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Research Article

Quotient of Ideals of an Intuitionistic Fuzzy Lattice

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The concept of intuitionistic fuzzy ideal of an intuitionistic fuzzy lattice is introduced, and its certain characterizations are provided. We defined the quotient (or residual) of ideals of an intuitionistic fuzzy sublattice and studied their properties.

1. Introduction

The concept of intuitionistic fuzzy sets was introduced by Atanassov [1, 2] as a generalization of that of fuzzy sets and it is a very effective tool to study the case of vagueness. Further many researches applied this notion in various branches of mathematics especially in algebra and defined intuitionistic fuzzy subgroups (IFG), intuitionistic fuzzy subrings (IFR), and intuitionistic fuzzy sublattice (IFL), and so forth. In the last five years there are so many articles appeared in this direction. Kim [3], Kim and Jun [4], Kim and Lee [5], introduced different types of IFI's in Semigroups. Torkzadeh and Zahedi [6] defined intuitionistic fuzzy commutative hyper K-ideals, Akram and Dudek [7] defined intuitionistic fuzzy Lie ideals of Lie algebras, and Hur et al. [8] introduced intuitionistic fuzzy prime ideals of a Ring.

The concept of ideal of a fuzzy subring was introduced by Mordeson and Malik in [9]. After that N Ajmal and A.S Prajapathi introduced the concept of residual of ideals of an L-Ring in [10]. Motivated by this, in this paper we first defined the intuitionistic fuzzy ideal of an IFL and certain characterizations are given. Lastly we defined quotients (residuals) of ideals of an intuitionistic fuzzy sublattice and studied their properties.

2. Preliminaries

We recall the following definitions and results which will be used in the sequel. Throughout this paper L stands for a lattice $(L, \vee, \wedge)$ with zero element "0" and unit element "1".

Definition 1 (see [1]). Let X be a nonempty set. An *intuitionistic fuzzy set* [IFS] A of X is an object of the following form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$, where $\mu_A : X \to [01]$ and $\nu_A : X \to [01]$ define the *degree of membership* and the *degree of non membership* of the element $x \in X$, respectively, and $\forall x \in X$, $0 \le \mu_A(x) + \nu_A(x) \le 1$.

The set of all IFS's on X is denoted by IFS (X).

Definition 2 (see [1]). If $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in S\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in S\}$ are any two IFS of X then

(i) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B$ and $\nu_A(x) \ge \nu_B(x)$ $\forall x \in X$;

(ii) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$;

(iii) $\bar{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle \mid x \in X\}$;

(iv) $[A] = \{\langle x, \mu_A(x), \mu_A{}^c(x)\rangle \mid x \in X\}$, where $\mu_A{}^c(x) = 1 - \mu_A(x)$;

(v) $\langle A\rangle = \{\langle x, \nu_A^c(x), \nu_A(x)\rangle \mid x \in X\}$, where $\nu_A{}^c(x) = 1 - \nu_A(x)$;

(vi) $A \cap B = \{\langle x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\nu_A(x), \nu_B(x)\}\rangle \mid x \in X\} = \{\langle x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)\rangle \mid x \in X\}$;

(vii) $A \cup B = \{\langle x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\nu_A(x), \nu_B(x)\}\rangle \mid x \in X\} = \{\langle x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)\rangle \mid x \in X\}$.

Definition 3 (see [11]). Let L be a lattice and $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in L\}$ be an IFS of L. Then A is called an *intuitionistic fuzzy sublattice* [IFL] of L if the following conditions are satisfied.

(i) $\mu_A(x \vee y) \geq \min\{\mu_A(x), \mu_A(y)\}$;

(ii) $\mu_A(x \wedge y) \geq \min\{\mu_A(x), \mu_A(y)\}$;

(iii) $\nu_A(x \vee y) \leq \max\{\nu_A(x), \nu_A(y)\}$;

(iv) $\nu_A(x \wedge y) \leq \max\ \{\nu_A(x), \nu_A(y)\}, \forall x, y \in L$.

The set of all intuitionist fuzzy sublattices (IFL's) of L is denoted as IFL (L).

Definition 4 (see [11]). An IFS A of L is called an *intuitionistic fuzzy ideal* (IFI) of L if the following conditions are satisfied.

(i) $\mu_A(x \vee y) \geq \min\{\mu_A(x), \mu_A(y)\}$;

(ii) $\mu_A(x \wedge y) \geq \max\{\mu_A(x), \mu_A(y)\}$;

(iii) $\nu_A(x \vee y) \leq \max\{\nu_A(x), \nu_A(y)\}$;

(iv) $\nu_A(x \wedge y) \leq \min\{\nu_A(x), \nu_A(y)\}, \forall x, y \in L$.

The set of all IFI's of L is denoted as IFI (L).

