Research Article

Relative Smooth Topological Spaces

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In 1992, Ramadan introduced the concept of a smooth topological space and relativeness between smooth topological space and fuzzy topological space in Chang's (1968) view points. In this paper we give a new definition of smooth topological space. This definition can be considered as a generalization of the smooth topological space which was given by Ramadan. Some general properties such as relative smooth continuity and relative smooth compactness are studied.

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1. Introduction

Let X be a nonempty set and let $\mathcal{L}$, $\mathcal{L}'$ be two lattice which will be copies of $[0, 1]$ or $\{0, 1\}$. The family of all fuzzy sets on X will be denoted by $\mathcal{L}^X$ Zadeh [1].

In the consideration of the nature an observer can be modeled by an operator which evaluates each proposition by a number in the closed interval $[0, 1]$; see Anvari and Molaei [2] and Molaei [3]. We assume that μ as a function from X to $\mathcal{L}$ is an observer of X on lattice $\mathcal{L}$ and denote $[\mu] = \{\lambda \in \mathcal{L}^X : \lambda \subseteq \mu\}$, where $\lambda \subseteq \mu$ implies that $\lambda(x) \leq \mu(x)$ for all $x \in X$.

Definition 1.1. Let $\mu \in \mathcal{L}^X$. A relative smooth topological space or μ -smooth topological space or μ -STS for short is a triple $(X, \mu, \mathcal{T}_\mu)$, where $\mathcal{T}_\mu : [\mu] \rightarrow \mathcal{L}'$ is a mapping satisfying the following properties:

- (i) $\mathcal{T}_\mu(\mu) = \mathcal{T}_\mu(\chi_\phi) = 1$, where χ is the characteristic function;
- (ii) if $\lambda_1, \lambda_2 \in [\mu]$, then $\mathcal{T}_\mu(\lambda_1 \cap \lambda_2) \geq \mathcal{T}_\mu(\lambda_1) \wedge \mathcal{T}_\mu(\lambda_2)$, where $\wedge$ is the minimum operator in $\mathcal{L}'$;
- (iii) $\mathcal{T}_\mu(\cup\{\lambda_i : i \in I\}) \geq \wedge\{\mathcal{T}_\mu(\lambda_i) : i \in I\}$.

We call $\mathcal{T}_\mu$ a smooth topology from view point of μ or a μ -smooth topology or a fuzzy family of μ -open sets on X .

Remark 1.2. If $\mu = \chi_X$ then the χ_X -STS $(X, \chi_X, \mathcal{T}_{\chi_X})$ coincides with the smooth topological space (X, τ) defined by Ramadan [4], and if we take $\mathcal{L} = [0, 1]$, $\mathcal{L}' = \{0, 1\}$, and

$\mu = \chi_X$ then the χ_X -STS coincides with the known definition of fuzzy topological space $(\bar{X}, \tau)$ defined by Chang [5]. If $\mathcal{L} = \mathcal{L}' = \{0, 1\}$, and $\mu = \chi_X$ then $\mathcal{T}_{\chi_X}$ is a classical topology.

Definition 1.3. Let $\mu \in \mathcal{L}^X$. A μ -smooth cotopological space is a triple $(X, \mu, \mathcal{F}_\mu)$, where $\mathcal{F}_\mu : [\mu] \rightarrow \mathcal{L}'$ is a mapping satisfying the following properties:

- (i) $\mathcal{F}_\mu(\mu) = \mathcal{F}_\mu(\chi_\phi) = 1$;
- (ii) if $\eta_1, \eta_2 \in [\mu]$, then $\mathcal{F}_\mu(\eta_1 \cup \eta_2) \geq \mathcal{F}_\mu(\eta_1) \wedge \mathcal{F}_\mu(\eta_2)$;
- (iii) $\mathcal{F}_\mu(\cap\{\eta_i : i \in I\}) \geq \wedge\{\mathcal{F}_\mu(\eta_i) : i \in I\}$.

We call $\mathcal{F}_\mu$ a μ -smooth co-topology or a fuzzy family of μ -closed sets on X .

