Research Article

Quotient of Ideals of an Intuitionistic Fuzzy Lattice

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The concept of intuitionistic fuzzy ideal of an intuitionistic fuzzy lattice is introduced, and its certain characterizations are cited.

1. Introduction

The concept of intuitionistic fuzzy sets was introduced by Atanassov [1, 2] as a generalization of that of fuzzy sets and it is a very effective tool to study the case of vagueness. Further many researches applied this notion in various branches of mathematics especially in algebra and defined intuitionistic fuzzy subgroups (IFG), intuitionistic fuzzy subrings (IFR), and intuitionistic fuzzy sublattice (IFL), and so forth. In the last five years there are so many articles appeared in this direction. Kim [3], Kim and Jun [4], Kim and Lee [5], introduced different types of IFI’s in Semigroups. Torkzadeh and Zahedi [6] defined intuitionistic fuzzy commutative hyper K-ideals, Akram and Dudek [7] defined intuitionistic fuzzy Lie ideals of Lie algebras, and Hur et al. [8] introduced intuitionistic fuzzy prime ideals of a Ring.

The concept of ideal of a fuzzy subring was introduced by Mordeson and Malik in [9]. After that N Ajmal and A.S Prajapathi introduced the concept of residual of ideals of an L-Ring in [10]. Motivated by this, in this paper we first defined the intuitionistic fuzzy ideal of an IFL and certain characterizations are given. Lastly we defined quotients (residuals) of ideals of an intuitionistic fuzzy sublattice and studied their properties.

2. Preliminaries

We recall the following definitions and results which will be used in the sequel. Throughout this paper $L$ stands for a lattice $(L, \lor, \land)$ with zero element “0” and unit element “1”.

Definition 1 (see [1]). Let $X$ be a nonempty set. An intuitionistic fuzzy set [IFS] $A$ of $X$ is an object of the following form $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}$, where $\mu_A : X \to [01]$ and $\nu_A : X \to [01]$ define the degree of membership and the degree of non membership of the element $x \in X$, respectively, and $\forall x \in X, 0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

The set of all IFS’s on $X$ is denoted by IFS ($X$).

Definition 2 (see [1]). If $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in S \}$ and $B = \{ \langle x, \mu_B(x), \nu_B(x) \rangle \mid x \in S \}$ any two IFS of $X$ then

(i) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ $\forall x \in X$;

(ii) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$;

(iii) $A = \{ \langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X \}$;

(iv) $[A] = \{ \langle x, \mu_A(x), \mu_A^c(x) \rangle \mid x \in X \}$, where $\mu_A^c(x) = 1 - \mu_A(x)$;

(v) $\langle A \rangle = \{ \langle x, \nu_A(x), \mu_A(x) \rangle \mid x \in X \}$, where $\nu_A^c(x) = 1 - \nu_A(x)$;

(vi) $A \cap B = \{ \langle x, \min \{ \mu_A(x), \mu_B(x) \}, \max \{ \nu_A(x), \nu_B(x) \} \rangle \mid x \in X \}$;

(vii) $A \cup B = \{ \langle x, \max \{ \mu_A(x), \mu_B(x) \}, \min \{ \nu_A(x), \nu_B(x) \} \rangle \mid x \in X \}$.

Definition 3 (see [11]). Let $L$ be a lattice and $A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in L \}$ be an IFS of $L$. Then $A$ is called an intuitionistic fuzzy sublattice [IFL] of $L$ if the following conditions are satisfied.
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Definition 4 (see [11]). An IFS $A$ of $L$ is called an intuitionistic fuzzy ideal (IFI) of $L$ if the following conditions are satisfied.

(i) $\mu_A(x \vee y) \geq \min \{\mu_A(x), \mu_A(y)\}$;
(ii) $\mu_A(x \wedge y) \geq \min \{\mu_A(x), \mu_A(y)\}$;
(iii) $\nu_A(x \vee y) \leq \max \{\nu_A(x), \nu_A(y)\}$;
(iv) $\nu_A(x \wedge y) \leq \max \{\nu_A(x), \nu_A(y)\}, \forall x, y \in L$.

