Research Article

Fuzzy Boundary and Fuzzy Semiboundary

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We present several properties of fuzzy boundary and fuzzy semiboundary which have been supported by examples. Properties of fuzzy semi-interior, fuzzy semiclosure, fuzzy boundary, and fuzzy semiboundary have been obtained in product-related spaces. We give necessary conditions for fuzzy continuous (resp., fuzzy semicontinuous, fuzzy irresolute) functions. Moreover, fuzzy continuous (resp., fuzzy semicontinuous, fuzzy irresolute) functions have been characterized via fuzzy-derived (resp., fuzzy-semiderived) sets.

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1. INTRODUCTION

After Zadeh's [1] introduction of fuzzy sets, Chang [2] defined and studied the notion of a fuzzy topological space in 1968. Since then, much attention has been paid to generalize the basic concepts of classical topology in fuzzy setting and thus a modern theory of fuzzy topology has been developed.

In recent years, fuzzy topology has been found to be very useful in solving many practical problems. Du et al. [3] fuzzified the very successful 9-intersection Egenhofer model [4, 5] for depicting topological relations in geographic information systems (GIS) query. In [6, 7], El-Naschie showed that the notion of fuzzy topology might be relevant to quantum particle physics and quantum gravity in connection with string theory and e^∞ theory. Tang [8] used a slightly changed version of Chang's fuzzy topological space to model spatial objects for GIS databases and Structured Query Language (SQL) for GIS.

Levine [9] introduced the concepts of semiopen sets and semicontinuous mappings in topological spaces. Interestingly, his work found applications in the field of digital topology [10]. For example, it was found that digital line is a $T_{1/2}$ -space [11], which is a weaker separation axiom based upon semiopen sets. Fuzzy digital topology [12] was introduced by Rosenfeld, which demonstrated the need for the fuzzification of weaker forms of notions of classical topology. Azad [13] carried out this fuzzification in 1981, and presented some

general properties of fuzzy spaces. Several properties of fuzzy semiopen (resp., fuzzy semiclosed), fuzzy regular open (resp., closed) sets have been discussed. Moreover, he defined fuzzy semicontinuous (resp., semiopen, semiclosed) functions and studied the properties of fuzzy semicontinuous function in product-related spaces. Finally, he defined and characterized fuzzy almost continuous mappings. In this direction much work followed subsequently, for example, [14–23].

Though Pu and Liu [24] defined the notion of fuzzy boundary in fuzzy topological spaces in 1980, yet there is very little work available on this notion in present literature. One reason, inter alia, of Tang's [8] use of a limited version of Chang's fuzzy topological space was the nonavailability of sufficient material about properties of fuzzy boundary. So, we study this concept further and establish several of its properties, thus providing sufficient material for researchers to utilize these concepts fruitfully. Ahmad and Athar [25] defined the concept of fuzzy semiboundary and characterized fuzzy semicontinuous functions in terms of fuzzy semiboundary.

In this paper, we present several properties of fuzzy boundary and fuzzy semiboundary which have been supported by examples. Properties of fuzzy semi-interior, fuzzy semiclosure, fuzzy boundary, and fuzzy semiboundary have been obtained in product-related spaces. We give necessary conditions for fuzzy continuous (resp., fuzzy semicontinuous, fuzzy irresolute) functions. Moreover, fuzzy continuous

(resp., fuzzy semicontinuous, fuzzy irresolute) functions have been characterized via fuzzy-derived (resp., fuzzy semiderived) sets.

2. PRELIMINARIES

First, we briefly recall certain definitions and results; for those not described; we refer to [1, 2, 13, 22].

A fuzzy set λ in a set X is a function from X to $[0, 1]$, that is, $\lambda : X \rightarrow [0, 1]$.

Definition 1 (see [1]). Let λ and μ be fuzzy sets in X . Then, for all $x \in X$,

$$\begin{aligned} \lambda = \mu &\iff \lambda(x) = \mu(x), \\ \lambda \leq \mu &\iff \lambda(x) \leq \mu(x), \\ \psi = \lambda \vee \mu &\iff \psi(x) = \max\{\lambda(x), \mu(x)\}, \\ \delta = \lambda \wedge \mu &\iff \delta(x) = \min\{\lambda(x), \mu(x)\}, \\ \eta = \lambda^c &\iff \eta(x) = 1 - \lambda(x). \end{aligned} \quad (1)$$

More generally, for a family $\Lambda = \{\lambda_i \mid i \in I\}$ of fuzzy sets in X , the union $\psi = \vee_i \lambda_i$ and intersection $\delta = \wedge_i \lambda_i$ are defined by

$$\begin{aligned} \psi(x) &= \sup_i \{\lambda_i(x) \mid x \in X\}, \\ \delta(x) &= \inf_i \{\lambda_i(x) \mid x \in X\}. \end{aligned} \quad (2)$$

The empty fuzzy set $\tilde{0}$ is defined as $\tilde{0}(x) = 0$, for all $x \in X$, and the symbol $\tilde{1}$ denotes the fuzzy set $\tilde{1}(x) = 1$, for all $x \in X$.

Definition 2 (see [2]). Let $f : X \rightarrow Y$ be a function. Let β be a fuzzy set in Y with membership function $\beta(y)$. Then, the inverse of β , written as $f^{-1}(\beta)$, is a fuzzy set in X whose membership function is defined by

$$f^{-1}(\beta)(x) = \beta(f(x)), \quad \forall x \in X. \quad (3)$$

Conversely, let λ be a fuzzy set in X with membership function $\lambda(x)$. The image of λ , written as $f(\lambda)$, is a fuzzy set in Y whose membership function is given by

$$f(\lambda)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \{\lambda(x)\}, & \text{if } f^{-1}(y) \text{ is nonempty,} \\ 0, & \text{otherwise,} \end{cases} \quad (4)$$

for all $y \in Y$, where $f^{-1}(y) = \{x \mid f(x) = y\}$.

Definition 3 (see [2]). A fuzzy topology is a family τ of fuzzy sets in X , which satisfies the following conditions:

- (1) $\tilde{0}, \tilde{1} \in \tau$;
- (2) if $\lambda, \mu \in \tau$, then $\lambda \wedge \mu \in \tau$;
- (3) if $\lambda_i \in \tau$ for each $i \in I$, then $\vee_i \lambda_i \in \tau$.

τ is called a fuzzy topology for X , and the pair (X, τ) is a fuzzy topological space. Every member of τ is called τ -open

fuzzy set (or simply fuzzy open set). A fuzzy set is τ -closed if and only if its complement is τ -open.

As in general topology, the indiscrete fuzzy topology contains only $\tilde{0}$ and $\tilde{1}$, while the discrete fuzzy topology contains all fuzzy sets. In the sequel, we write an fts X (or (X, τ)) in place of “a space X with fuzzy topology τ .”

Definition 4 (see [2]). The closure and interior of a fuzzy set λ in an fts (X, τ) are denoted and defined as

$$\begin{aligned} \text{Cl } \lambda &= \inf\{\mu \mid \lambda \leq \mu, \mu^c \in \tau\}, \\ \text{Int } \lambda &= \sup\{\omega \mid \omega \leq \lambda, \omega \in \tau\}. \end{aligned} \quad (5)$$

We mention below some properties of closure and interior of a fuzzy set which will be used in the sequel.