Theorem 1.4. Let $(X, \mu, \mathcal{T}_\mu)$ be a μ -STS and $\mathcal{F}_\mu : [\mu] \rightarrow \mathcal{L}'$ be a mapping defined by $\mathcal{F}_\mu(\eta) = \mathcal{T}_\mu(\eta')$, where $\eta' = \mu - \eta$. Then $\mathcal{F}_\mu$ is a fuzzy family of μ -closed sets.

Proof.

- (i) It is clear.
- (ii) It flows from

$$\begin{aligned} (\eta_1 \cup \eta_2)' &= \mu - (\eta_1 \cup \eta_2) = \mu - \sup\{\eta_1, \eta_2\} \\ &= \inf\{\mu - \eta_1, \mu - \eta_2\} = \eta_1' \cap \eta_2'. \end{aligned} \quad (1)$$

$$\text{So, } \mathcal{F}_\mu(\eta_1 \cup \eta_2) = \mathcal{T}_\mu((\eta_1 \cup \eta_2)') = \mathcal{T}_\mu(\eta_1' \cap \eta_2').$$

(iii) It flows from

$$\begin{aligned} (\cap_{i \in I} \eta_i)' &= \mu - \cap_{i \in I} \eta_i = \mu - \inf\{\eta_i : i \in I\} \\ &= \sup\{\mu - \eta_i : i \in I\} = \cup_{i \in I} \eta_i'. \end{aligned} \quad (2)$$

So, $\mathcal{F}_\mu(\cap\{\eta_i : i \in I\}) = \mathcal{T}_\mu(\cup\{\eta_i' : i \in I\})$. $\square$

Theorem 1.5. Let $\mathcal{F}_\mu$ be a fuzzy family of μ -closed sets and define $\mathcal{T}_{\mathcal{F}_\mu} : [\mu] \rightarrow \mathcal{L}'$ by $\mathcal{T}_{\mathcal{F}_\mu}(\eta) = \mathcal{F}_\mu(\eta')$. Then $\mathcal{T}_{\mathcal{F}_\mu}$ is a μ -STS on X .

Proof. The proof is similar to the previous theorem. $\square$

Corollary 1.6. Let $\mathcal{T}_\mu$ be a μ -STS and $\mathcal{F}_\mu$ a fuzzy family of μ -closed sets. Then $\mathcal{T}_{\mathcal{F}_\mu} = \mathcal{T}_\mu$ and $\mathcal{F}_{\mathcal{T}_{\mathcal{F}_\mu}} = \mathcal{F}_\mu$.

Proof. Suppose $\lambda, \eta \in [\mu]$ then we have $\mathcal{T}_{\mathcal{F}_\mu}(\lambda) = \mathcal{F}_{\mathcal{T}_\mu}(\lambda') = \mathcal{T}_\mu(\lambda)$ and $\mathcal{F}_{\mathcal{T}_{\mathcal{F}_\mu}}(\eta) = \mathcal{T}_{\mathcal{F}_\mu}(\eta') = \mathcal{F}_\mu(\eta)$. $\square$

Example 1.7. Let X be the set of all differentiable real-valued functions on $(1, \infty)$, with positive derivative of order one and let $\mathcal{L}$ be the set of real-valued functions defined on $(1, \infty)$. Let $\mu : X \rightarrow \mathcal{L}$ be defined by $\mu(f) = f' + \text{Exp}$, where Exp is the exponential function. For nonnegative integer n define $\lambda^n : X \rightarrow \mathcal{L}$ by

$$(\lambda^n(f))(x) = f'(x) + \sum_{i=1}^n \frac{x^{i-1}}{(i-1)!}. \quad (3)$$

If we take $\mathcal{L}' = [0, 1]$ and define $\mathcal{T}_\mu : [\mu] \rightarrow [0, 1]$ by $\mathcal{T}_\mu(\chi_\emptyset) = \mathcal{T}_\mu(\mu) = 1$; $\mathcal{T}_\mu(\lambda^n) = 1 - 1/n$ for $n = 1, 2, \dots$. Then $(X, \mu, \mathcal{T}_\mu)$ is a μ -STS. Since $\lambda^n \cap \lambda^m = \lambda^m$, $\lambda^n \cup \lambda^m = \lambda^n$, where $n > m$ and $\cup \lambda^{n_i} = \mu$ whenever n_i tends to $+\infty$, so

$$\mathcal{T}_\mu(\lambda^n \cap \lambda^m) = \mathcal{T}_\mu(\lambda^m) \geq \mathcal{T}_\mu(\lambda^n) \wedge \mathcal{T}_\mu(\lambda^m), \quad (4)$$

and for $I \subseteq \mathbb{N}$ we find

$$\mathcal{T}_\mu(\cup\{\lambda^n : n \in I\}) \geq \wedge\{\mathcal{T}_\mu(\lambda^n) : n \in I\}. \quad (5)$$