Definition 5 (see [12]). Let $A, B \in$ IFS (L). Then we define an IFS (L),

(i) $A + B = \{\langle z, \mu_{A+B}(z), \nu_{A+B}(z)\rangle \mid z \in L\}$, where

$$\begin{aligned} \mu_{A+B}(z) &= \sup_{z=x\vee y}\{\min\{\mu_A(x), \mu_B(y)\}\}, \\ \nu_{A+B}(z) &= \inf_{z=x\vee y}\{\max\{\nu_A(x), \nu_B(y)\}\}. \end{aligned} \tag{1}$$

(ii) $AB = \{\langle z, \mu_{AB}(z), \nu_{AB}(z)\rangle \mid z \in L\}$, where

$$\begin{aligned} \mu_{AB}(z) &= \sup_{z=x\wedge y}\{\min\{\mu_A(x), \mu_B(y)\}\} \\ \nu_{AB}(z) &= \inf_{z=x\wedge y}\{\max\{\nu_A(x), \nu_B(y)\}\}. \end{aligned} \tag{2}$$

(iii) $A \cdot B = \{\langle z, \mu_{A\cdot B}(z), \nu_{A\cdot B}(z)\rangle \mid z \in L\}$ where

$$\begin{aligned} \mu_{A\cdot B}(z) &= \sup_{z=\vee_{i=1}^{n}(x_i\wedge y_i)}(\min_i\{\min\{\mu_A(x_i), \mu_B(y_i)\}\}), \\ \nu_{A\cdot B}(z) &= \inf_{z=\vee_{i=1}^{n}(x_i\wedge y_i)}(\max_i\{\max\{\nu_A(x_i), \nu_B(y_i)\}\}) \end{aligned} \tag{3}$$

Lemma 1 (see [13]). *Let A, B, and C be IFS (L), then the following assertions hold.*

(1) $AB = BA$, $A + B = B + A$, $A \cdot B = B \cdot A$.

(2) $AB \subseteq A \cdot B$.

(3) $C(A + B) \subseteq CA + CB$.

(4) $(C + B)A \subseteq CA + BA$.

(5) $(A \cap B)C \subseteq AC \cap BC$.

(6) $A \subseteq B \Rightarrow AC \subseteq BC$ *and* $A \cdot C \subseteq B \cdot C$.

Lemma 2 (see [13]). *Let A be an IFL of L, then*

(1) $A + A = A$.

(2) $AA = A$.

3. Ideal of an Intuitionistic Fuzzy Lattice

In this section we define the ideal of an IFL, and give some characterization of these ideals in terms of operations on IFS (L). We also used $\vee$ and $\wedge$ to represent maximum and minimum, respectively, which is clear from the context.

Definition 6. Let A be an IFL of L and B an IFS of L with $B \subseteq A$. Then B is called an *intuitionistic fuzzy ideal* (IFI) *of A* if the following conditions are satisfied.

(i) $\mu_B(x \vee y) \geq \mu_B(x) \wedge \mu_B(y)$.

(ii) $\mu_B(x \wedge y) \geq [\mu_A(x) \wedge \mu_B(y)] \vee [\mu_B(x) \wedge \mu_A(y)]$.

(iii) $\nu_B(x \vee y) \leq \nu_B(x) \vee \nu_B(y)$.

(iv) $\nu_B(x\wedge y) \leq [\nu_A(x)\vee\nu_B(y)]\wedge[\nu_B(x)\vee\nu_A(y)]\ \forall x, y \in L$.

If B IFI of A, then we write $B \lhd A$

Example 1. Consider the lattice $L = \{1, 2, 5, 10\}$ under divisibility.

Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in L\}$ be an IFL of L defined by $\langle 1,.5,.1\rangle$, $\langle 2,.4,.5\rangle$, $\langle 5,.4,.3\rangle$, $\langle 10,.7,.3\rangle$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in L\}$ be an IFS of L given by $\langle 1,.5,.3\rangle$, $\langle 2,.4,.5\rangle$, $\langle 5,.3,.4\rangle$, $\langle 10,.3,.4\rangle$. Clearly $B \lhd A$.

Definition 7. Let A be an IFL and B is also an IFL with $B \subseteq A$. Then B is called an *intuitionistic fuzzy sublattice of A*.

Lemma 3. *The intersection of two IFI's of A is again an IFI of A.*

Proof. Let B, C be IFI's of A. Then we can prove that $B \cap C$ is also an IFI of A. Since $B \subseteq A$ and $C \subseteq A$, we have $B \cap C \subseteq A$ Also

$$\begin{aligned} \mu_{B\cap C}(x \vee y) &= \min\{\mu_B(x \vee y), \mu_C(x \vee y)\} \\ &\geq \min\{\mu_B(x) \wedge \mu_B(y), \mu_C(x) \wedge \mu_C(y)\}, \\ &\qquad \text{since } B \text{ and } C \text{ are IFI's of } A. \\ &\geq \min\{\mu_B(x) \wedge \mu_C(x), \mu_B(y) \wedge \mu_C(y)\} \\ &\geq \min\{\mu_{B\cap C}(x), \mu_{B\cap C}(y)\} \\ &\geq \mu_{B\cap C}(x) \wedge \mu_{B\cap C}(y), \\ \mu_{B\cap C}(x \wedge y) &= \min\{\mu_B(x \wedge y), \mu_C(x \wedge y)\} \\ &\geq \min\{[\mu_B(x) \wedge \mu_A(y)] \vee [\mu_B(y) \wedge \mu_A(x)], \\ &\qquad [\mu_C(x) \wedge \mu_A(y)] \vee [\mu_A(x) \wedge \mu_C(y)]\} \\ &\qquad \text{since } B \text{ and } C \text{ are IFI's of } A. \\ &\geq \min\{\mu_B(x), \mu_C(x)\} \wedge \mu_A(y) \\ &\quad \vee \min\{\mu_B(y), \mu_C(y)\} \wedge \mu_A(x) \\ &\geq [\mu_{B\cap C}(x) \wedge \mu_A(y)] \vee [\mu_{B\cap C}(y) \wedge \mu_A(x)]. \end{aligned} \tag{4}$$