Lemma 1 (see [26]). For fuzzy sets λ and μ in an fts X , one has the following:

- (1) λ is fuzzy closed (resp., fuzzy open) $\Leftrightarrow \text{Cl } \lambda = \lambda$ (resp., $\text{Int } \lambda = \lambda$);
- (2) $\lambda \leq \mu \Rightarrow \text{Cl } \lambda \leq \text{Cl } \mu$ ($\text{Int } \lambda \leq \text{Int } \mu$);
- (3) $\text{Cl } \text{Cl } \lambda = \text{Cl } \lambda$ ($\text{Int } \text{Int } \lambda = \text{Int } \lambda$);
- (4) $\text{Cl } \lambda \vee \text{Cl } \mu = \text{Cl } (\lambda \vee \mu)$;
- (5) $\text{Cl } \lambda \wedge \text{Cl } \mu \geq \text{Cl } (\lambda \wedge \mu)$;
- (6) $\text{Int } \lambda \vee \text{Int } \mu \leq \text{Int } (\lambda \vee \mu)$;
- (7) $\text{Int } \lambda \wedge \text{Int } \mu = \text{Int } (\lambda \wedge \mu)$;
- (8) $\text{Int } \lambda^c = (\text{Cl } \lambda)^c$;
- (9) $\text{Cl } \lambda^c = (\text{Int } \lambda)^c$.

Definition 5 (see [2]). A function $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be fuzzy continuous if and only if the inverse of each δ -fuzzy open set is τ -fuzzy open.

Theorem 1 (see [19, 26]). Let $f : X \rightarrow Y$ be a function from an fts (X, τ) to another fts (Y, δ) . Then, the following conditions are equivalent:

- (1) f is fuzzy continuous,
- (2) the inverse of each δ -closed fuzzy set is τ -closed,
- (3) for each fuzzy set λ in X , $f(\text{Cl } \lambda) \leq \text{Cl } f(\lambda)$,
- (4) for each fuzzy set λ in Y , $f^{-1}(\text{Int } \nu) \leq \text{Int } f^{-1}(\nu)$.

3. FUZZY BOUNDARY

Definition 6 (see [24]). Let λ be a fuzzy set in an fts X . Then, the fuzzy boundary of λ is defined as $\text{Bd } \lambda = \text{Cl } \lambda \wedge \text{Cl } \lambda^c$. Obviously, $\text{Bd } \lambda$ is a fuzzy closed set.

Remark 1. In classical topology, for an arbitrary set A of a topological space X , we have $A \cup \text{Bd } A = \text{Cl } A$, but in fuzzy topology we have $\lambda \vee \text{Bd } \lambda \leq \text{Cl } \lambda$, for an arbitrary fuzzy set λ in X , the converse of which is not true as shown by Pu and Liu [24]. Moreover, we have the following proposition.

Proposition 1. For fuzzy sets λ and μ in an fts X , the following conditions hold.

- (1) $\text{Bd } \lambda = \text{Bd } \lambda^c$.
- (2) If λ is fuzzy closed, then $\text{Bd } \lambda \leq \lambda$.
- (3) If λ is fuzzy open, then $\text{Bd } \lambda \leq \lambda^c$.
- (4) Let $\lambda \leq \mu$ and $\mu \in \text{FC}(X)$ (resp., $\mu \in \text{FO}(X)$). Then, $\text{Bd } \lambda \leq \mu$ (resp., $\text{Bd } \lambda \leq \mu^c$), where $\text{FC}(X)$ (resp., $\text{FO}(X)$) denotes the class of fuzzy closed (resp., fuzzy open) sets in X .
- (5) $(\text{Bd } \lambda)^c = \text{Int } \lambda \vee \text{Int } \lambda^c$.

Proof. (1) $\text{Bd } \lambda = \text{Cl } \lambda \wedge \text{Cl } \lambda^c = \text{Cl } \lambda^c \wedge \text{Cl } \lambda = \text{Cl } \lambda^c \wedge \text{Cl } (\lambda^c)^c = \text{Bd } \lambda^c$.

(2) $\text{Bd } \lambda = \text{Cl } \lambda \wedge \text{Cl } \lambda^c \leq \text{Cl } \lambda = \lambda$, hence $\text{Bd } \lambda \leq \lambda$.

(3) λ is fuzzy open implies λ^c is fuzzy closed. By (2), $\text{Bd } \lambda^c \leq \lambda^c$ and by (1), we get $\text{Bd } \lambda \leq \lambda^c$.

(4) Since $\lambda \leq \mu$ implies $\text{Cl } \lambda \leq \text{Cl } \mu$, we have $\text{Bd } \lambda = \text{Cl } \lambda \wedge \text{Cl } \lambda^c \leq \text{Cl } \mu \wedge \text{Cl } \lambda^c \leq \text{Cl } \mu = \mu$, since $\mu \in \text{FC}(X)$.

(5) $(\text{Bd } \lambda)^c = (\text{Cl } \lambda \wedge \text{Cl } \lambda^c)^c = (\text{Cl } \lambda)^c \vee (\text{Cl } \lambda^c)^c = \text{Int } \lambda^c \vee \text{Int } \lambda$. $\square$

The converse of (2) and (3) of Proposition 1 is, in general, not true as is shown by the following example.

Example 1. Let $X = \{a, b, c\}$ be a set and τ , the fuzzy topology generated by fuzzy sets $\mu = \{a_4, b_8, c_2\}$, $\nu = \{a_6, b_9, c_1\}$, and $\omega = \{a_5, b_7, c_3\}$. Then, $\tau = \{\tilde{0}, \mu, \nu, \omega, \{a_4, b_8, c_1\}, \{a_4, b_7, c_2\}, \{a_5, b_7, c_1\}, \{a_4, b_7, c_1\}, \{a_5, b_8, c_3\}, \{a_5, b_8, c_1\}, \{a_6, b_9, c_2\}, \{a_5, b_8, c_2\}, \{a_6, b_9, c_3\}, \{a_5, b_7, c_2\}, \tilde{1}\}$. Choose $\lambda = \{a_5, b_7, c_9\}$ and $\psi = \{a_5, b_3, c_3\}$. Calculations give

$$\begin{aligned} \text{Bd } \lambda &= \{a_5, b_3, c_7\} \leq \lambda, \quad \text{but } \lambda \text{ is not fuzzy closed,} \\ \text{Bd } \psi &= \{a_5, b_3, c_7\} \leq \psi^c, \quad \text{but } \psi \text{ is not fuzzy open.} \end{aligned} \quad (6)$$

The following proposition gives some more properties of fuzzy boundary.

Proposition 2. Let λ be a fuzzy set in an fts X . Then,

- (1) $\text{Bd } \lambda = \text{Cl } \lambda - \text{Int } \lambda$;
- (2) $\text{Bd } \text{Int } \lambda \leq \text{Bd } \lambda$;
- (3) $\text{Bd } \text{Cl } \lambda \leq \text{Bd } \lambda$;
- (4) $\text{Int } \lambda \leq \lambda - \text{Bd } \lambda$.

Proof. (1) Since $(\text{Cl } \lambda)^c = \text{Int } \lambda$, therefore we have $\text{Bd } \lambda = \text{Cl } \lambda \wedge \text{Cl } \lambda^c = \text{Cl } \lambda - (\text{Cl } \lambda)^c = \text{Cl } \lambda - \text{Int } \lambda$. Thus $\text{Bd } \lambda = \text{Cl } \lambda - \text{Int } \lambda$. This proves (1),

(2) $\text{Bd } \text{Int } \lambda = \text{Cl } \text{Int } \lambda - \text{Int } \text{Int } \lambda = \text{Cl } \text{Int } \lambda - \text{Int } \lambda \leq \text{Cl } \lambda - \text{Int } \lambda = \text{Bd } \lambda$,

(3) $\text{Bd } \text{Cl } \lambda = \text{Cl } \text{Cl } \lambda - \text{Int } \text{Cl } \lambda = \text{Cl } \lambda - \text{Int } \text{Cl } \lambda \leq \text{Cl } \lambda - \text{Int } \lambda = \text{Bd } \lambda$,

(4)

$$\begin{aligned} \lambda - \text{Bd } \lambda &= \lambda \wedge (\text{Bd } \lambda)^c \\ &= \lambda \wedge (\text{Cl } \lambda \wedge \text{Cl } \lambda^c)^c \\ &= \lambda \wedge (\text{Int } \lambda^c \vee \text{Int } \lambda) \\ &= (\lambda \wedge \text{Int } \lambda^c) \vee (\lambda \wedge \text{Int } \lambda) \\ &= (\lambda \wedge \text{Int } \lambda^c) \vee \text{Int } \lambda \\ &\geq \text{Int } \lambda. \end{aligned} \quad (7) \quad \square$$

The following example shows that the equality does not hold in Proposition 2(2)–(4).