Definition 1.8. Let $\mathcal{T}_\mu^1$ and $\mathcal{T}_\mu^2$ be two μ -smooth topological spaces on X . We say that $\mathcal{T}_\mu^1$ is finer than $\mathcal{T}_\mu^2$ or $\mathcal{T}_\mu^2$ is coarser than $\mathcal{T}_\mu^1$ and denoted by $\mathcal{T}_\mu^1 > \mathcal{T}_\mu^2$ if $\mathcal{T}_\mu^1(\lambda) \geq \mathcal{T}_\mu^2(\lambda)$ for every $\lambda \in [\mu]$.

Theorem 1.9. Let $\{\mathcal{T}_\mu^i : i \in I\}$ be a family of μ -STS on X . Then $\mathcal{T}_\mu = \wedge_{i \in I} \mathcal{T}_\mu^i$ is also μ -STS on X , where

$$(\wedge_{i \in I} \mathcal{T}_\mu^i)(\lambda) = \wedge_{i \in I} \mathcal{T}_\mu^i(\lambda). \quad (6)$$

Proof.

(i) It is clear.

(ii) For every $\lambda, \eta \in [\mu]$,

$$\begin{aligned} \mathcal{T}_\mu(\lambda \cap \eta) &= \wedge_{i \in I} \mathcal{T}_\mu^i(\lambda \cap \eta) \geq \wedge_{i \in I} (\mathcal{T}_\mu^i(\lambda) \wedge \mathcal{T}_\mu^i(\eta)) \\ &= \wedge_{i \in I} (\mathcal{T}_\mu^i(\lambda)) \wedge (\wedge_{i \in I} \mathcal{T}_\mu^i(\eta)) \\ &= \mathcal{T}_\mu(\lambda) \wedge \mathcal{T}_\mu(\eta). \end{aligned} \quad (7)$$

(iii) For Γ ,

$$\begin{aligned} \mathcal{T}_\mu(\cup\{\lambda_j : j \in \Gamma\}) &= \wedge_{i \in I} \mathcal{T}_\mu^i(\cup\{\lambda_j : j \in \Gamma\}) \\ &\geq \wedge_{i \in I} \wedge\{\mathcal{T}_\mu^i(\lambda_j) : j \in \Gamma\} \\ &= \wedge(\wedge_{i \in I} \{\mathcal{T}_\mu^i(\lambda_j) : j \in \Gamma\}) \\ &= \wedge\{\mathcal{T}_\mu(\lambda_j) : j \in \Gamma\}. \end{aligned} \quad (8)$$

$\square$

Let A be a subset of X and $\lambda \in [\mu]$. The restriction of λ on A is denoted by $\lambda|_A$.

Theorem 1.10. Let $(X, \mu, \mathcal{T}_\mu)$ be a μ -STS and $A \subset X$. Define a mapping $\mathcal{T}_{\mu|_A} : [\mu] \rightarrow \mathcal{L}'$ by $\mathcal{T}_{\mu|_A}(\lambda) = \vee\{\mathcal{T}_\mu(\eta) : \eta \in [\mu], \eta|_A = \lambda\}$. Then $\mathcal{T}_{\mu|_A}$ is a μ -STS on A .

Proof.

(i) It is clear that $\mathcal{T}_{\mu|_A}(\chi_\emptyset) = \mathcal{T}_{\mu|_A}(\mu) = 1$.