The set of all intuitionist fuzzy sublattices (IFLs) of $L$ is denoted as IFL ($L$).

Definition 5 (see [12]). Let $A, B \in$ IFS ($L$). Then we define an IFS ($L$),

(i) $A + B = \{\langle z, \mu_{A+B}(z), \nu_{A+B}(z) \rangle \mid z \in L\}$, where

\[
\mu_{A+B}(z) = \sup_{x \in A, y \in B} \{\mu_A(x), \mu_B(y)\},
\nu_{A+B}(z) = \inf_{x \in A, y \in B} \{\nu_A(x), \nu_B(y)\}.
\]

(ii) $AB = \{\langle z, \mu_{AB}(z), \nu_{AB}(z) \rangle \mid z \in L\}$, where

\[
\mu_{AB}(z) = \sup_{x \in A, y \in B} \{\mu_A(x), \mu_B(y)\},
\nu_{AB}(z) = \inf_{x \in A, y \in B} \{\nu_A(x), \nu_B(y)\}.
\]

(iii) $A \cdot B = \{\langle z, \mu_{A\cdot B}(z), \nu_{A\cdot B}(z) \rangle \mid z \in L\}$, where

\[
\mu_{A\cdot B}(z) = \sup_{x \in A, y \in B} \{\mu_A(x), \mu_B(y)\},
\nu_{A\cdot B}(z) = \inf_{x \in A, y \in B} \{\nu_A(x), \nu_B(y)\}.
\]

3. Ideal of an Intuitionistic Fuzzy Lattice

In this section we define the ideal of an IFL, and give some characterization of these ideals in terms of operations on IFS ($L$). We also used $\vee$ and $\wedge$ to represent maximum and minimum, respectively, which is clear from the context.

Definition 6. Let $A$ be an IFL of $L$ and $B$ an IFS of $L$ with $B \subseteq A$. Then $B$ is called an intuitionistic fuzzy ideal (IFI) of $A$ if the following conditions are satisfied.

(i) $\mu_B(x \vee y) \geq \mu_B(x) \wedge \mu_B(y)$.
(ii) $\mu_B(x \wedge y) \geq [\mu_B(x) \wedge \mu_B(y)] \vee [\mu_B(x) \wedge \mu_B(y)]$.
(iii) $\nu_B(x \vee y) \leq \nu_B(x) \vee \nu_B(y)$.
(iv) $\nu_B(x \wedge y) \leq [\nu_B(x) \vee \nu_B(y)] \wedge [\nu_B(x) \vee \nu_B(y)] \forall x, y \in L$.

If $B$ IFL of $A$, then we write $B \ll A$.

Example 1. Consider the lattice $L = \{1, 2, 5, 10\}$ under divisibility.

Let $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in L\}$ be an IFL of $L$ defined by $(1,5,1), (2,4,5), (5,4,3), (10,7,3)$ and $B = \{(x, \mu_B(x), \nu_B(x)) \mid x \in L\}$ be an IFS of $L$ given by $(1,5,3), (2,4,5), (5,3,4), (10,3,4)$. Clearly $B \ll A$.

Definition 7. Let $A$ be an IFL and $B$ is also an IFL with $B \subseteq A$. Then $B$ is called an intuitionistic fuzzy sublattice of $A$.

Lemma 1. The intersection of two IFLs of $A$ is again an IFL of $A$.

Proof. Let $B, C$ be IFLs of $A$. Then we can prove that $B \cap C$ is also an IFL of $A$. Since $B \subseteq A$ and $C \subseteq A$, we have $B \cap C \subseteq A$. Also

\[
\mu_{B \cap C}(x \vee y) = \min \{\mu_B(x \vee y), \mu_C(x \vee y)\} \geq \min \{\mu_B(x \wedge y), \mu_C(x \wedge y)\},
\]

since $B$ and $C$ are IFLs of $A$.

\[
\mu_{B \cap C}(x \wedge y) = \min \{\mu_B(x \wedge y), \mu_C(x \wedge y)\} \geq \min \{\mu_{B \cap C}(x), \mu_{B \cap C}(y)\}.
\]

Lemma 2. The intersection of two IFLs of $A$ is again an IFL of $A$.