Also

$$\begin{aligned}
\nu_{B\cap C}(x\vee y) &= \max\{\nu_B(x\vee y),\nu_C(x\vee y)\}\\
&\le \max\{\nu_B(x)\vee\nu_B(y),\nu_C(x)\vee\nu_C(y)\},\\
&\qquad \text{since } B \text{ and } C \text{ are IFI's of } A.\\
&\le \max\{\nu_B(x)\vee\nu_C(x),\nu_B(y)\vee\nu_C(y)\}\\
&\le \max\{\nu_{B\cap C}(x),\nu_{B\cap C}(y)\}\\
&\le \nu_{B\cap C}(x)\vee\nu_{B\cap C}(y),\\
\underset{B\cap C}{\nu}(x\wedge y) &= \max\{\nu_B(x\wedge y),\nu_C(x\wedge y)\}\\
&\le \max\{[\nu_B(x)\vee\nu_A(y)]\wedge[\nu_B(y)\vee\nu_A(x)],\\
&\qquad [\nu_C(x)\vee\nu_A(y)]\wedge[\nu_A(x)\vee\nu_C(y)]\},\\
&\qquad \text{since } B \text{ and } C \text{ are IFI's of } A.\\
&\le \max\{\nu_B(x),\nu_C(x)\}\vee\nu_A(y)\\
&\quad \wedge\max\{\nu_B(y),\nu_C(y)\}\vee\nu_A(x)\\
&\le [\nu_{B\cap C}(x)\vee\nu_A(y)]\wedge[\nu_{B\cap C}(y)\vee\nu_A(x)].
\end{aligned}\tag{5}$$

Hence, $B\cap C$ is an IFI of A. □

Theorem 1. *Let A an IFL and B an IFS of L with $B\subseteq A$. Then B is an IFI of A if and only if*

(1) $\mu_B(x\vee y)\ge\mu_B(x)\wedge\mu_B(y)$,

(2) $\nu_B(x\vee y)\le\nu_B(x)\vee\nu_B(y)$,

(3) $AB\subseteq B$.

Proof. Suppose that conditions (1), (2), and (3) hold. Then we prove that B is an IFI of A.

We have

$$\begin{aligned}
\mu_B(x\wedge y) &\ge \mu_{AB}(x\wedge y),\quad \text{since } B\supseteq AB\\
&= \sup_{x\wedge y=x_i\wedge y_i}(\mu_A(x_i)\wedge\mu_B(y_i))\\
&\ge \mu_A(x)\wedge\mu_B(y).
\end{aligned}\tag{6}$$

Similarly

$$\mu_B(x\wedge y)\ge\mu_B(x)\wedge\mu_A(y),\quad \text{since } B\supseteq AB=BA.\tag{7}$$

Hence

$$\mu_B(x\wedge y)\ge[\mu_A(x)\wedge\mu_B(y)]\vee[\mu_B(x)\wedge\mu_A(y)].\tag{a}$$

Also

$$\begin{aligned}
\nu_B(x\wedge y) &\le \nu_{AB}(x\wedge y),\quad \text{since } B\supseteq AB\\
&= \inf_{x\wedge y=x_i\wedge y_i}(\nu_A(x_i)\vee\nu_B(y_i))\\
&\le \nu_A(x)\vee\nu_B(y).
\end{aligned}\tag{8}$$

Similarly

$$\nu_B(x\wedge y)\le\nu_B(x)\vee\nu_A(y),\quad \text{since } B\supseteq AB=BA.\tag{9}$$

Hence

$$\nu_B(x\wedge y)\le[\nu_A(x)\vee\nu_B(y)]\wedge[\nu_B(x)\vee\nu_A(y)].\tag{b}$$

So from (1), (2), (a), and (b) B is an IFI of A.

Conversely suppose B is an IFI of A. Then obviously conditions (1) and (2) holds. Also we have

$$\begin{aligned}
&\mu_B(x\wedge y)\ge\mu_A(x)\wedge\mu_B(y),\\
&\nu_B(x\wedge y)\le\nu_A(x)\vee\nu_B(y),\quad \forall x,y\in L.
\end{aligned}\tag{10}$$

So $\forall z\in L$ with $z=x\wedge y$

$$\begin{aligned}
&\mu_B(z)\ge\bigvee_{z=x\wedge y}[\mu_A(x)\wedge\mu_B(y)]=\mu_{AB}(z),\\
&\nu_B(z)\le\bigwedge_{z=x\wedge y}[\nu_A(x)\vee\nu_B(y)]=\nu_{AB}(z).
\end{aligned}\tag{11}$$

Hence $AB\subseteq B$. □

Theorem 2. *Let A be an IFL of L and B an IFS with $B\subseteq A$. Then B is an IFI of A if and only if*

(1) $\mu_B(x\vee y)\ge\mu_B(x)\wedge\mu_B(y)$,

(2) $\nu_B(x\vee y)\le\nu_B(x)\vee\nu_B(y)$,

(3) $A\cdot B\subseteq B$.

Proof. Suppose conditions (1), (2), and (3) holds. We prove B is an IFI of A.