(ii) $\forall \lambda_1, \lambda_2 \in \mathcal{L}^A, \lambda_1, \lambda_2 \subset \mu$.

$$\begin{aligned} \mathcal{T}_{\mu|_A}(\lambda_1) \wedge \mathcal{T}_{\mu|_A}(\lambda_2) &= \vee\{\mathcal{T}_\mu(\eta_1) : \eta_1 \in [\mu], \eta_1|_A = \lambda_1\} \\ &\quad \wedge \vee\{\mathcal{T}_\mu(\eta_2) : \eta_2 \in [\mu], \eta_2|_A = \lambda_2\} \\ &= \vee\left\{\mathcal{T}_\mu(\eta_1) \wedge \mathcal{T}_\mu(\eta_2) : \eta_1, \eta_2 \in [\mu], \eta_1 \cap \eta_2|_A \right. \\ &\quad \left. = \lambda_1 \cap \lambda_2\right\} \\ &\leq \vee\{\mathcal{T}_\mu(\eta_1 \cap \eta_2) : \eta_1, \eta_2 \in [\mu], \eta_1 \cap \eta_2|_A = \lambda_1 \cap \lambda_2\} \\ &= \mathcal{T}_{\mu|_A}(\lambda_1 \cap \lambda_2). \end{aligned} \quad (9)$$

(iii) $\forall i \in I, \mathcal{T}_{\mu|_A}\{\lambda_i : i \in I\} = \vee\{\mathcal{T}_\mu(\eta_i) : \eta_i \in [\mu], \eta_i|_A = \lambda_i\}$. So

$$\begin{aligned} \mathcal{T}_{\mu|_A}(\cup\{\lambda_i : i \in I\}) &= \vee\{\mathcal{T}_\mu(\cup\{\eta_i : i \in I\}) : \eta_i \in [\mu], \cup\eta_i|_A = \cup\lambda_i\} \\ &\leq \vee\left\{\wedge\{\mathcal{T}_\mu(\eta_i) : i \in I\} : \eta_i \in [\mu], \cup\eta_i|_A = \cup\lambda_i\right\} \\ &= \wedge\mathcal{T}_{\mu|_A}\{\lambda_i : i \in I\}. \end{aligned} \quad (10)$$

$\square$

Definition 1.11. The μ -STS $(A, \mu, \mathcal{T}_{\mu|_A})$ is called a subspace of $(X, \mu, \mathcal{T}_\mu)$ and $\mathcal{T}_{\mu|_A}$ is called the induced μ -STS on A from $\mathcal{T}_\mu$.

Theorem 1.12. Let $(A, \mu, \mathcal{T}_{\mu|_A})$ be a μ -smooth subspace of $(X, \mu, \mathcal{T}_\mu)$ and $\lambda \in \mathcal{L}^A, \lambda \in [\mu]$. Then

- (a) $\mathcal{F}_{\mathcal{T}_{\mu|_A}}(\lambda) = \vee\{\mathcal{F}_{\mathcal{T}_\mu}(\eta) : \eta \in [\mu], \eta|_A = \lambda\}$,
- (b) if $B \subset A \subset X$, then $\mathcal{T}_{\mu|_B} = (\mathcal{T}_{\mu|_A})_{\mu|_B}$.

Proof.

(a) we have

$$\begin{aligned}
 \mathcal{F}_{\mathcal{T}_{\mu|_A}}(\lambda) &= \mathcal{T}_{\mu|_A}(\lambda') \\
 &= \vee \{ \mathcal{T}_{\mu}(\eta) : \eta \in [\mu], \eta|_A = \lambda' \} \\
 &= \vee \{ \mathcal{T}_{\mu}(\eta) : \eta' \in [\mu], \eta'|_A = \lambda \} \\
 &= \vee \{ \mathcal{F}_{\mathcal{T}_{\mu}}(\eta') : \eta' \in [\mu], \eta'|_A = \lambda \} \\
 &= \vee \{ \mathcal{F}_{\mathcal{T}_{\mu}}(\xi) : \xi \in [\mu], \xi|_A = \lambda \}.
 \end{aligned} \tag{11}$$

(b) we have

$$\begin{aligned}
 \mathcal{T}_{\mu|_B}(\lambda) &= \vee \{ \mathcal{T}_{\mu}(\eta) : \eta \in [\mu], \eta|_B = \lambda \} \\
 &= \vee \left\{ \vee \{ \mathcal{T}_{\mu}(\eta) : \eta \in [\mu], \eta|_A = \xi \} : \xi \in [\mu]|_A, \right. \\
 &\quad \left. \xi|_B = \lambda \right\} \\
 &= \vee \{ \mathcal{T}_{\mu|_A}(\xi) : \xi \in [\mu]|_A, \xi|_B = \lambda \} \\
 &= (\mathcal{T}_{\mu|_A})_{\mu|_B}.
 \end{aligned} \tag{12}$$

□

2. Relative Smooth Continuous Maps

The concept of continuity has been studied by Chang, Ramadan [4, 5] but here we shall study this concept from a different point of view.