Proof. Let $B, C$ be IFLs of $A$. Then we can prove that $B \cap C$ is also an IFL of $A$. Since $B \subseteq A$ and $C \subseteq A$, we have $B \cap C \subseteq A$. Also

\[
\mu_{B \cap C}(x \vee y) = \min \{\mu_B(x \vee y), \mu_C(x \vee y)\} \geq \min \{\mu_B(x \wedge y), \mu_C(x \wedge y)\},
\]

since $B$ and $C$ are IFLs of $A$.

\[
\mu_{B \cap C}(x \wedge y) = \min \{\mu_B(x \wedge y), \mu_C(x \wedge y)\} \geq \min \{\mu_{B \cap C}(x), \mu_{B \cap C}(y)\}.
\]
Also
\[ \nu_{B \cap C}(x \land y) = \max \{ \nu_B(x \land y), \nu_C(x \land y) \} \]
\[ \leq \max \{ \nu_B(x) \lor \nu_B(y), \nu_C(x) \lor \nu_C(y) \}, \]
since $B$ and $C$ are IFIs of $A$.
\[ \leq \max \{ \nu_{B \cap C}(x), \nu_{B \cap C}(y) \} \]
\[ \leq \nu_{B \cap C}(x) \lor \nu_{B \cap C}(y), \]
\[ \nu_{B \cap C}(x \land y) = \max \{ \nu_B(x \land y), \nu_C(x \land y) \} \]
\[ \leq \max \{ \nu_B(x) \lor \nu_B(y), \nu_C(x) \lor \nu_C(y) \}, \]
\[ \leq \max \{ \nu_{B \cap C}(x), \nu_{B \cap C}(y) \} \]
\[ \leq \nu_{B \cap C}(x) \lor \nu_{B \cap C}(y), \]
\[ \nu_B(x \land y) \leq \nu_{B \cap C}(x \land y) \]
\[ \leq \nu_B(x) \lor \nu_B(y). \]
\[ \nu_B(x \land y) \leq \nu_{B \cap C}(x \land y), \quad \text{since } B \supseteq AB = BA. \quad (5) \]

Hence, $B \cap C$ is an IFI of $A$. \qed

**Theorem 1.** Let $A$ an IFIL and $B$ an IFIS of $L$ with $B \subseteq A$. Then $B$ is an IFI of $A$ if and only if

1. $\mu_B(x \land y) \geq \mu_B(x) \land \mu_B(y)$,
2. $\nu_B(x \land y) \leq \nu_B(x) \lor \nu_B(y)$,
3. $AB \subseteq B$.

**Proof.** Suppose that conditions (1), (2), and (3) hold. Then we prove that $B$ is an IFI of $A$.

We have
\[ \mu_B(x \land y) \geq \mu_{AB}(x \land y), \quad \text{since } B \supseteq AB \]
\[ = \sup_{x \land y = x \land y} (\mu_A(x) \land \mu_B(y)) \]
\[ \geq \mu_A(x) \land \mu_B(y). \]
\[ \mu_B(x \land y) \geq \mu_B(x) \land \mu_B(y), \quad \text{since } B \supseteq AB = BA. \quad (6) \]

Hence
\[ \mu_B(x \land y) \geq [\mu_A(x) \land \mu_B(y)] \lor [\mu_B(x) \land \mu_A(y)]. \quad (a) \]

Similarly
\[ \nu_B(x \land y) \leq \nu_{AB}(x \land y), \quad \text{since } B \supseteq AB \]
\[ = \inf_{x \land y = x \land y} (\nu_A(x) \lor \nu_B(y)) \]
\[ \leq \nu_A(x) \lor \nu_B(y). \]
\[ \nu_B(x \land y) \leq \nu_{A \cdot B}(x \land y), \quad \text{since } B \supseteq AB = BA. \quad (7) \]
\[ \nu_B(x \land y) \leq [\nu_A(x) \lor \nu_B(y)] \land [\nu_B(x) \lor \nu_A(y)]. \quad (b) \]

So from (1), (2), and (b) $B$ is an IFI of $A$.