We have

$$\begin{aligned}
\mu_B(x\wedge y) &\ge \mu_{A\cdot B}(x\wedge y),\quad \text{since } B\supseteq A\cdot B\\
&= \sup_{x\wedge y=\vee_{i=1}^{n}(x_i\wedge y_i)}\left[\bigwedge_{i=1}^{n}(\mu_A(x_i)\wedge\mu_B(y_i))\right]\\
&\ge \mu_A(x)\wedge\mu_B(y).
\end{aligned}\tag{12}$$

Similarly, we can obtain

$$\mu_B(x\wedge y)\ge\mu_B(x)\wedge\mu_A(y),\quad \text{since } B\supseteq A\cdot B=B\cdot A.\tag{13}$$

Hence

$$\mu_B(x\wedge y)\ge[\mu_A(x)\wedge\mu_B(y)]\vee[\mu_B(x)\wedge\mu_A(y)].\tag{a.1}$$

Also

$$\begin{aligned}
\nu_B(x\wedge y) &\le \nu_{A\cdot B}(x\wedge y),\quad \text{since } B\supseteq A\cdot B\\
&= \inf_{x\wedge y=\vee_{i=1}^{n}(x_i\wedge y_i)}\left[\bigvee_{i=1}^{n}(\nu_A(x_i)\vee\nu_B(y_i))\right]\\
&\le \nu_A(x)\vee\nu_B(y).
\end{aligned}\tag{14}$$

Similarly

$$\nu_B(x \wedge y) \leq \nu_B(x) \vee \nu_A(y), \quad \text{since } B \supseteq A \cdot B = B \cdot A. \tag{15}$$

Hence

$$\nu_B(x \wedge y) \leq [\nu_A(x) \vee \nu_B(y)] \wedge [\nu_B(x) \vee \nu_A(y)]. \tag{b.1}$$

So from (1), (2), (a.1), and (b.1) B is an IFI of A.

Conversely suppose that B is an IFI of A. Then obviously conditions (1) and (2) hold.

Let $z \in L$ and $z = \bigvee_{i=1}^{n}(x_i \wedge y_i)$, where $x_i \in A$, $y_i \in B$.

We have

$$\begin{aligned}\mu_B(z) = \mu_B\left[\bigvee_{i=1}^{n}(x_i \wedge y_i)\right] &\geq \bigwedge_{i=1}^{n}\mu_B(x_i \wedge y_i)\\ &\geq \bigwedge_{i=1}^{n}\{\mu_A(x_i) \wedge \mu_B(y_i)\}, \quad \text{since } B \text{ IFI of } A.\end{aligned} \tag{16}$$

Thus

$$\mu_B(z) \geq \bigvee\left[\bigwedge_{i=1}^{n}\{\mu_A(x_i) \wedge \mu_B(y_i)\}\right] = \mu_{A\cdot B}(z). \tag{17}$$

Also

$$\begin{aligned}\nu_B(z) = \nu_B\left[\bigvee_{i=1}^{n}(x_i \wedge y_i)\right] &\leq \bigvee_{i=1}^{n}\nu_B(x_i \wedge y_i)\\ &\leq \bigvee_{i=1}^{n}\{\nu_A(x_i) \vee \nu_B(y_i)\}, \quad \text{since } B \text{ IFI of } A.\end{aligned} \tag{18}$$

Thus

$$\nu_B(z) \leq \wedge\left[\bigvee_{i=1}^{n}\{\nu_A(x_i) \vee \nu_B(y_i)\}\right] = \nu_{A\cdot B}(z). \tag{19}$$

Hence $A \cdot B \subseteq B$. □

Theorem 3. *Let A be an IFL of L and B, C are IFI's of A. Then $B + C$ is an IFI of A.*

Proof. We have

$$\begin{aligned}\mu_{B+C}(x \vee y) &\geq \mu_{B+C}(x) \wedge \mu_{B+C}(y),\\ \nu_{B+C}(x \vee y) &\leq \nu_{B+C}(x) \vee \nu_{B+C}(y)\end{aligned} \tag{20}$$

(by [11, Theorem 5.2]).

And

$$\begin{aligned}A(B + C) &\subseteq AB + AC \subseteq B + C,\\ (B + C)\,A &\subseteq BA + CA \subseteq B + C.\end{aligned} \tag{21}$$

(by Lemma 1 and Theorem 1).

Hence $B + C$ is an IFI of A. □

4. Quotient of Ideals

Here first we define the residual of ideals of an IFL and prove that the residual of ideals is again an IFI of the IFL. Moreover we establish that it is the largest ideal with respect to some property on the operation $\cdot$.