Example 2. Choose $\alpha = \{a_{11}, b_{41}, c_{71}\}$, $\beta = \{a_{61}, b_{81}, c_{51}\}$, and $\gamma = \{a_{61}, b_{31}, c_{81}\}$ in Example 1. Then, calculations give

$$\begin{aligned} \text{Bd } \alpha &= \tilde{1} \neq \tilde{0} = \text{Bd } \text{Int } \alpha, \\ \text{Bd } \beta &= \{a_{51}, b_{21}, c_{71}\} \neq \tilde{0} = \text{Bd } \text{Cl } \beta, \\ \gamma - \text{Bd } \gamma &= \{a_{41}, b_{31}, c_{21}\} \neq \text{Int } \gamma = \tilde{0}. \end{aligned} \quad (8)$$

Remark 2. In general topology, the following conditions hold:

$$\begin{aligned} \text{Bd } A \cap \text{Int } A &= \phi, \\ \text{Int } A \cup \text{Bd } A &= \text{Cl } A, \\ \text{Int } A \cup \text{Int } A^c \cup \text{Bd } A &= X, \end{aligned} \quad (9)$$

whereas, in fuzzy topology, we give counter-examples to show that these may not hold in general.

Example 3. In the fts (X, τ) of Example 1, we choose fuzzy set $\lambda = \{a_4, b_7, c_7\}$, then calculations give

$$\begin{aligned} \text{Bd } \lambda \wedge \text{Int } \lambda &= \{a_6, b_3, c_8\} \wedge \{a_4, b_7, c_2\} \neq \tilde{0}, \\ \text{Int } \lambda \vee \text{Bd } \lambda &= \{a_6, b_7, c_8\} \neq \text{Cl } \lambda = \tilde{1}, \\ \text{Int } \lambda \vee \text{Int } \lambda^c \vee \text{Bd } \lambda &= \{a_6, b_7, c_8\} \neq \tilde{1}. \end{aligned} \quad (10)$$

It is easily seen that $\text{Int } \lambda \vee \text{Bd } \lambda \leq \text{Cl } \lambda$.

Theorem 2. Let λ and μ be fuzzy sets in an fts X . Then, $\text{Bd } (\lambda \vee \mu) \leq \text{Bd } \lambda \vee \text{Bd } \mu$.

Proof.

$$\begin{aligned} \text{Bd } (\lambda \vee \mu) &= \text{Cl } (\lambda \vee \mu) \wedge \text{Cl } (\lambda \vee \mu)^c \\ &\leq (\text{Cl } \lambda \vee \text{Cl } \mu) \wedge (\text{Cl } \lambda^c \wedge \text{Cl } \mu^c) \\ &= [\text{Cl } \lambda \wedge (\text{Cl } \lambda^c \wedge \text{Cl } \mu^c)] \vee [\text{Cl } \mu \wedge (\text{Cl } \lambda^c \wedge \text{Cl } \mu^c)] \\ &= (\text{Bd } \lambda \wedge \text{Cl } \mu^c) \vee (\text{Bd } \mu \wedge \text{Cl } \lambda^c) \\ &\leq \text{Bd } \lambda \vee \text{Bd } \mu. \end{aligned} \quad (11) \quad \square$$

In Theorem 2, the equality does not hold as is shown by the following.

Example 4. In Example 1, choose fuzzy sets $\alpha = \{a_0, b_0, c_2\}$ and $\beta = \{a_4, b_8, c_1\}$. Then, calculations give

$$\text{Bd } \alpha \vee \text{Bd } \beta = \{a_6, b_2, c_9\} \neq \{a_6, b_2, c_8\} = \text{Bd}(\alpha \vee \beta). \quad (12)$$

The following examples show that

$$\text{Bd}(\gamma \wedge \delta) \not\leq \text{Bd } \gamma \wedge \text{Bd } \delta, \quad \text{Bd } \psi \wedge \text{Bd } \xi \not\leq \text{Bd}(\psi \wedge \xi). \quad (13)$$

For this, choose $\gamma = \{a_4, b_1, c_9\}$, $\delta = \{a_4, b_1, c_1\}$, $\psi = \{a_7, b_4, c_4\}$, and $\xi = \{a_4, b_1, c_8\}$. Then, calculations give

$$\begin{aligned} \text{Bd}(\gamma \wedge \delta) &= \{a_4, b_1, c_9\} \not\leq \{a_4, b_1, c_8\} = \text{Bd } \gamma \wedge \text{Bd } \delta, \\ \text{Bd } \psi \wedge \text{Bd } \xi &= \{a_4, b_1, c_8\} \not\leq \{a_4, b_1, c_7\} = \text{Bd}(\psi \wedge \xi). \end{aligned} \quad (14)$$

However, we have the following theorem.

Theorem 3. For any fuzzy sets λ and μ in an fts X , one has

$$\text{Bd}(\lambda \wedge \mu) \leq (\text{Bd } \lambda \wedge \text{Cl } \mu) \vee (\text{Bd } \mu \wedge \text{Cl } \lambda). \quad (15)$$

Proof.

$$\begin{aligned} \text{Bd}(\lambda \wedge \mu) &= \text{Cl}(\lambda \wedge \mu) \wedge \text{Cl}(\lambda \wedge \mu)^c \\ &\leq (\text{Cl } \lambda \wedge \text{Cl } \mu) \wedge (\text{Cl } \lambda^c \vee \text{Cl } \mu^c) \\ &= [(\text{Cl } \lambda \wedge \text{Cl } \mu) \wedge \text{Cl } \lambda^c] \vee [(\text{Cl } \lambda \wedge \text{Cl } \mu) \wedge \text{Cl } \mu^c] \\ &= (\text{Bd } \lambda \wedge \text{Cl } \mu) \vee (\text{Bd } \mu \wedge \text{Cl } \lambda). \end{aligned} \quad (16)$$

□

Corollary 1. For any fuzzy sets λ and μ in an fts X , one has

$$\text{Bd}(\lambda \wedge \mu) \leq \text{Bd } \lambda \vee \text{Bd } \mu. \quad (17)$$

Example 5. To show that the equality in Theorem 3, in general, does not hold, choose $\lambda = \{a_6, b_6, c_7\}$ and $\mu = \{a_6, b_2, c_9\}$ in the fts X defined in Example 1. Then, calculations give

$$(\text{Bd } \lambda \wedge \text{Cl } \mu) \vee (\text{Bd } \mu \wedge \text{Cl } \lambda) = \mu \not\leq \{a_6, b_2, c_8\} = \text{Bd}(\lambda \wedge \mu). \quad (18)$$

In general topology, it is known that

$$\text{Bd } \text{Bd } \text{Bd } A = \text{Bd } \text{Bd } A, \quad (19)$$

for any subset A of a space X . However, in fuzzy topology, we have the following proposition.

Proposition 3. For any fuzzy set λ in an fts X , one has

- (1) $\text{Bd } \text{Bd } \lambda \leq \text{Bd } \lambda$;
- (2) $\text{Bd } \text{Bd } \text{Bd } \lambda \leq \text{Bd } \text{Bd } \lambda$.