Conversely suppose $B$ is an IFI of $A$. Then obviously conditions (1) and (2) holds. Also we have
\[ \mu_B(x \land y) \geq \mu_A(x) \land \mu_B(y), \]
\[ \nu_B(x \land y) \leq \nu_A(x) \lor \nu_B(y), \quad \forall x, y \in L. \quad (10) \]

So $\forall z \in L$ with $z = x \land y$
\[ \mu_B(z) \geq \bigvee_{z = x \land y} [\mu_A(x) \land \mu_B(y)] = \mu_{AB}(z), \]
\[ \nu_B(z) \leq \bigwedge_{z = x \land y} [\nu_A(x) \lor \nu_B(y)] = \nu_{AB}(z). \quad (11) \]

Hence $AB \subseteq B$. \qed

**Theorem 2.** Let $A$ be an IFL of $L$ and $B$ an IFIS with $B \subseteq A$. Then $B$ is an IFI of $A$ if and only if

1. $\mu_B(x \land y) \geq \mu_B(x) \land \mu_B(y)$,
2. $\nu_B(x \land y) \leq \nu_B(x) \lor \nu_B(y)$,
3. $A \cdot B \subseteq B$.

**Proof.** Suppose conditions (1), (2), and (3) holds. We prove $B$ is an IFI of $A$.

We have
\[ \mu_B(x \land y) \geq \mu_{A \cdot B}(x \land y), \quad \text{since } B \supseteq A \cdot B \]
\[ = \sup_{x \land y = x \land y} \left( \bigwedge_{i=1}^{n} (\mu_A(x_i) \land \mu_B(y_i)) \right) \]
\[ \geq \mu_A(x) \land \mu_B(y). \]
\[ \mu_B(x \land y) \geq [\mu_A(x) \land \mu_B(y)] \lor [\mu_B(x) \land \mu_A(y)]. \quad (a.1) \]

Similarly, we can obtain
\[ \mu_B(x \land y) \geq \mu_B(x) \land \mu_A(y), \quad \text{since } B \supseteq A \cdot B = B \cdot A. \quad (12) \]

Hence
\[ \mu_B(x \land y) \geq [\mu_A(x) \land \mu_B(y)] \lor [\mu_B(x) \land \mu_A(y)]. \quad (a.1) \]

Also
\[ \nu_B(x \land y) \leq \nu_{A \cdot B}(x \land y), \quad \text{since } B \supseteq A \cdot B \]
\[ = \inf_{x \land y = x \land y} \left( \bigvee_{i=1}^{n} (\nu_A(x_i) \lor \nu_B(y_i)) \right) \]
\[ \leq \nu_A(x) \lor \nu_B(y). \]
\[ \nu_B(x \land y) \leq \nu_{A \cdot B}(x \land y), \quad \text{since } B \supseteq A \cdot B \]
\[ = \inf_{x \land y = x \land y} \left( \bigvee_{i=1}^{n} (\nu_A(x_i) \lor \nu_B(y_i)) \right) \]
\[ \leq \nu_A(x) \lor \nu_B(y). \]
Similarly
\[ \nu_B(x \land y) \leq \nu_B(x) \lor \nu_A(y), \quad \text{since } B \supseteq A \cdot B = B \cdot A. \] (15)

Hence
\[ \nu_B(x \land y) \leq [\nu_A(x) \lor \nu_B(y)] \land [\nu_B(x) \lor \nu_A(y)]. \] (b.1)

So from (1), (2), (a.1), and (b.1) B is an IFI of A.