Definition 8. Let A be an IFL of L and B, C be IFI's of A. Then the *quotient (residual) of* B by C denoted as B/C is defined by

$$B/C = \cup\{D/D \lhd A,\ DC \subseteq B\}. \tag{22}$$

Theorem 4. *Let A be an IFL of L and B,C are IFI's of A. Then the quotient B/C is an IFI of A. Also $B \subseteq B/C \subseteq A$.*

Proof. Let $\eta = \{D/D \lhd A,\ DC \subseteq B\}$. Suppose $D, D' \in \eta$. Then D and D' are IFI's of A such that $DC \subseteq B$ and $D'C \subseteq B$. Then by Theorem 3 $D + D'$ is an IFI of A. So by Lemmas 1 and 2 $(D + D')C \subseteq DC + D'C \subseteq B + B = B$. Thus $D + D' \in \eta$. Now

$$\begin{aligned}\mu_{B/C}(x) \wedge \mu_{B/C}(y) &= \left[\bigvee_{D\in\eta}\mu_D(x)\right] \wedge \left[\bigvee_{D'\in\eta}\mu_{D'}(y)\right]\\ &= \bigvee\{\mu_D(x) \wedge \mu_{D'}(y)/D,\ D' \in \eta\}\\ &\leq \bigvee\{\mu_{D+D'}(x \vee y)/D,\ D' \in \eta\}\\ &\leq \mu_{B/C}(x \vee y), \quad \text{since } \mathrm{D} + D' \in \eta.\end{aligned} \tag{23}$$

That is,

$$\mu_{B/C}(x \vee y) \geq \mu_{B/C}(x) \wedge \mu_{B/C}(y). \tag{24}$$

Also

$$\begin{aligned}\mu_{B/C}(x \wedge y) &= \bigvee_{D\in\eta}\mu_D(x \wedge y)\\ &\geq \bigvee_{D\in\eta}\{\mu_D(x) \wedge \mu_A(y)\}, \quad \text{since } D \lhd A\\ &= \left[\bigvee_{D\in\eta}\mu_D(x)\right] \wedge \mu_A(y)\\ &= \mu_{B/C}(x) \wedge \mu_A(y).\end{aligned} \tag{25}$$

Similarly

$$\mu_{B/C}(x \wedge y) \geq \mu_{B/C}(y) \wedge \mu_A(x). \tag{26}$$

Thus

$$\mu_{B/C}(x \wedge y) \geq [\mu_{B/C}(x) \wedge \mu_A(y)] \vee [\mu_{B/C}(y) \wedge \mu_A(x)]. \tag{27}$$

Now

$$\begin{aligned}\nu_{B/C}(x)\vee\nu_{B/C}(y) &= \left[\bigwedge_{D\in\eta}\nu_D(x)\right]\vee\left[\bigwedge_{D'\in\eta}\nu_{D'}(y)\right]\\ &= \bigwedge\{\nu_D(x)\vee\nu_{D'}(y)/D,\ D'\in\eta\} \qquad (28)\\ &\geq \bigwedge\{\nu_{D+D'}(x\vee y)/D,\ D'\in\eta\}\\ &\geq \nu_{B/C}(x\vee y),\quad \text{since } D+D'\in\eta.\end{aligned}$$

That is

$$\nu_{B/C}(x\vee y)\leq\nu_{B/C}(x)\vee\nu_{B/C}(y). \qquad (29)$$

Also

$$\begin{aligned}\nu_{B/C}(x\wedge y) &= \bigwedge_{D\in\eta}\nu_D(x\wedge y)\\ &\leq \bigwedge_{D\in\eta}\{\nu_D(x)\vee\nu_A(y)\},\quad \text{since } D\lhd A\\ &= \left[\bigwedge_{D\in\eta}\nu_D(x)\right]\vee\nu_A(y)\\ &= \nu_{B/C}(x)\vee\nu_A(y).\end{aligned} \qquad (30)$$

Similarly

$$\nu_{B/C}(x\wedge y)\leq\nu_{B/C}(y)\vee\nu_A(x). \qquad (31)$$

Thus

$$\nu_{B/C}(x\wedge y)\leq[\nu_{B/C}(x)\vee\nu_A(y)]\wedge[\nu_{B/C}(y)\vee\nu_A(x)]. \qquad (32)$$

From (24), (27), (29), and (32) B/C is an IFI of A.

Clearly $B/C\subseteq A$.

Since B is an IFI of A, $BA\subseteq B$ (by Theorem 1).

Since $C\subseteq A$, by Lemma 1 $BC\subseteq BA\subseteq B$. Hence $B\in\eta$. So $B\subseteq B/C$.

Thus we have

$$B\subseteq B/C\subseteq A. \qquad (33)$$

□

Theorem 5. *Let A be an IFL and B, C be IFI's of A. Then B/C is the largest IFI of A with the property $(B/C)\cdot C\subseteq B$.*

Proof. Let $\eta=\{D/D\lhd A \text{ and } DC\subseteq B\}$. We have $B/C=\bigcup_{D\in\eta}D$ Let $x\in L$ such that $x=\bigvee_{i=1}^{n}(a_i\wedge b_i)$.

Then

$$\mu_B(a_i\wedge b_i)\geq\mu_{DC}(a_i\wedge b_i)\geq\mu_D(a_i)\wedge\mu_C(b_i),\quad\forall D\in\eta. \qquad (34)$$

So

$$\begin{aligned}\mu_B(a_i\wedge b_i) &\geq \bigvee_{D\in\eta}[\mu_D(a_i)\wedge\mu_C(b_i)]\\ &= \left[\bigvee_{D\in\eta}\mu_D(a_i)\right]\wedge\mu_C(b_i)\\ &= \mu_{B/C}(a_i)\wedge\mu_C(b_i).\end{aligned} \qquad (35)$$