Definition 2.1. Let $h : \mathcal{L} \rightarrow \mathcal{L}'$ be a linear isomorphism of vector lattices (or an order preserving one-to-one mapping when $\mathcal{L}$ and $\mathcal{L}'$ are copies of $[0, 1]$) and $(X, \mu, \mathcal{T}_{\mu}), (Y, \gamma, \mathcal{T}_{\gamma})$ μ -STS and γ -STS, respectively. A function $f : X \rightarrow Y$ is called (h, μ, γ) -smooth fuzzy continuous if $\mathcal{T}_{\mu}(f^{-1}(\eta)) \geq \mathcal{T}_{\gamma}(\eta)$ for all $\eta \in \mathcal{T}_{\gamma}$, where $f^{-1}(\eta)(x) = (h \circ \eta)(f(x)) \cap \mu(x)$ for all $x \in X$. $f^{-1}(\gamma) \cap \mu$ is called the inverse image of γ relative to μ .

Remark 2.2. When $\mathcal{L} = \mathcal{L}' = [0, 1]$ and $\mu = \chi_X$ then the $(X, \chi_X, \tau_{\chi_X}), \chi_X$ -RST coincides with the fuzzy topological space $(\bar{X}, \tau)$ defined by Chang [5–7].

Theorem 2.3. Let $\mathcal{L} = \mathcal{L}' = [0, 1]$ and $f : X \rightarrow Y$ be (I, χ_X, χ_Y) -fuzzy continuous, where $I : \mathcal{L} \rightarrow \mathcal{L}'$ is the identity function. Then f is continuous in Chang's view.

Proof. In Remark 2.2 we considered $(X, \chi_X, \mathcal{T}_{\chi_X})$ and $(Y, \chi_Y, \mathcal{T}_{\chi_Y})$ as fuzzy topological spaces. Now let γ be an open set of smooth topology $\mathcal{T}_{\chi_Y}$. Then

$$\begin{aligned}
 \mathcal{T}_{\chi_X}(f^{-1}(\gamma)) &= \mathcal{T}_{\chi_X}((I^{-1} \circ \gamma \circ f) \cap \chi_X) \\
 &= \mathcal{T}_{\chi_X}(\gamma \circ f) \geq \mathcal{T}_{\chi_X}(\gamma).
 \end{aligned} \tag{13}$$

So f is a fuzzy continuous function. □

Theorem 2.4. Let $\mu \in \mathcal{L}^X, \gamma \in \mathcal{L}'^Y$ where $\mathcal{L}$ and $\mathcal{L}'$ are copies of $[0, 1]$ and $f : X \rightarrow Y$ a (h, μ, γ) -fuzzy continuous functions where $f^{-1}(\gamma) = \mu$. Then for every γ -closed fuzzy set $\eta, f^{-1}(\eta)$ is a μ -closed fuzzy set.

Proof. Let η be γ -closed set. Then η' is a γ -open set and we have

$$\begin{aligned}
 (f^{-1}(\eta))' &= \mu - \inf\{h^{-1} \circ \eta \circ f, \mu\} \\
 &= \sup\{\mu - h^{-1} \circ \eta \circ f, 0\} \\
 &= \mu - h^{-1} \circ \eta \circ f = \mu - f^{-1}(\eta).
 \end{aligned} \tag{14}$$

Hence

$$\begin{aligned}
 \mathcal{T}_{\mu}(f^{-1}(\eta')) &= \mathcal{T}_{\mu}(f^{-1}(\gamma - \eta)) \\
 &= \mathcal{T}_{\mu}(h^{-1} \circ (\gamma - \eta) \circ f \cap \mu) \\
 &= \mathcal{T}_{\mu}((h^{-1} \circ \gamma \circ f - h^{-1} \circ \eta \circ f) \cap \mu) \\
 &= \mathcal{T}_{\mu}(f^{-1}(\gamma) - f^{-1}(\eta)) \\
 &= \mathcal{T}_{\mu}(\mu - f^{-1}(\eta)) \\
 &= \mathcal{F}_{\mathcal{T}_{\mu}}(f^{-1}(\eta)).
 \end{aligned} \tag{15}$$

So $f^{-1}(\eta)$ is a μ -closed fuzzy set. □

Theorem 2.5. Let $(X_i, \mu_i, \mathcal{T}_{\mu_i})$ be relative smooth topological spaces for $i = 1, 2, 3$. If $f : X_1 \rightarrow X_2$ and $g : X_2 \rightarrow X_3$ are relative smooth continuous maps and $\mu_1 = f^{-1}(\mu_2)$ then so is $g \circ f$.