Conversely suppose that B is an IFI of A. Then obviously conditions (1) and (2) hold.

Let \( z \in L \) and \( z = \bigvee_{i=1}^{n} (x_i \land y_i) \), where \( x_i \in A, y_i \in B \).

We have
\[
\mu_B(z) = \mu_B \left[ \bigvee_{i=1}^{n} (x_i \land y_i) \right] \geq \left[ \bigvee_{i=1}^{n} \mu_B(x_i \land y_i) \right] \\
\geq \bigvee_{i=1}^{n} \left[ \mu_A(x_i) \land \mu_B(y_i) \right], \quad \text{since } B \text{ IFI of } A.
\]

Thus
\[
\mu_B(z) \geq \bigvee_{i=1}^{n} \left[ \mu_A(x_i) \land \mu_B(y_i) \right] = \mu_{A \cdot B}(z). \] (17)

Also
\[
\nu_B(z) = \nu_B \left[ \bigvee_{i=1}^{n} (x_i \land y_i) \right] \leq \bigvee_{i=1}^{n} \nu_B(x_i \land y_i) \\
\leq \bigvee_{i=1}^{n} \left[ \nu_A(x_i) \lor \nu_B(y_i) \right], \quad \text{since } B \text{ IFI of } A.
\]

Thus
\[
\nu_B(z) \leq \bigvee_{i=1}^{n} \left[ \nu_A(x_i) \lor \nu_B(y_i) \right] = \nu_{A \cdot B}(z). \] (19)

Hence \( A \cdot B \subseteq B \).

\( \square \)

**Theorem 3.** Let A be an IFL of L and B, C are IFI’s of A. Then \( B + C \) is an IFI of A.

**Proof.** We have
\[
\mu_{B+C}(x \lor y) \geq \mu_{B+C}(x) \land \mu_{B+C}(y),
\]
\[
\nu_{B+C}(x \lor y) \leq \nu_{B+C}(x) \lor \nu_{B+C}(y)
\]

(by [11, Theorem 5.2]).

And
\[
A(B + C) \subseteq AB + AC \subseteq B + C,
\]
\[
(B + C) A \subseteq BA + CA \subseteq B + C.
\]

(by Lemma 1 and Theorem 1).

Hence \( B + C \) is an IFI of A.

\( \square \)

**4. Quotient of Ideals**

Here first we define the residual of ideals of an IFL and prove that the residual of ideals is again an IFI of the IFL. Moreover we establish that it is the largest ideal with respect to some property on the operation \( \cdot \).

**Definition 8.** Let A be an IFL of L and B, C be IFI’s of A. Then the quotient (residual) of B by C denoted as \( B/C \) is defined by
\[
B/C = \bigcup \{ D/D \land A, \; DC \subseteq B \}. \] (22)

**Theorem 4.** Let A be an IFL of L and B, C are IFI’s of A. Then the quotient \( B/C \) is an IFI of A. Also \( B \subseteq B/C \subseteq A \).

**Proof.** Let \( \eta = \{ D/D \land A, \; DC \subseteq B \} \). Suppose \( D, D' \in \eta \). Then \( D \) and \( D' \) are IFI’s of A such that \( DC \subseteq B \) and \( D' C \subseteq B \). Then by Theorem 3 \( D + D' \) is an IF of A. So by Lemmas 1 and 2 \((D + D') C \subseteq DC + D' C \subseteq B + B = B\). Thus \( D + D' \in \eta \). Now
\[
\mu_{B/C}(x) \land \mu_{B/C}(y) = \bigvee_{D \in \eta} \mu_{D}(x) \land \bigvee_{D' \in \eta} \mu_{D'}(y) \\
= \bigvee \{ \mu_{D}(x) \land \mu_{D'}(y)/D, \; D' \in \eta \} \\
\leq \bigvee \{ \mu_{D+D'}(x \lor y)/D, \; D' \in \eta \} \\
\leq \mu_{B/C}(x \lor y), \quad \text{since } D + D' \in \eta.
\]