Proof. (1) $\text{Bd } \text{Bd } \lambda = \text{Cl } \text{Bd } \lambda \wedge \text{Cl}(\text{Bd } \lambda)^c \leq \text{Cl } \text{Bd } \lambda = \text{Bd } \lambda$.
(2)

$$\begin{aligned} \text{Bd } \text{Bd } \text{Bd } \lambda &= \text{Cl } \text{Bd } \text{Bd } \lambda \wedge \text{Cl}(\text{Bd } \text{Bd } \lambda)^c \\ &= \text{Bd } \text{Bd } \lambda \wedge \text{Cl}(\text{Bd } \text{Bd } \lambda)^c \\ &\leq \text{Bd } \text{Bd } \lambda. \end{aligned} \quad (20)$$

□

Remark 3. We could not find an example to show that the equality in (2) does not hold. However, the equality in (1), in general, does not hold as is shown by the following example.

Example 6. Choose $\lambda = \{a_2, b_5, c_9\}$ in fts X of Example 1. Then, $\text{Bd } \lambda = \tilde{1}$ but $\text{Bd } \text{Bd } \lambda = \tilde{0}$.

Definition 7 (see [27]). If λ is a fuzzy set of X and μ is a fuzzy set of Y , then

$$(\lambda \times \mu)(x, y) = \min \{\lambda(x), \mu(y)\}, \quad \text{for each } (x, y) \in X \times Y. \quad (21)$$

Definition 8 (see [13]). An fts (X, τ_X) is product related to another fts (Y, τ_Y) if for any fuzzy set ν of X and ξ of Y whenever $\lambda^c \not\geq \nu$ and $\mu^c \not\geq \xi$ imply $\lambda^c \times \tilde{1} \vee \tilde{1} \times \mu^c \geq \nu \times \xi$, where $\lambda \in \tau_X$ and $\mu \in \tau_Y$, there exist $\lambda_1 \in \tau_X$ and $\mu_1 \in \tau_Y$ such that $\lambda_1^c \geq \nu$ and $\mu_1^c \geq \xi$ and $\lambda_1^c \times \tilde{1} \vee \tilde{1} \times \mu_1^c = \lambda^c \times \tilde{1} \vee \tilde{1} \times \mu^c$.

Theorem 4 (see [13]). Let X and Y be product-related fts's. Then, for a fuzzy set λ of X and a fuzzy set μ of Y , one has

- (1) $\text{Cl}(\lambda \times \mu) = \text{Cl } \lambda \times \text{Cl } \mu$,
- (2) $\text{Int}(\lambda \times \mu) = \text{Int } \lambda \times \text{Int } \mu$.

Lemma 2. For fuzzy sets λ, μ, ν , and ω in a set X , one has

$$(\lambda \wedge \mu) \times (\nu \wedge \omega) = (\lambda \times \omega) \wedge (\mu \times \nu). \quad (22)$$

Proof.

$$\begin{aligned} &((\lambda \wedge \mu) \times (\nu \wedge \omega))(x, y) \\ &= \min((\lambda \wedge \mu)(x), (\nu \wedge \omega)(y)) \\ &= \min(\min(\lambda(x), \mu(x)), \min(\nu(y), \omega(y))) \\ &= \min(\min(\lambda(x), \omega(y)), \min(\mu(x), \nu(y))) \\ &= \min((\lambda \times \omega)(x, y), (\mu \times \nu)(x, y)) \\ &= ((\lambda \times \omega) \wedge (\mu \times \nu))(x, y). \end{aligned} \quad (23)$$

□

Theorem 5. Let $X_i, i = 1, 2, \dots, n$, be a family of product-related fuzzy topological spaces. If each λ_i is a fuzzy set in X_i , then

$$\begin{aligned} \text{Bd} \prod_{i=1}^n \lambda_i &= [\text{Bd } \lambda_1 \times \text{Cl } \lambda_2 \times \dots \times \text{Cl } \lambda_n] \\ &\vee [\text{Cl } \lambda_1 \times \text{Bd } \lambda_2 \times \text{Cl } \lambda_3 \times \dots \times \text{Cl } \lambda_n] \\ &\vee \dots \vee [\text{Cl } \lambda_1 \times \text{Cl } \lambda_2 \times \dots \times \text{Bd } \lambda_n]. \end{aligned} \quad (24)$$

Proof. We use Proposition 2(1), Theorem 4, and Lemma 2 to prove this.

It suffices to prove this for $n = 2$. Consider

$$\begin{aligned}
 \text{Bd}(\lambda_1 \times \lambda_2) &= \text{Cl}(\lambda_1 \times \lambda_2) - \text{Int}(\lambda_1 \times \lambda_2) \\
 &= (\text{Cl}\lambda_1 \times \text{Cl}\lambda_2) - (\text{Int}\lambda_1 \times \text{Int}\lambda_2) \\
 &= (\text{Cl}\lambda_1 \times \text{Cl}\lambda_2) - ((\text{Int}\lambda_1 \wedge \text{Cl}\lambda_1) \times (\text{Int}\lambda_2 \wedge \text{Cl}\lambda_2)) \\
 &= (\text{Cl}\lambda_1 \times \text{Cl}\lambda_2) - ((\text{Int}\lambda_1 \times \text{Cl}\lambda_2) \wedge (\text{Cl}\lambda_1 \times \text{Int}\lambda_2)) \\
 &\quad \text{(by Lemma 2)} \\
 &= [(\text{Cl}\lambda_1 \times \text{Cl}\lambda_2) - (\text{Int}\lambda_1 \times \text{Cl}\lambda_2)] \\
 &\quad \vee [(\text{Cl}\lambda_1 \times \text{Cl}\lambda_2) - (\text{Cl}\lambda_1 \times \text{Int}\lambda_2)] \\
 &= [(\text{Cl}\lambda_1 - \text{Int}\lambda_1) \times \text{Cl}\lambda_2] \vee [\text{Cl}\lambda_1 \times (\text{Cl}\lambda_2 - \text{Int}\lambda_2)] \\
 &= (\text{Bd}\lambda_1 \times \text{Cl}\lambda_2) \vee (\text{Cl}\lambda_1 \times \text{Bd}\lambda_2).
 \end{aligned} \tag{25}$$

□

Theorem 6. Let $f : X \rightarrow Y$ be a fuzzy continuous function. Then,

$$\text{Bd } f^{-1}(\mu) \leq f^{-1}(\text{Bd } \mu), \tag{26}$$

for any fuzzy set μ in Y .

Proof. Let f be fuzzy continuous and μ a fuzzy set in Y . Then,

$$\begin{aligned}
 \text{Bd } f^{-1}(\mu) &= \text{Cl } f^{-1}(\mu) \wedge \text{Cl}(f^{-1}(\mu))^c \\
 &\leq \text{Cl } f^{-1}(\text{Cl } \mu) \wedge \text{Cl } f^{-1}(\text{Cl } \mu^c) \\
 &= f^{-1}(\text{Cl } \mu) \wedge f^{-1}(\text{Cl } \mu^c) \\
 &= f^{-1}(\text{Cl } \mu \wedge \text{Cl } \mu^c) \\
 &= f^{-1}(\text{Bd } \mu).
 \end{aligned} \tag{27}$$

Therefore, $\text{Bd } f^{-1}(\mu) \leq f^{-1}(\text{Bd } \mu)$. □

Before closing this section, we give an interesting characterization of fuzzy continuous functions in terms of fuzzy-derived set and fuzzy closure. It shows that fuzzy continuity, in essence, amounts to preservation of fuzzy closedness. For this we first recall following definitions.

Definition 9 (see [24]). A fuzzy set in X is called a fuzzy point if and only if it takes the value 0 for all $x \in X$ except one, say, $e \in X$. If its value at e is α ($0 < \alpha \leq 1$), one denotes this fuzzy point by e_α . The point $e \in X$ is called support of fuzzy point e_α and is denoted as $\text{supp}(e)$.