Hence

$$\begin{aligned}\mu_B(x) &= \mu_B\left(\bigvee_{i=1}^{n}(a_i\wedge b_i)\right)\geq\bigwedge_{i=1}^{n}\mu_B(a_i\wedge b_i),\\ &\qquad\text{since } B \text{ is an IFI of } A\\ &\geq\bigwedge_{i=1}^{n}[\mu_{B/C}(a_i)\wedge\mu_C(b_i)].\end{aligned} \qquad (36)$$

Consequently

$$\begin{aligned}\mu_B(x) &\geq \bigvee\left\{\bigwedge_{i=1}^{n}[\mu_{B/C}(a_i)\wedge\mu_C(b_i)]/x=\bigvee_{i=1}^{n}(a_i\wedge b_i)\right\}\\ &= \mu_{(B/C)\cdot C}(x).\end{aligned} \qquad (37)$$

Also

$$\nu_B(a_i\wedge b_i)\leq\nu_{DC}(a_i\wedge b_i)\leq\nu_D(a_i)\vee\nu_C(b_i),\quad\forall D\in\eta. \qquad (38)$$

So

$$\begin{aligned}\nu_B(a_i\wedge b_i) &\leq \bigwedge_{D\in\eta}[\nu_D(a_i)\vee\nu_C(b_i)]\\ &= \left[\bigwedge_{D\in\eta}\nu_D(a_i)\right]\vee\nu_C(b_i)\\ &= \nu_{B/C}(a_i)\wedge\nu_C(b_i).\end{aligned} \qquad (39)$$

Hence

$$\begin{aligned}\nu_B(x) &= \nu_B\left(\bigvee_{i=1}^{n}(a_i\wedge b_i)\right)\leq\bigvee_{i=1}^{n}\nu_B(a_i\wedge b_i),\\ &\qquad\text{since } B \text{ is an IFI of } A\\ &\leq\bigvee_{i=1}^{n}[\nu_{B/C}(a_i)\wedge\nu_C(b_i)].\end{aligned} \qquad (40)$$

Consequently

$$\begin{aligned}\nu_B(x) &\leq \bigwedge\left\{\bigvee_{i=1}^{n}[\nu_{B/C}(a_i)\wedge\nu_C(b_i)]/x=\bigvee_{i=1}^{n}(a_i\wedge b_i)\right\}\\ &= \nu_{(B/C)\cdot C}(x).\end{aligned} \qquad (41)$$

Thus from (37) and (41) $(B/C)\cdot C\subseteq B$.

If D is an ideal of A such that $D\cdot C\subseteq B$ then $DC\subseteq D\cdot C\subseteq B$. So $D\in\eta$. Hence $D\subseteq B/C$. Thus B/C is the largest IFI of A such that $(B/C)\cdot C\subseteq B$. □

Theorem 6. *Let A be an IFL and B, C, D be IFI's of A. Then the following holds.*

(1) *If $B\subseteq C$ then $B/D\subseteq C/D$ and $D/C\subseteq D/B$.*

(2) *If $B\subseteq C$ then $C/B=A$.*

(3) *$B/B=A$.*

Proof. (1) Let $B \subseteq C$. Write $\eta = \{E/E \lhd A \text{ and } ED \subseteq B\}$ and $\xi = \{E/E \lhd A \text{ and } ED \subseteq C\}$. If $E \in \eta$ then $E \lhd A$ and $ED \subseteq B \subseteq C$. Thus $E \in \xi$ and hence $\eta \subseteq \xi$. So $B/D = \bigcup_{E\in\eta} E \subseteq \bigcup_{E\in\xi} E = C/D$.

Similarly, let $\eta_1 = \{E/E \lhd A \text{ and } EC \subseteq D\}$ and $\xi_1 = \{E/E \lhd A \text{ and } EB \subseteq D\}$. If $E \in \eta_1$ then $EC \subseteq D$. But $B \subseteq C$. So $EB \subseteq EC \subseteq D$. Thus $E \in \xi_1$ and hence $\eta_1 \subseteq \xi_1$. So $D/C = \bigcup_{E\in\eta_1} E \subseteq \bigcup_{E\in\xi_1} E = D/B$.

(2) Let $\eta = \{E/E \lhd A \text{ and } EB \subseteq C\}$. Since $B \lhd A$, we have $AB \subseteq B \subseteq C$, and $A \lhd A$. Thus $A \in \eta$ and hence $A \subseteq \bigcup_{E\in\eta} E = C/B \subseteq A$, since C/B is an IFI of A. Therefore $C/B = A$.

(3) We have $B \subseteq B$. So from (2) $B/B = A$. □

Corollary 1. *Let A be an IFL of L and B, and C be IFI's of A. Then*

(1) $(B/C)/B = A$,

(2) $(B/B)/C = A$,

(3) $B/(B \cap C) = A$.

Proof. (1) Since $B \subseteq B/C$, by Theorem 6 (2), $(B/C)/B = A$.

(2) By Theorem 6 (3) $B/B = A$. Since $C \subseteq A = B/B$ by Theorem 6 (2), $(B/B)/C = A$.

(3) Since $B \lhd A$ and $C \lhd A$. So $B \cap C \lhd A$ and $B \cap C \subseteq B$. Hence by Theorem 6 (2), $B/(B \cap C) = A$. □

Theorem 7. *Let A be an IFL of L and B_i $i = 1, 2 \ldots\ldots m$, C, are IFI's of A. Then*

$$\left(\bigcap_{i=1}^{m} B_i\right)/C = \bigcap_{i=1}^{m}(B_i/C). \tag{42}$$

Proof. Since $\bigcap_{i=1}^{m} B_i \subseteq B_i$ by Theorem 6 (1) $(\bigcap_{i=1}^{m} B_i)/C \subseteq B_i/C$, $\forall i$.