That is,
\[
\mu_{B/C}(x \lor y) \geq \mu_{B/C}(x) \land \mu_{B/C}(y). \] (24)

Also
\[
\mu_{B/C}(x \land y) = \bigvee_{D \in \eta} \mu_{D}(x \land y) \\
\geq \bigvee_{D \in \eta} \{ \mu_{D}(x) \land \mu_{A}(y) \}, \quad \text{since } D \land A
\]
\[
= \bigvee_{D \in \eta} \mu_{D}(x) \land \mu_{A}(y) \\
= \mu_{B/C}(x) \land \mu_{A}(y).
\]

Similarly
\[
\mu_{B/C}(x \land y) \geq \mu_{B/C}(y) \land \mu_{A}(x). \] (26)

Thus
\[
\mu_{B/C}(x \land y) \geq [\mu_{B/C}(x) \land \mu_{A}(y)] \lor [\mu_{B/C}(y) \land \mu_{A}(x)]. \] (27)
Now
\[ \nu_{B/C}(x) \lor \nu_{B/C}(y) = \left[ \bigwedge_{D \in \eta} \nu_{D}(x) \right] \lor \left[ \bigwedge_{D' \in \eta} \nu_{D'}(y) \right] \]
\[ = \bigwedge_{D \in \eta} \left[ \nu_{D}(x) \lor \nu_{D'}(y) / D, D' \in \eta \right] \] \hspace{1cm} (28)
\[ \geq \bigwedge_{D \in \eta} \left[ \nu_{D \lor D'}(x \lor y) / D, D' \in \eta \right] \]
\[ \geq \nu_{B/C}(x \lor y), \quad \text{since } D + D' \in \eta. \]
That is
\[ \nu_{B/C}(x \lor y) \leq \nu_{B/C}(x) \lor \nu_{B/C}(y). \] \hspace{1cm} (29)

Also
\[ \nu_{B/C}(x \lor y) = \bigwedge_{D \in \eta} \nu_{D}(x \lor y) \]
\[ \leq \bigwedge_{D \in \eta} \{ \nu_{D}(x) \lor \nu_{A}(y) \}, \quad \text{since } D \leq A \] \hspace{1cm} (30)
\[ = \left[ \bigwedge_{D \in \eta} \nu(x) \right] \lor \nu_{A}(y) \]
\[ = \nu_{B/C}(x) \lor \nu_{A}(y). \]

Similarly
\[ \nu_{B/C}(x \lor y) \leq \nu_{B/C}(x) \lor \nu_{A}(x). \] \hspace{1cm} (31)

Thus
\[ \nu_{B/C}(x \lor y) \leq \nu_{B/C}(x) \lor \nu_{A}(y) \land \nu_{B/C}(y) \lor \nu_{A}(x). \] \hspace{1cm} (32)

From (24), (27), (29), and (32) B/C is an IFI of A.

Clearly B/C \leq A.

Since B is an IFI of A, BA \leq B (by Theorem 1).

Since C \leq A, by Lemma 1 BC \leq BA \leq B. Hence B \in \eta.

So B \leq B/C.

Thus we have
\[ B \leq B/C \leq A. \] \hspace{1cm} (33)

**Theorem 5.** Let A be an IFI and B, C be IFI’s of A. Then B/C is the largest IFI of A with the property (B/C) \cdot C \subseteq B.

**Proof.** Let \( \eta = \{ D/D \leq A \text{ and } DC \subseteq B \} \). We have B/C = \( \bigcup_{D \in \eta} D \). Let \( x \in L \) such that \( x = \bigvee_{i=1}^{n} (a_i \land b_i) \).

