volatility. It should be noted that although we need to evaluate a double integral in this case, our proposed method works very well in evaluating the option prices, in terms of both accuracy and efficiency.

6. Conclusion

By a series of similarity transformations and changing variables, we have derived the analytical kernels of the pricing formulae of the CEV knockout options with time-dependent parameters for a parametric class of moving barriers. These results enable us to develop a simple and efficient method for computing accurate estimates of the single-barrier option prices (both call and put options) as well as their upper and lower bounds in the CEV model environment when the model parameters are time-dependent. By means of the multistage approximation scheme, the upper and lower bounds for the exact barrier option prices can be efficiently improved in a systematic manner. In view of the CEV model being empirically considered to be a better candidate in equity option pricing than the traditional Black-Scholes model, more comparative pricing and precise risk management in equity options can be achieved by incorporating term structures of interest rates, volatility, and dividend into the CEV option valuation model. Extension to the CEV double-knockout options with time-dependent parameters can also be straightforwardly achieved by solving (2.13) for $L_1 \leq x \leq L_2$ and $\tau \geq 0$ with absorbing boundary conditions at both endpoints. Such a boundary value problem is well documented in most standard textbooks on partial differential equations, but the solution will involve explicit searching for eigenvalues numerically. Moreover, it is natural that this new approach can be easily applied to capture the valuation of standard CEV options with specified moving knockout barriers. Finally, we would like to point out that the results for the down-and-out CEV option can be generalized to price the CEV American put option and lookback options with time-dependent parameters too. This research is now in progress and results will be published elsewhere.

References

- [1] F. Black and M. Scholes, "The pricing of options and corporate liability," *Journal of Political Economy*, vol. 81, pp. 637–654, 1973.
- [2] R. C. Merton, "Theory of rational option pricing," *The Rand Journal of Economics*, vol. 4, pp. 141–183, 1973.
- [3] M. Rubinstein and E. Reiner, "Breaking down the barriers," *Risk*, vol. 8, pp. 28–35, 1991.
- [4] M. Rubinstein and E. Reiner, "Unscrambling the binary code," *Risk*, vol. 9, pp. 37–42, 1991.
- [5] D. R. Rich, "The mathematical foundations of barrier option-pricing theory," *Advances in Futures and Options Research*, vol. 7, pp. 267–311, 1994.
- [6] R. Heynen and H. Kat, "Crossing the barrier," *Risk*, vol. 7, pp. 46–51, 1994.
- [7] Y. K. Kwok, L. Wu, and H. Yu, "Pricing multi-asset options with an external barrier," *International Journal of Theoretical and Applied Finance*, vol. 1, pp. 523–541, 1998.
- [8] C. F. Lo, P. H. Yuen, and C. H. Hui, "Pricing barrier options with square root process," *International Journal of Theoretical and Applied Finance*, vol. 4, no. 5, pp. 805–818, 2001.
- [9] N. Kunitomo and M. Ikeda, "Pricing options with curved boundaries," *Mathematical Finance*, vol. 2, pp. 275–298, 1992.
- [10] H. Geman and M. Yor, "Pricing and hedging double-barrier options: a probabilistic approach," *Mathematical Finance*, vol. 6, no. 4, pp. 365–378, 1996.
- [11] C. H. Hui, "One-touch double barrier binary option values," *Applied Financial Economics*, vol. 6, no. 4, pp. 343–346, 1996.

- [12] C. H. Hui, "Time-dependent barrier option values," *Journal of Futures Markets*, vol. 17, no. 6, pp. 667–688, 1997.
- [13] C. H. Hui, C. F. Lo, and P. H. Yuen, "Comment on: 'pricing double barrier options using Laplace transforms' by Antoon Pelsser," *Finance and Stochastics*, vol. 4, no. 1, pp. 105–107, 2000.
- [14] A. Pelsser, "Pricing double barrier options using Laplace transforms," *Finance and Stochastics*, vol. 4, no. 1, pp. 95–104, 2000.
- [15] F. Black, "Studies of stock price volatility changes," in *Proceedings of the Meetings of the American Statistical Association, Business and Economics Statistics Division*, pp. 177–181, 1976.
- [16] A. A. Christie, "The stochastic behavior of common stock variances," *Journal of Financial Economics*, vol. 10, no. 4, pp. 407–432, 1982.
- [17] R. Schmalensee and R. R. Trippi, "Common stock volatility expectations implied by option premia," *Journal of Finance*, vol. 33, pp. 129–147, 1978.
- [18] S. Hauser and B. Lauterbach, "Tests of warrant pricing models: the trading profits perspective," *Journal of Derivatives*, pp. 71–79, 1996.
- [19] B. Lauterbach and P. Schultz, "Pricing warrants: an empirical study of the Black-Scholes model and its alternatives," *Journal of Finance*, vol. 45, pp. 1181–1209, 1990.
- [20] S. Beckers, "The constant elasticity of variance model and its implications for option pricing," *Journal of Finance*, vol. 35, pp. 661–673, 1980.
- [21] L. Cathcart and L. El-Jahel, "Valuation of defaultable bonds," *Journal of Fixed Income*, vol. 2, pp. 65–78, 1998.
- [22] J. C. Cox and S. A. Ross, "The valuation of options for alternative stochastic processes," *Journal of Financial Economics*, vol. 3, no. 1-2, pp. 145–166, 1976.
- [23] J. Cox, "Notes on option pricing I: constant elasticity of variance diffusions," Working Paper, Stanford University, 1975.
- [24] C. F. Lo, P. H. Yuen, and C. H. Hui, "Constant elasticity of variance option pricing model with time-dependent parameters," *International Journal of Theoretical and Applied Finance*, vol. 3, no. 4, pp. 661–674, 2000.
- [25] C. F. Lo and C. H. Hui, "Lie-algebraic approach for pricing moving barrier options with time-dependent parameters," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1455–1464, 2006.
- [26] C. W. Gardiner, *Handbook of Stochastic Methods: For Physics, Chemistry and the Natural Science*, vol. 13 of *Springer Series in Synergetics*, Springer, Berlin, Germany, 2nd edition, 1985.
- [27] A. Perelomov, *Generalized Coherent States and Their Applications*, Texts and Monographs in Physics, Springer, Berlin, Germany, 1986.
- [28] N. N. Lebedev, *Special Functions and Their Applications*, Dover, New York, NY, USA, 1972.
- [29] E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-Order Differential Equations*, Clarendon Press, Oxford, UK, 1946.
- [30] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1964.