Hence

$$\left(\bigcap_{i=1}^{m} B_i\right)/C \subseteq \bigcap_{i=1}^{m}(B_i/C). \tag{43}$$

Let

$$\begin{aligned} \eta_1 &= \{E/E \lhd A,\ EC \subseteq B_1\}, \\ \eta_2 &= \{E/E \lhd A,\ EC \subseteq B_2\}, \\ \eta_3 &= \{E/E \lhd \mathrm{A},\ EC \subseteq B_1 \cap B_2\}. \end{aligned} \tag{44}$$

Then $\forall x \in L$

$$\begin{aligned} \mu_{B_1/C\cap B_2/C}(x) &= \mu_{B_1/C}(x) \wedge \mu_{B_2/C}(x) \\ &= \left(\bigvee_{E\in\eta_1} \mu_E(x)\right) \wedge \left(\bigvee_{E'\in\eta_2} \mu_{E'}(x)\right) \\ &= \vee\{[\mu_E(x) \wedge \mu_{E'}(x)]/E \in \eta_1,\ E' \in \eta_2\}. \end{aligned} \tag{a.2}$$

Similarly

$$\begin{aligned} \nu_{B_1/C\cap B_2/C}(x) &= \nu_{B_1/C}(x) \vee \nu_{B_2/C}(x) \\ &= \left(\bigwedge_{E\in\eta_1} \nu_E(x)\right) \vee \left(\bigwedge_{E'\in\eta_2} \nu_{E'}(x)\right) \\ &= \wedge\{[\nu_E(x) \vee \nu_{E'}(x)]/E \in \eta_1,\ E' \in \eta_2\}. \end{aligned} \tag{b.2}$$

Now let $E \in \eta_1$ and $E' \in \eta_2$. Then $EC \subseteq B_1$ and $E'C \subseteq B_2$. Also $E \cap E'$ IFI of A.

So that

$$(E \cap E')C \subseteq EC \cap E'C \subseteq B_1 \cap B_2. \tag{45}$$

Thus $E \cap E' \in \eta_3$. So $\eta_1 \cap \eta_2 \subseteq \eta_3$.

Hence

$$(B_1 \cap B_2)/C = \bigcup_{E\in\eta_3} E \supseteq \bigcup_{E\in\eta_1, E'\in\eta_2} (E \cap E'). \tag{46}$$

So

$$\begin{aligned} \mu_{B_1\cap B_2/C}(x) &\geq \bigvee \mu_{E\cap E'}(x) \\ &= \bigvee[\mu_E(x) \wedge \mu_{E'}(x)] \\ &= \mu_{B_1/C\cap B_2/C}(x), \quad \text{from (a.2)}, \\ \nu_{B_1\cap B_2/C}(x) &\leq \bigwedge \nu_{E\cap E'}(x) \\ &= \bigwedge[\nu_E(x) \vee \nu_{E'}(x)] \\ &= \nu_{B_1/C\cap B_2/C}(x), \quad \text{from (b.2)}. \end{aligned} \tag{47}$$

Hence

$$B_1 \cap B_2/C \supseteq B_1/C \cap B_2/C. \tag{48}$$

From (43) and (48) $(B_1 \cap B_2)/C = B_1/C \cap B_2/C$. This completes the proof. □

Next, we denote the set of all IFI's $\{Bi\}$ $i = 1, 2 \ldots, m$ of an IFL A that satisfies the property $\mu_{B_i}(0) = \mu_{Bj}(0)$ and $\nu_{B_i}(0) = \nu_{Bj}(0)$ $\forall i, j$ by IFI (A^*). Then we have the following results.

Lemma 4. *Let A be an IFL of L and B, $C \in$ IFI (A^*). Then*

(1) *$B \subseteq B + C$ and $C \subseteq B + C$.*

(2) *$B/C = B/B + C$.*

(3) *$B + C/B = A$ and $B + C/B \cap C = A$.*

Proof. (1) We have

$$\begin{aligned}
\mu_{B+C}(x) &= \bigvee_{x=y\vee z} (\mu_B(y) \wedge \mu_C(z)) \\
&\geq \mu_B(x) \wedge \mu_C(0) \quad \text{as } x = x \vee 0 \\
&= \mu_B(x) \wedge \mu_B(0) \quad \text{since } \mu_B(0) = \mu_C(0) \\
&= \mu_B(x) \quad \text{since } \mu_B(0) \geq \mu_B(x), \\
\nu_{B+C}(x) &= \bigwedge_{x=y\vee z} (\nu_B(y) \vee \nu_C(z)) \\
&\leq \nu_B(x) \vee \nu_C(0) \quad \text{as } x = x \vee 0 \\
&= \nu_B(x) \vee \nu_B(0) \quad \text{since } \nu_B(0) = \nu_C(0) \\
&= \nu_B(x) \quad \text{since } \nu_B(0) \leq \nu_B(x).
\end{aligned} \tag{49}$$

So $B \subseteq B + C$. Similarly $\subseteq B + C$.