Definition 10 (see [24]). A fuzzy point e_α is said to be quasicoincident (also written as q -coincident or Q -coincident) with a fuzzy set λ , denoted by $e_\alpha q \lambda$, if for some $x \in X$, $\alpha > \lambda^c(x)$, or $\alpha + \lambda(x) > 1$.

Definition 11 (see [24]). A fuzzy set λ in an fts X is called a Q -neighborhood of a fuzzy point e , if there exists a $\mu \in \tau$ such that $eq\mu \leq \lambda$.

Definition 12 (see [24]). A fuzzy point e is called an adherent point of a fuzzy set λ , if every Q -neighborhood of e is quasicoincident with λ .

Definition 13 (see [24]). A fuzzy point e is called an accumulation point of a fuzzy set λ , if e is an adherent point of λ and every Q -neighborhood of e and λ is quasicoincident at some point different from $\text{supp}(e)$, whenever $e \in \lambda$. The union of all the accumulation points of λ is called the derived set of λ , denoted as λ^d . It is evident that $\lambda^d \leq \text{Cl } \lambda$.

Proposition 4 (see [24]). Let λ be a fuzzy set in an fts X , then

- (1) $\text{Cl } \lambda = \lambda \vee \lambda^d$;
- (2) λ is fuzzy closed if and only if $\lambda^d \leq \lambda$.

We use Proposition 4 and prove the following theorem.

Theorem 7. Let $f : X \rightarrow Y$ be a function. Then, the following conditions are equivalent:

- (1) $f : X \rightarrow Y$ is fuzzy continuous;
- (2) $f(\lambda^d) \leq \text{Cl } f(\lambda)$, for any fuzzy set λ in X .

Proof. (1) $\Rightarrow$ (2) Let f be fuzzy continuous and λ a fuzzy set in X . Since $\text{Cl } f(\lambda)$ is fuzzy closed in Y , then $f^{-1}(\text{Cl } f(\lambda))$ is fuzzy closed in X . $\lambda \leq f^{-1}(\text{Cl } f(\lambda))$ gives $\text{Cl } \lambda \leq \text{Cl } f^{-1}(\text{Cl } f(\lambda)) = f^{-1}(\text{Cl } f(\lambda))$. Therefore, $f(\lambda^d) \leq f(\text{Cl } \lambda) \leq f f^{-1}(\text{Cl } f(\lambda)) \leq \text{Cl } f(\lambda)$. Consequently, $f(\lambda^d) \leq \text{Cl } f(\lambda)$.

(2) $\Rightarrow$ (1) Suppose $f(\lambda^d) \leq \text{Cl } f(\lambda)$, $\lambda \in I^X$. Let μ be any fuzzy closed set in Y . We show that $f^{-1}(\mu)$ is fuzzy closed in X . By our hypothesis, $f([f^{-1}(\mu)]^d) \leq \text{Cl } f(f^{-1}(\mu)) \leq \text{Cl } \mu = \mu$, $f([f^{-1}(\mu)]^d) \leq \mu$ gives $[f^{-1}(\mu)]^d \leq f^{-1}(f[f^{-1}(\mu)]^d) \leq f^{-1}(\mu)$, or $[f^{-1}(\mu)]^d \leq f^{-1}(\mu)$ implies that $f^{-1}(\mu)$ is fuzzy closed in X . Thus, f is fuzzy continuous. □

4. FUZZY SEMIBOUNDARY

First, we recall some notions.

Definition 14 (see [13]). Let λ be a fuzzy set in an fts (X, τ) . Then, λ is called a fuzzy semiopen set of X if there exists a $\nu \in \tau$ such that $\nu \leq \lambda \leq \text{Cl } \nu$.

Definition 15 (see [22]). Let λ be a fuzzy set in an fts X . Then, semiclosure (briefly $s\text{Cl}$) and semi-interior (briefly $s\text{Int}$) of λ are given as

$$\begin{aligned}
 s\text{Cl } \lambda &= \wedge \{\beta \mid \lambda \leq \beta, \beta \text{ fuzzy semiclosed}\}, \\
 s\text{Int } \lambda &= \vee \{\beta \mid \beta \leq \lambda, \beta \text{ fuzzy semiopen}\}.
 \end{aligned} \tag{28}$$

Remark 4. In the following theorems, we note that almost all the properties related to fuzzy semi-interior, fuzzy semiclosure, and fuzzy semiboundary are analogous to their counterparts in fuzzy topology and hence some of the proofs are not given.

Theorem 8. For fuzzy sets λ and μ in an fts X , one has

- (1) $s\text{Int}(\lambda \vee \mu) \geq s\text{Int } \lambda \vee s\text{Int } \mu$;
- (2) $s\text{Int}(\lambda \wedge \mu) = s\text{Int } \lambda \wedge s\text{Int } \mu$;
- (3) $s\text{Cl}(\lambda \vee \mu) = s\text{Cl } \lambda \vee s\text{Cl } \mu$;

- (4) $sCl(\lambda \wedge \mu) \leq sCl\lambda \wedge sCl\mu$;
- (5) $sCl sCl\lambda = sCl\lambda$;
- (6) $sInt sInt\lambda = sInt\lambda$;
- (7) $(sInt\lambda)^c = sCl(\lambda^c)$;
- (8) $(sCl\lambda)^c = sInt(\lambda^c)$.

Proof. (1) $sInt\lambda$ and $sInt\mu$ are both fuzzy semiopen and $\lambda \leq \lambda \vee \mu$, $\mu \leq \lambda \vee \mu$ imply $sInt\lambda \leq sInt(\lambda \vee \mu)$ and $sInt\mu \leq sInt(\lambda \vee \mu)$. Combining, $sInt\lambda \vee sInt\mu \leq sInt(\lambda \vee \mu)$ or

$$sInt(\lambda \vee \mu) \geq sInt\lambda \vee sInt\mu. \quad (29)$$

(2) $\lambda \wedge \mu \leq \lambda$ and $\lambda \wedge \mu \leq \mu$ imply $sInt(\lambda \wedge \mu) \leq sInt\lambda$, $sInt(\lambda \wedge \mu) \leq sInt\mu$ and therefore $sInt(\lambda \wedge \mu) \leq sInt\lambda \wedge sInt\mu$. Conversely $sInt\lambda \leq \lambda$ and $sInt\mu \leq \mu$ implies $sInt\lambda \wedge sInt\mu \leq \lambda \wedge \mu$ and $sInt\lambda \wedge sInt\mu$ is fuzzy semiopen. But $sInt(\lambda \wedge \mu)$ is the largest fuzzy semiopen set contained in $\lambda \wedge \mu$, hence $sInt\lambda \wedge sInt\mu \leq sInt(\lambda \wedge \mu)$. This gives the equality.

(3) It follows easily from (2).

(4) Since $\lambda \wedge \mu \leq \lambda$, $\lambda \wedge \mu \leq \mu$

$$\begin{aligned} \Rightarrow sCl(\lambda \wedge \mu) &\leq sCl\lambda, \quad sCl(\lambda \wedge \mu) \leq sCl\mu \\ \Rightarrow sCl(\lambda \wedge \mu) &\leq sCl\lambda \wedge sCl\mu. \end{aligned} \quad (30)$$

(5)–(8) Proofs are straightforward. $\square$

The inequalities (1) and (4) of Theorem 8 are irreversible as is shown by the following example.