Proof. Using the relative smooth continuity of g and f it follows that

$$\begin{aligned}
 \mathcal{T}_{\mu_1}((g \circ f)^{-1}(\eta)) &= \mathcal{T}_{\mu_1}(f^{-1}(g^{-1}(\eta))) \\
 &\geq \mathcal{T}_{\mu_2}(g^{-1}(\eta)) \geq \mathcal{T}_{\mu_3}(\eta).
 \end{aligned} \tag{16}$$

Since for every $x \in X$,

$$\begin{aligned}
 (g \circ f)^{-1}(\eta)(x) &= \eta(g \circ f)(x) \cap \mu_1(x) \\
 &= \eta(g \circ f)(x) \cap \mu_1(x) \cap \mu_2(f(x)) \\
 &= g^{-1}(\eta(f(x)) \cap \mu_1(x)) = f^{-1}(g^{-1}(\eta)(x)) \\
 &= (f^{-1} \circ g^{-1})(\eta)(x).
 \end{aligned} \tag{17}$$

□

Theorem 2.6. Let $(X, \mu, \mathcal{T}_{\mu})$ and $(Y, \nu, \mathcal{T}_{\nu})$ be two relative smooth topological spaces, $f : X \rightarrow Y$ a relative smooth continuous map, $A \subset X$, and $f^{-1}(\nu) = \mu$. Then the $f|_A : (A, \mu|_A, \mathcal{T}_{\mu|_A}) \rightarrow (Y, \nu, \mathcal{T}_{\nu})$ is also relative smooth continuous.

Proof. For each $\eta \in [\nu]$,

$$\begin{aligned} \mathcal{T}_{\mu|_A}((f|_A)^{-1}(\eta)) &= \vee \{ \mathcal{T}_\mu(\lambda) : \lambda \in [\mu], \lambda|_A = (f|_A)^{-1}(\eta) \} \\ &\geq \mathcal{T}_\mu(f^{-1}(\eta)) \\ &\geq \mathcal{T}_\nu(\eta). \end{aligned} \quad (18)$$

3. The Representation of a Relative Smooth Topology

Now we study the representation of a relative smooth topology $\mathcal{T}_\mu$.

Let $(X, \mu, \mathcal{T}_\mu)$ be a μ -STS, $\alpha \in \mathcal{L}'$. Then we define

$$\mathcal{T}_\mu^\alpha = \{ \lambda \in [\mu] : \mathcal{T}_\mu(\lambda) \geq \alpha \}. \quad (19)$$

Theorem 3.1. Let $(X, \mu, \mathcal{T}_\mu)$ be a μ -STS. Then for every $\alpha > 0$, $\mathcal{T}_\mu^\alpha$ is a relative topological space. Moreover $\alpha_1 \leq \alpha_2$ implies $\mathcal{T}_\mu^{\alpha_1} \geq \mathcal{T}_\mu^{\alpha_2}$.

Proof. It is clear that $\chi_\emptyset, \mu \in \mathcal{T}_\mu^\alpha$. When $\lambda, \eta \in \mathcal{T}_\mu^\alpha$, we have

$$\mathcal{T}_\mu(\lambda) \geq \alpha, \quad \mathcal{T}_\mu(\eta) \geq \alpha, \quad (20)$$

and so

$$\mathcal{T}_\mu(\lambda \cap \eta) \geq \mathcal{T}_\mu(\lambda) \wedge \mathcal{T}_\mu(\eta) \geq \alpha. \quad (21)$$

This implies that $\lambda \cap \eta \in \mathcal{T}_\mu^\alpha$. When $\lambda_j \in \mathcal{T}_\mu^\alpha$ for each $j \in I$ we have

$$\mathcal{T}_\mu(\cup \lambda_j) \geq \wedge \mathcal{T}_\mu(\lambda_j) \geq \alpha. \quad (22)$$

Hence $\cup \lambda_j \in \mathcal{T}_\mu^\alpha$. So $\mathcal{T}_\mu^\alpha$ is a relative topology.