(2) We have $B + C \lhd A$ (by Theorem 3) and $C \subseteq B + C$ (by (1)). So by Theorem 6 (1),

$$B/B + C \subseteq B/C. \tag{50}$$

Write $\eta = \{E/E \lhd A \text{ and } EC \subseteq B\}$ and $\xi = \{E/E \lhd A \text{ and } E(B + C) \subseteq B\}$.

Let $E \in \eta$, then $E \lhd A$ and $E \subseteq A$. So $EB \subseteq AB$ (by Lemma 1). But $AB \subseteq B$, since $B \lhd A$. Hence $EB \subseteq B$ and also $EC \subseteq B$. So By Lemma 1, $E(B + C) \subseteq EB + EC \subseteq B + B = B$. Therefore $E \in \xi$. So $\eta \subseteq \xi$. Thus

$$B/C = \bigcup_{E\in\eta} E \subseteq \bigcup_{E\in\xi} E = B/B + C. \tag{51}$$

From (50) and (51) $B/C = B/B + C$.

(3) We have $B + C \lhd A$ and $B \subseteq B + C$. So by Theorem 6 (2), $B + C/B = A$.

Also we have $B \cap C \lhd A$ and $B \cap C \subseteq B + C$. Hence by Theorem 6 (2) $B + C/B \cap C = A$. □

Theorem 8. *Let A be an IFL of L and $\{Bi\}\, i = 1,\ 2 \ldots m \in IFI$ (A^*) and C any IFI of A. Then $C/\sum_{i=1}^{m} B_i = \bigcap_{i=1}^{m} (C/B_i)$.*

Proof. We have $B_1 + B_2 \lhd A$ and $B_1 \subseteq B_1 + B_2$, $B_2 \subseteq B_1 + B_2$.

So by Theorem 6 (1), $C/B_1 + B_2 \subseteq C/B_1$ and $C/B_1 + B_2 \subseteq C/B_2$.

Therefore

$$C/B_1 + B_2 \subseteq C/B_1 \bigcap C/B_2. \tag{52}$$

Let

$$\begin{aligned}
\eta_1 &= \{E/E \lhd A,\ EB_1 \subseteq C\}, \\
\eta_2 &= \{E/E \lhd A,\ EB_2 \subseteq C\}, \\
\eta_3 &= \{E/E \lhd A,\ E(B_1 + B_2) \subseteq C\}.
\end{aligned} \tag{53}$$

Then $\forall x \in L$

$$\begin{aligned}
\mu_{C/B_1\cap C/B_2}(x) &= \mu_{C/B_1}(x) \wedge \mu_{C/B_2}(x) \\
&= \left(\bigvee_{E\in\eta_1} \mu_E(x)\right) \wedge \left(\bigvee_{E'\in\eta_2} \mu_{E'}(x)\right) \\
&= \bigvee \{[\mu_E(x) \wedge \mu_{E'}(x)]/E \in \eta_1,\ E' \in \eta_2\}.
\end{aligned} \tag{a.3}$$

Similarly

$$\begin{aligned}
\nu_{C/B_1\cap C/B_2}(x) &= \nu_{C/B_1}(x) \vee \nu_{C/B_2}(x) \\
&= \left(\bigwedge_{E\in\eta_1} \nu_E(x)\right) \vee \left(\bigwedge_{E'\in\eta_2} \nu_{E'}(x)\right) \\
&= \wedge \{[\nu_E(x) \vee \nu_{E'}(x)]/E \in \eta_1,\ E' \in \eta_2\}.
\end{aligned} \tag{b.3}$$

Now let $E \in \eta_1$ and $E' \in \eta_2$. Then $EB_1 \subseteq C$ and $E'B_2 \subseteq C$. Also $E \cap E'$ IFI of A, so that

$$\begin{aligned}
(\mathrm{E} \cap E')(\mathrm{B}_1 + \mathrm{B}_2) \subseteq (\mathrm{E} \cap E')\mathrm{B}_1 + (\mathrm{E} \cap E')\mathrm{B}_2 \ \subseteq \mathrm{EB}_1 \\
+ E'\mathrm{B}_2 \subseteq \mathrm{C} + \mathrm{C} = \mathrm{C}.
\end{aligned} \tag{54}$$

So $E \cap E' \in \eta_3$. Hence $\eta_1 \cap \eta_2 \subseteq \eta_3$. Thus $C/B_1 + B_2 = \bigcup_{E\in\eta_3} E \supseteq \bigcup_{E\in\eta_1, E'\in\eta_2} (E \cap E')$.

So

$$\begin{aligned}
\mu_{C/B_1+B_2}(x) &\geq \vee \mu_{E\cap E'}(x) \\
&= \vee [\mu_E(x) \wedge \mu_{E'}(x)] \\
&= \mu_{C/B_1\cap C/B_2}(x) \quad \text{from (a.3)}, \\
\nu_{C/B_1+B_2}(x) &\leq \wedge \nu_{E\cap E'}(x) \\
&= \wedge [\nu_E(x) \vee \nu_{E'}(x)] \\
&= \nu_{C/B_1\cap C/B_2}(x) \quad \text{from (b.3)}.
\end{aligned} \tag{55}$$

Therefore

$$C/B_1 + B_2 \supseteq C/B_1 \bigcap C/B_2. \tag{56}$$

From (52) and (56) $C/B_1 + B_2 = C/B_1 \cap C/B2$, hence the result. □

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