Example 7. Let $X = \{a, b\}$ be a set and $I = \{0, .5, 1\}$ the lattice of membership grades for fuzzy sets in X . Let τ be the fuzzy topology on X given as

$$\tau = \{\tilde{0}, \{a_{.5}, b_1\}, \{a_{.0}, b_1\}, \{a_1, b_{.5}\}, \{a_{.5}, b_{.5}\}, \{a_0, b_{.5}\}, \tilde{1}\}. \quad (31)$$

We choose fuzzy sets $\alpha = \{a_{.5}, b_0\}$, $\beta = \{a_0, b_{.5}\}$, $\gamma = \{a_{.5}, b_1\}$, and $\delta = \{a_1, b_{.5}\}$. Then, calculations give

$$\begin{aligned} sInt(\alpha \vee \beta) &= \{a_{.5}, b_{.5}\} \not\leq \{a_0, b_{.5}\} = sInt\alpha \vee sInt\beta, \\ sCl\gamma \wedge sCl\delta &= \{a_1, b_{.5}\} \not\leq \{a_{.5}, b_{.5}\} = sCl(\alpha \wedge \delta). \end{aligned} \quad (32)$$

Definition 16 (see [25]). Let λ be a fuzzy set in an fts X . Then, the fuzzy semiboundary of λ is defined as $sBd\lambda = sCl\lambda \wedge sCl\lambda^c$. Obviously, $sBd\lambda$ is a fuzzy semiclosed set.

Remark 5. In fuzzy topology, we have $\lambda \vee sBd\lambda \leq sCl\lambda$, for an arbitrary fuzzy set λ in X , the equality does not hold as the following example shows.

Example 8. Let $X = \{a, b\}$ be a set with fuzzy topology $\tau = \{\tilde{0}, \{a_{.4}, b_{.8}\}, \{a_{.6}, b_{.9}\}, \{a_{.5}, b_{.7}\}, \{a_{.5}, b_{.2}\}, \{a_{.8}, b_{.7}\}, \{a_{.3}, b_{.2}\}, \{a_{.4}, b_{.7}\}, \{a_{.4}, b_{.2}\}, \{a_{.6}, b_{.7}\}, \{a_{.5}, b_{.8}\}, \{a_{.8}, b_{.8}\}, \{a_{.6}, b_{.8}\}, \{a_{.8}, b_{.9}\}, \tilde{1}\}$. Choose $\lambda = \{a_{.4}, b_{.7}\}$, then calculations give

$$sCl\lambda = \{a_{.5}, b_{.8}\} \neq \{a_{.5}, b_{.7}\} = \lambda \vee sBd\lambda. \quad (33)$$

In the following theorem, (1)–(5) are analogs of Proposition 1 and hence we omit their proofs.

Proposition 5. For a fuzzy set λ in an fts X , the following conditions hold.

- (1) $sBd\lambda = sBd\lambda^c$.
- (2) If λ is fuzzy semiclosed, then $sBd\lambda \leq \lambda$.
- (3) If λ is fuzzy semiopen, then $sBd\lambda \leq \lambda^c$.
- (4) Let $\lambda \leq \mu$ and $\mu \in FSC(X)$ (resp., $\mu \in FSO(X)$). Then, $sBd\lambda \leq \mu$ (resp., $sBd\lambda \leq \mu^c$), where $FSC(X)$ (resp., $FSO(X)$) denotes the class of fuzzy semiclosed (resp., fuzzy semiopen) sets in X .
- (5) $(sBd\lambda)^c = sInt\lambda \vee sInt\lambda^c$.
- (6) $sBd\lambda \leq Bd\lambda$.
- (7) $sCl sBd\lambda \leq Bd\lambda$.

Proof. (6) Since $sCl\lambda \leq Cl\lambda$ and $sCl\lambda^c \leq Cl\lambda^c$, then we have

$$sBd\lambda = sCl\lambda \wedge sCl\lambda^c \leq Cl\lambda \wedge Cl\lambda^c = Bd\lambda. \quad (34)$$

(7)

$$\begin{aligned} sCl sBd\lambda &= sCl(sCl\lambda \wedge sCl\lambda^c) \leq sCl sCl\lambda \wedge sCl sCl\lambda^c \\ &= sCl\lambda \wedge sCl\lambda^c = sBd\lambda \leq Bd\lambda. \end{aligned} \quad (35)$$

$\square$

The converse of (2) and (3) and reverse inequalities of (6) and (7) in Proposition 5 are, in general, not true as is shown by the following example.

Example 9. Choose $\lambda = \{a_{.7}, b_{.7}\}$, $\psi = \{a_{.3}, b_{.3}\}$, $\omega = \{a_{.7}, b_{.9}\}$, $\xi = \{a_{.6}, b_1\}$ in the fts (X, τ) defined in Example 8. Then, calculations give

$$\begin{aligned} sBd\lambda &= \{a_{.4}, b_{.3}\} \leq \lambda, \quad \text{but } \lambda \text{ is not fuzzy semiclosed,} \\ sBd\psi &= \{a_{.4}, b_{.3}\} \leq \psi^c, \quad \text{but } \psi \text{ is not fuzzy semiopen,} \\ Bd\omega &= \{a_{.4}, b_{.1}\} \not\leq \{a_{.3}, b_{.1}\} = sBd\omega, \\ Bd\xi &= \{a_{.4}, b_{.1}\} \not\leq \{a_{.4}, b_0\} = sCl sBd\xi. \end{aligned} \quad (36)$$

The following is analog of Proposition 2 and hence we omit its proof.

Proposition 6. Let λ be a fuzzy set in an fts X . Then, one has

- (1) $sBd\lambda = sCl\lambda - sInt\lambda$,
- (2) $sBd sInt\lambda \leq sBd\lambda$,
- (3) $sBd sCl\lambda \leq sBd\lambda$,
- (4) $sInt\lambda \leq \lambda - sBd\lambda$.

To show that the inequalities (2), (3), and (4) of Proposition 6 are, in general, irreversible, we have the following example.

Example 10. Choose $\alpha = \{a_{.7}, b_{.5}\}$ and $\mu = \{a_{.6}, b_{.3}\}$ in the fts X defined in Example 8. Then, calculations give

$$\begin{aligned} sBd\alpha &= \{a_{.5}, b_{.8}\} \not\leq \{a_{.5}, b_{.2}\} = sBd sInt\alpha, \\ sBd\alpha &= \{a_{.5}, b_{.8}\} \not\leq \{a_{.4}, b_{.2}\} = sBd sCl\alpha, \\ \mu - sBd\mu &= \{a_{.5}, b_{.3}\} \not\leq \{a_{.5}, b_{.2}\} = sInt\mu. \end{aligned} \quad (37)$$

Remark 6. In general topology, the following conditions hold:

$$\begin{aligned} \text{sBd } A \cap \text{sInt } A &= \phi, \\ \text{sInt } A \cup \text{sBd } A &= \text{Cl } A, \\ \text{sInt } A \cup \text{sInt } A^c \cup \text{sBd } A &= X, \end{aligned} \quad (38)$$

whereas, in fuzzy topology, we give counter-examples to show that these may not hold in general.

Example 11. In the fts X of Example 8, we choose fuzzy set $\lambda = \{a_7, b_5\}$, then calculations give

$$\begin{aligned} \text{sBd } \lambda \wedge \text{sInt } \lambda &= \{a_5, b_8\} \wedge \{a_5, b_2\} \neq \tilde{0}, \\ \text{sInt } \lambda \vee \text{sBd } \lambda &= \{a_5, b_8\} \neq \{a_7, b_8\} = \text{sCl } \lambda, \\ \text{sInt } \lambda \vee \text{sInt } \lambda^c \vee \text{sBd } \lambda &= \{a_5, b_8\} \neq \tilde{1}. \end{aligned} \quad (39)$$

It is easily seen that $\text{sInt } \lambda \vee \text{sBd } \lambda \leq \text{sCl } \lambda$.

Theorem 9. Let λ and μ be fuzzy sets in an fts X . Then, $\text{sBd}(\lambda \vee \mu) \leq \text{sBd } \lambda \vee \text{sBd } \mu$.

The reverse inequality in Theorem 9 is, in general, not true as is shown by the following example.