The second part is trivial to verify, since for $\lambda \in \mathcal{T}_\mu^{\alpha_2}$, $\mathcal{T}_\mu \geq \alpha_2 \geq \alpha_1$, $\lambda \in \mathcal{T}_\mu^{\alpha_1}$, so $\mathcal{T}_\mu^{\alpha_1} \geq \mathcal{T}_\mu^{\alpha_2}$. $\square$

Theorem 3.2. Let $\mathcal{T}_\mu^\alpha, \alpha \in (0, 1]$ be a family of μ -fuzzy topology on X such that $\alpha_1 \geq \alpha_2$ implies $\mathcal{T}_\mu^{\alpha_1} \leq \mathcal{T}_\mu^{\alpha_2}$. Let τ be the $\mathcal{L}'$ -fuzzy set built by $\tau_\mu(\lambda) = \vee \{ \alpha : \lambda \in \mathcal{T}_\mu^\alpha \}$. Then τ_μ is a μ -smooth topology.

Proof.

(a) $\tau_\mu(\chi_\emptyset) = \tau_\mu(\mu) = 1$ by the definition.

(b) For every $\lambda, \eta \in [\mu]$ and $\alpha > 0$ if $\lambda, \eta \in \mathcal{T}_\mu^\alpha$ then $\lambda \cap \eta \in \mathcal{T}_\mu^\alpha$. Therefore

$$\{ \alpha : \lambda \cap \eta \in \mathcal{T}_\mu^\alpha \} \supset \{ \alpha : \lambda \in \mathcal{T}_\mu^\alpha, \eta \in \mathcal{T}_\mu^\alpha \} \quad (23)$$

implies that

$$\tau_\mu(\lambda \cap \eta) \geq \tau_\mu(\lambda) \wedge \tau_\mu(\eta). \quad (24)$$

(c) If every $\lambda_i \in \mathcal{T}_\mu^\alpha$ then $\cup \lambda_i \in \mathcal{T}_\mu^\alpha$. Since

$$\{ \alpha : \cup \lambda_i \in \mathcal{T}_\mu^\alpha \} \supset \{ \alpha : \cup \lambda_i \in \mathcal{T}_\mu^\alpha, \forall i \in I \}, \quad (25)$$

then

$$\tau_\mu(\cup_{i \in I} \lambda_i) \geq \wedge_{i \in I} \tau_\mu(\lambda_i). \quad (26)$$

$\square$

For τ_μ being a relative $\mathcal{L}'$ -fuzzy set, with $\mathcal{L}' = [0, 1]$, we can state a representation theorem.

Theorem 3.3. Let $\mathcal{T}_\mu$ be a relative smooth topology and $\mathcal{T}_\mu^\alpha$ the α cut of $\mathcal{T}_\mu$. From the families of relative fuzzy topologies $\mathcal{T}_\mu^\alpha$ one built $\mathcal{T}_{1\mu}(\lambda) = \vee \{ \alpha : \lambda \in \mathcal{T}_\mu^\alpha \}$. Then $\mathcal{T}_{1\mu} = \mathcal{T}_\mu$.

Proof. The proof is trivial from the preceding results and the well-known fact that

$$\vee \{ \alpha : \lambda \in \mathcal{T}_\mu^\alpha \} = \vee \{ \alpha : \mathcal{T}_\mu(\lambda) \geq \alpha \} = \mathcal{T}_\mu(\lambda). \quad (27)$$

$\square$

Definition 3.4. Let τ be a Chang fuzzy topology on X . Then a μ -smooth topology $\mathcal{T}_\mu$ on X is said to be compatible with τ if $\tau = \{ \lambda \in \mathcal{L}^X : \mathcal{T}_\mu(\lambda \cap \mu) > 0 \}$.