Example 12. In Example 8, choose fuzzy sets $\alpha = \{a_4, b_7\}$ and $\beta = \{a_6, b_2\}$. Then, calculations give

$$\text{sBd } \alpha \vee \text{sBd } \beta = \{a_5, b_3\} \not\leq \{a_4, b_3\} = \text{sBd}(\alpha \vee \beta). \quad (40)$$

The following example shows that

$$\text{sBd}(\gamma \wedge \delta) \not\leq \text{sBd } \gamma \wedge \text{sBd } \delta, \quad \text{sBd } \lambda \wedge \text{sBd } \delta \not\leq \text{sBd}(\gamma \wedge \delta). \quad (41)$$

For this choose $\gamma = \{a_4, b_8\}$ and $\delta = \{a_6, b_3\}$. Then, calculations give

$$\begin{aligned} \text{sBd}(\gamma \wedge \delta) &= \{a_4, b_3\} \not\leq \{a_5, b_2\} = \text{sBd } \gamma \wedge \text{sBd } \delta, \\ \text{sBd } \lambda \wedge \text{sBd } \delta &= \{a_5, b_2\} \not\leq \{a_4, b_3\} = \text{sBd}(\gamma \wedge \delta), \end{aligned} \quad (42)$$

However, we have the following theorem which is an analog of Theorem 3.

Theorem 10. For any fuzzy sets λ and μ in an fts X , one has

$$\text{sBd}(\lambda \wedge \mu) \leq (\text{sBd } \lambda \wedge \text{sCl } \mu) \vee (\text{sBd } \mu \wedge \text{sCl } \lambda). \quad (43)$$

Corollary 2. For any fuzzy sets λ and μ in an fts X , one has

$$\text{sBd}(\lambda \wedge \mu) \leq \text{sBd } \lambda \vee \text{sBd } \mu. \quad (44)$$

Example 13. To show that the reverse inequality in Theorem 10 is, in general, not true, choose fuzzy sets γ and δ as given in Example 8. Then, calculations give

$$\begin{aligned} (\text{sBd } \gamma \wedge \text{sCl } \delta) \vee (\text{sBd } \gamma \wedge \text{sCl } \delta) &= \{a_5, b_3\} \not\leq \{a_4, b_3\} \\ &= \text{sBd}(\gamma \wedge \delta). \end{aligned} \quad (45)$$

The analog of Proposition 3 is the following theorem, the proof of which is similar.

Proposition 7. For any fuzzy set λ in an fts X , one has

- (1) $\text{sBd } \text{sBd } \lambda \leq \text{sBd } \lambda$,
- (2) $\text{sBd } \text{sBd } \text{sBd } \lambda \leq \text{sBd } \text{sBd } \lambda$.

Remark 7. As in the case of Proposition 3(2), we also do not know whether the equality in Proposition 7(2) holds or not. However, the reverse inequality of (1) is, in general, not true as is shown by the following example.

Example 14. Choose $\lambda = \{a_6, b_4\}$ in fts X of Example 8. Then,

$$\text{sBd } \lambda = \{a_5, b_8\} \not\leq \{a_5, b_2\} = \text{sBd } \text{sBd } \lambda. \quad (46)$$

Lemma 3 (see [13]). If λ is a fuzzy set of X and μ is a fuzzy set of Y , then

$$\tilde{1} - (\lambda \times \mu) = \lambda^c \times \tilde{1} \vee \tilde{1} \times \mu^c. \quad (47)$$

Using Lemma 3, we have the following one.

Lemma 4. Let λ be a fuzzy semiclosed (resp., fuzzy semiopen [13]) set of an fts X and μ a fuzzy semiclosed (resp., fuzzy semiopen) set of an fts Y . Then, $\lambda \times \mu$ is a fuzzy closed (resp., fuzzy semiopen) set of the fuzzy product space $X \times Y$.

Using Lemma 4, we have the following theorem.

Theorem 11. If λ is a fuzzy set of fts X and μ of fts Y , then

- (1) $\text{sCl } \lambda \times \text{sCl } \mu \geq \text{sCl}(\lambda \times \mu)$,
- (2) $\text{sInt } \lambda \times \text{sInt } \mu \leq \text{sInt}(\lambda \times \mu)$.

Moreover, we have the following one.

Theorem 12. Let X and Y be product-related fts's. Then, for a fuzzy set λ of X and a fuzzy set μ of Y , one has

- (1) $\text{sCl}(\lambda \times \mu) = \text{sCl } \lambda \times \text{sCl } \mu$,
- (2) $\text{sInt}(\lambda \times \mu) = \text{sInt } \lambda \times \text{sInt } \mu$.

Proof. (1) For fuzzy sets λ_i 's of X and μ_j 's of Y , we first note that

- (i) $\inf \{\lambda_i, \mu_j\} = \min\{\inf \lambda_i, \inf \mu_j\}$,
- (ii) $\inf \{\lambda_i \times \tilde{1}\} = (\inf \lambda_i) \times \tilde{1}$,
- (iii) $\inf \{\tilde{1} \times \mu_j\} = \tilde{1} \times (\inf \mu_j)$.

In view of Theorem 11, it is sufficient to show that $\text{sCl}(\lambda \times \mu) \geq \text{sCl } \lambda \times \text{sCl } \mu$. Let $\lambda_i \in \text{FSO}(X)$ and $\mu_j \in \text{FSO}(Y)$. Then,

$$\begin{aligned} \text{sCl}(\lambda \times \mu) &= \inf \{(\lambda_i \times \mu_j)^c \mid (\lambda_i \times \mu_j)^c \geq \lambda \times \mu\} \\ &= \inf \{\lambda_i^c \times \tilde{1} \vee \tilde{1} \times \mu_j^c \mid \lambda_i^c \times \tilde{1} \vee \tilde{1} \times \mu_j^c \geq \lambda \times \mu\} \\ &= \inf \{\lambda_i^c \times \tilde{1} \vee \tilde{1} \times \mu_j^c \mid \lambda_i^c \geq \lambda \text{ or } \mu_j^c \geq \mu\} \\ &= \min \left(\inf \{\lambda_i^c \times \tilde{1} \vee \tilde{1} \times \mu_j^c \mid \lambda_i^c \geq \lambda\}, \right. \\ &\quad \left. \inf \{\lambda_i^c \times \tilde{1} \vee \tilde{1} \times \mu_j^c \mid \mu_j^c \geq \mu\} \right). \end{aligned} \quad (48)$$

Since

$$\begin{aligned}
 \inf \{ \lambda_i^c \times \tilde{I} \vee \tilde{I} \times \mu_j^c \mid \lambda_i^c \geq \lambda \} &\geq \inf \{ \lambda_i^c \times \tilde{I} \mid \lambda_i^c \geq \lambda \} \\
 &= \inf \{ \lambda_i^c \mid \lambda_i^c \geq \lambda \} \times \tilde{I} \\
 &= (sCl\lambda) \times \tilde{I}, \\
 \inf \{ \lambda_i^c \times \tilde{I} \vee \tilde{I} \times \mu_j^c \mid \mu_j^c \geq \mu \} &\geq \inf \{ \mu_j^c \times \tilde{I} \mid \mu_j^c \geq \mu \} \\
 &= \tilde{I} \times \inf \{ \mu_j^c \mid \mu_j^c \geq \mu \} \\
 &= \tilde{I} \times (sCl\mu),
 \end{aligned} \tag{49}$$

we have

$$sCl(\lambda \times \mu) \geq \min(sCl\lambda \times \tilde{I}, \tilde{I} \times sCl\mu) = sCl\lambda \times sCl\mu. \tag{50}$$

(2) This follows from (1) using the facts that $(sInt\psi)^c = sCl\psi^c$ and $(sCl\psi)^c = sInt\psi^c$. $\square$

The analog of Theorem 5 is the following, the proof of which is similar.

Theorem 13. Let X_i , $i = 1, 2, \dots, n$, be a family of product-related fuzzy topological spaces. If each λ_i is a fuzzy set in X_i , then

$$\begin{aligned}
 sBd \prod_{i=1}^n \lambda_i &= [sBd\lambda_1 \times sCl\lambda_2 \times \dots \times sCl\lambda_n] \\
 &\vee [sCl\lambda_1 \times sBd\lambda_2 \times sCl\lambda_3 \times \dots \times sCl\lambda_n] \\
 &\vee \dots \vee [sCl\lambda_1 \times sCl\lambda_2 \times \dots \times sBd\lambda_n].
 \end{aligned} \tag{51}$$

The following theorem gives a necessary condition for fuzzy semicontinuous functions in terms of fuzzy boundary and fuzzy semiboundary.

Theorem 14. Let $f : X \rightarrow Y$ be a fuzzy semicontinuous function. Then, one has

$$sBd f^{-1}(\mu) \leq f^{-1}(Bd\mu), \tag{52}$$

for any fuzzy set μ in Y .

Proof. Let f be fuzzy semicontinuous and μ a fuzzy set in Y . Then, $Cl\mu$ is fuzzy closed in Y , which implies that $f^{-1}(Cl\mu)$ is fuzzy semiclosed in X . Therefore,

$$\begin{aligned}
 sBd f^{-1}(\mu) &= sCl f^{-1}(\mu) \wedge sCl (f^{-1}(\mu))^c \\
 &\leq sCl f^{-1}(Cl\mu) \wedge sCl f^{-1}(Cl\mu^c) \\
 &= f^{-1}(Cl\mu) \wedge f^{-1}(Cl\mu^c) \\
 &= f^{-1}(Cl\mu \wedge Cl\mu^c) \\
 &= f^{-1}(Bd\mu).
 \end{aligned} \tag{53}$$

Hence, $sBd f^{-1}(\mu) \leq f^{-1}(Bd\mu)$. $\square$

Definition 17 (see [28]). A function $f : X \rightarrow Y$ is said to be fuzzy irresolute if $f^{-1}(\beta)$ is fuzzy semiopen in X , for each fuzzy semiopen set β in Y .

The following theorem gives a necessary condition of fuzzy irresolute functions in terms of fuzzy boundary

and fuzzy semiboundary, the proof of which is similar to Theorem 14.

Theorem 15. Let $f : X \rightarrow Y$ be a fuzzy irresolute function. Then, one has

$$sBd f^{-1}(\mu) \leq f^{-1}(sBd\mu), \tag{54}$$

for any fuzzy set μ in Y .

Definition 18. A fuzzy set λ in an fts X is called a fuzzy semi-Q-neighborhood of a fuzzy point e , if there exists a fuzzy semiopen set μ in X , such that $eq\mu \leq \lambda$.

Theorem 16. A fuzzy point $e = x_\alpha \in sCl\lambda$ if each semi-Q-neighborhood of e is quasicoincident with λ .

Proof. $x_\alpha \in sCl\lambda$ if and only if for every fuzzy closed set $\psi \geq \lambda$, $x_\alpha \in \psi$. This gives $\psi(x_\alpha) \geq \lambda(x_\alpha)$. Equivalently, $x_\alpha \in sCl\lambda$ if and only if for every fuzzy semiopen set $\beta \leq \lambda^c$, $\beta(x_\alpha) \leq \lambda^c(x_\alpha)$. That is, for every fuzzy open set β satisfying $\beta(x_\alpha) \geq \lambda^c$, β is not contained in λ^c , or $\beta q \lambda^{cc} = \lambda$. Thus, $x_\alpha \in sCl\lambda$ if every fuzzy open Q-neighborhood β of x_α is quasicoincident with λ . $\square$

Definition 19. A fuzzy point e is called semiadherent point of a fuzzy set λ if every semi-Q-neighborhood of e is quasicoincident with λ .

Definition 20. A fuzzy point e is called a semiaccumulation point of a fuzzy set λ if e is a semiadherent point of λ and every semi-Q-neighborhood of e and λ is quasicoincident at some point different from $\text{supp}(e)$, whenever $e \in \lambda$. The union of all the semiaccumulation points of λ is called the fuzzy semiderived set of λ , denoted as λ^{sd} . It is evident that $\lambda^{sd} \leq sCl\lambda$.

Proposition 8. Let λ be a fuzzy set in X , then $sCl\lambda = \lambda \vee \lambda^{sd}$.

Proof. Let $\Omega = \{e \mid e \text{ is a semiadherent point of } \lambda\}$. Then, from Theorem 16, $sCl\lambda = \vee \Omega$. On the other hand, $e \in \Omega$ is either $e \in \lambda$ or $e \notin \lambda$; for the latter case, by Definition 20, $e \in \lambda^{sd}$, hence $sCl\lambda = \vee \Omega \leq \lambda \vee \lambda^{sd}$. The reverse inclusion is obvious. $\square$

Corollary 3. For any fuzzy set λ in an fts X , λ is fuzzy semiclosed if and only if $\lambda^{sd} \leq \lambda$.

Definition 21 (see [13]). Let $f : X \rightarrow Y$ be a function from an fts X to another fts Y . Then, f is said to be fuzzy semicontinuous function if $f^{-1}(\lambda)$ is fuzzy semiopen in X , for each fuzzy open set λ in Y .

We use Corollary 3 and characterize fuzzy semicontinuous functions in terms of fuzzy semiderived set as follows.

Theorem 17. Let $f : X \rightarrow Y$ be a function. Then, the following conditions are equivalent:

- (1) f is fuzzy semicontinuous,
- (2) $f(\lambda^{sd}) \leq Cl f(\lambda)$, for any fuzzy set λ in X .

Proof. (1) $\Rightarrow$ (2) Let f be fuzzy semicontinuous and λ a fuzzy set in X . Since $\text{Cl } f(\lambda)$ is fuzzy closed in Y , $f^{-1}(\text{Cl } f(\lambda))$ is fuzzy semiclosed in X such that $\lambda \leq f^{-1}(\text{Cl } f(\lambda))$. $\lambda \leq f^{-1}(s\text{Cl } f(\lambda))$ gives $s\text{Cl } \lambda \leq s\text{Cl } f^{-1}(\text{Cl } f(\lambda)) = f^{-1}(\text{Cl } f(\lambda))$. Therefore, $f(\lambda^{\text{sd}}) \leq f(s\text{Cl } \lambda) \leq f f^{-1}(\text{Cl } f(\lambda)) \leq \text{Cl } f(\lambda)$. Consequently, $f(\lambda^{\text{sd}}) \leq \text{Cl } f(\lambda)$.

(2) $\Rightarrow$ (1) Suppose $f(\lambda^{\text{sd}}) \leq \text{Cl } f(\lambda)$. Let μ be any fuzzy closed set in Y . We show that $f^{-1}(\mu)$ is fuzzy semiclosed in X . By our hypothesis, $f([f^{-1}(\mu)]^{\text{sd}}) \leq \text{Cl } f(f^{-1}(\mu)) \leq \text{Cl } \mu = \mu$, $f([f^{-1}(\mu)]^{\text{sd}}) \leq \mu$ gives $[f^{-1}(\mu)]^{\text{sd}} \leq f^{-1}(f[f^{-1}(\mu)]^{\text{sd}}) \leq f^{-1}(\mu)$, or $[f^{-1}(\mu)]^{\text{sd}} \leq f^{-1}(\mu)$ implies that $f^{-1}(\mu)$ is fuzzy semiclosed in X . Thus, f is fuzzy semicontinuous. $\square$

Finally, we characterize fuzzy irresolute functions via fuzzy semiderived set as follows.

Theorem 18. *Let $f : X \rightarrow Y$ be a function. Then, the following conditions are equivalent:*

- (1) f is fuzzy irresolute,
- (2) $f(\lambda^{\text{sd}}) \leq s\text{Cl } f(\lambda)$, for any fuzzy set λ in X .

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