Example 3.5. Let X be a nonempty set and $\mathcal{T}_\mu : [\mu] \rightarrow \mathcal{L}'$ be a mapping defined by $\mathcal{T}_\mu(\mu) = \mathcal{T}_\mu(\chi_\emptyset) = 1, \mathcal{T}_\mu(\lambda) = 0$ for every $\lambda \in [\mu] \setminus \{ \chi_\emptyset, \mu \}$.

It is clear that $\mathcal{T}_\mu$ is the only relative smooth topology on X compatible with the indiscrete fuzzy topology of Chang.

Example 3.6. Let X be a nonempty set and define a mapping $\mathcal{T}_\mu : [\mu] \rightarrow \mathcal{L}'$ by $\mathcal{T}_\mu(\mu) = \mathcal{T}_\mu(\chi_\emptyset) = 1, \mathcal{T}_\mu(\lambda) = \alpha$ for every $\lambda \in [\mu] \setminus \{ \chi_\emptyset, \mu \}$.

It is clear that $\mathcal{T}_\mu$ is a μ -smooth topology on X compatible with the discrete fuzzy topology of Chang.

4. Relative Smooth Compactness

Definition 4.1. Let $(X, \mu, \mathcal{T}_\mu)$ be a μ -STS. $\lambda \in [\mu], \mathcal{A}, \mathcal{B} \subset [\mu]$. $\mathcal{A}$ is called a relative cover of λ , if $\vee \mathcal{A} \geq \lambda$ particularly, $\mathcal{A}$ is called a cover $(X, \mu, \mathcal{T}_\mu)$ if $\mathcal{A}$ is a cover of μ . $\mathcal{A}$ is called a μ -open cover of λ , if $\mathcal{A}$ is a family of μ -open and $\mathcal{A}$ is a cover of λ .

For a cover $\mathcal{A}$ of λ , $\mathcal{B}$ is called a subcover of λ , if $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{B}$ is still a cover of λ .

Definition 4.2. Let $(X, \mu, \mathcal{T}_\mu)$ be a μ -STS. For every $\alpha \in [0, 1]$, a family $\mathcal{A} \subset [\mu]$ is called an α -cover, if for every $\lambda \in \mathcal{A}, \mathcal{T}_\mu(\lambda) \geq \alpha$; $\mathcal{A}$ is called a μ -open α -cover if $\mathcal{A}$ is a family of μ -open set and $\mathcal{A}$ is a α -cover; $\mathcal{A}_0 \subset [\mu]$ is called a sub- α -cover of $\mathcal{A}$ if $\mathcal{A}_0 \subset \mathcal{A}$ and $\mathcal{A}_0$ is an α -cover.

Definition 4.3. Let $\alpha \in [0, 1]$. A μ -STS $(X, \mu, \mathcal{T}_\mu)$ is called α -compact if every μ -open α -cover has a finite sub- α -cover.

Theorem 4.4. Let $f : (X, \mu, \mathcal{T}_\mu) \rightarrow (Y, \nu, \mathcal{T}_\nu)$ be an onto μ -smooth continuous mapping and $f^{-1}(\nu) = \mu$. If $(X, \mu, \mathcal{T}_\mu)$ is α -compact then so is $(Y, \nu, \mathcal{T}_\nu)$.

Proof. Let $\{\lambda_i : i \in I\}$ be a ν -open α -cover of ν . Now consider the family $\{f^{-1}(\lambda_i) : i \in I\}$, since f is μ -smooth continuous, we have

$$\begin{aligned} \lambda \in \mathcal{T}_\nu^\alpha &\iff \mathcal{T}_\nu \geq \alpha \\ &\iff \mathcal{T}_\mu(f^{-1}(\lambda)) \geq \alpha \\ &\iff f^{-1}(\lambda) \in \mathcal{T}_\mu^\alpha. \end{aligned} \quad (28)$$

It follows that $\{f^{-1}(\lambda_i) : i \in I\}$ is a μ -open α -cover of μ . Since $(X, \mu, \mathcal{T}_\mu)$ is α -compact there exists a finite subset I_0 of I such that $\{f^{-1}(\lambda_i) : i \in I_0\}$ is a μ -open α -cover of $(X, \mu, \mathcal{T}_\mu)$. Since f is onto, then $\{\lambda_i : i \in I_0\}$ is a ν -open α -cover of $(Y, \nu, \mathcal{T}_\nu)$, which concludes the proof. $\square$

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