Then
\[ \mu_B(a_i \land b_i) \geq \mu_{DC}(a_i \land b_i) \geq \mu_D(a_i) \land \mu_C(b_i), \quad \forall D \in \eta. \] \hspace{1cm} (34)

So
\[ \mu_B(a_i \land b_i) \geq \bigvee_{D \in \eta} \left[ \mu_D(a_i) \land \mu_C(b_i) \right] \]
\[ = \left[ \bigvee_{D \in \eta} \mu_D(a_i) \right] \land \mu_C(b_i) \] \hspace{1cm} (35)
\[ = \nu_{B/C}(a_i) \land \mu_C(b_i). \]

Hence
\[ \mu_B(x) = \mu_B \left( \bigvee_{i=1}^{n} (a_i \land b_i) \right) \geq \bigvee_{i=1}^{n} \mu_B(a_i \land b_i), \]
since B is an IFI of A \hspace{1cm} (36)
\[ \geq \bigvee_{i=1}^{n} [\mu_{B/C}(a_i) \land \mu_C(b_i)] \]
 Consequently
\[ \mu_B(x) \geq \bigvee_{i=1}^{n} \left[ \nu_{B/C}(a_i) \lor \nu_{C}(b_i) / x = \bigvee_{i=1}^{n} (a_i \land b_i) \right] \] \hspace{1cm} (37)

Also
\[ \nu_B(a_i \land b_i) \leq \nu_{DC}(a_i \land b_i) \leq \nu_D(a_i) \lor \nu_{C}(b_i), \quad \forall D \in \eta. \] \hspace{1cm} (38)

So
\[ \nu_B(a_i \land b_i) \leq \bigwedge_{D \in \eta} \left[ \nu_D(a_i) \lor \nu_{C}(b_i) \right] \]
\[ = \left[ \bigwedge_{D \in \eta} \nu_D(a_i) \right] \lor \nu_{C}(b_i) \] \hspace{1cm} (39)
\[ = \nu_{B/C}(a_i) \land \nu_{C}(b_i). \]

Hence
\[ \nu_B(x) = \nu_B \left( \bigvee_{i=1}^{n} (a_i \land b_i) \right) \leq \bigvee_{i=1}^{n} \nu_B(a_i \land b_i), \]
since B is an IFI of A \hspace{1cm} (40)
\[ \leq \bigvee_{i=1}^{n} \left[ \nu_{B/C}(a_i) \lor \nu_{C}(b_i) \right]. \]

Consequently
\[ \nu_B(x) \leq \bigwedge_{i=1}^{n} \left[ \nu_{B/C}(a_i) \lor \nu_{C}(b_i) / x = \bigvee_{i=1}^{n} (a_i \land b_i) \right] \] \hspace{1cm} (41)

Thus from (37) and (41) (B/C) \cdot C \subseteq B.

If D is an ideal of A such that \( D \cdot C \subseteq B \) then \( DC \subseteq D \cdot C \subseteq B \). So \( D \in \eta \). Hence \( D \subseteq B/C \). Thus B/C is the largest IFI of A such that \( (B/C) \cdot C \subseteq B \).

**Theorem 6.** Let A be an IFI and B, C, D be IFI’s of A. Then the following holds.

(1) If \( B \subseteq C \) then \( B/D \subseteq C/D \) and \( D/C \subseteq D/B \).

(2) If \( B \subseteq C \) then \( C/B = A \).

(3) \( B/B = A \).
Proof. (1) Let $B \subseteq C$. Write $\eta = \{E/E \not\subset A \text{ and } ED \subseteq B\}$ and $\xi = \{E/E \not\subset A \text{ and } ED \subseteq C\}$. If $E \in \eta$ then $E \not\subset A$ and $ED \subseteq B \subseteq C$. Thus $E \in \xi$ and hence $\eta \subseteq \xi$. So $B/D = \bigcup_{E \in \eta} E \subseteq \bigcup_{E \in \xi} E = C/D.

Similarly, let $\eta_1 = \{E/E \not\subset A \text{ and } EC \subseteq D\}$ and $\xi_1 = \{E/E \not\subset A \text{ and } EB \subseteq D\}$. If $E \in \eta_1$ then $EC \subseteq D$. But $B \subseteq C$. So $EB \subseteq EC \subseteq D$. Thus $E \in \xi_1$ and hence $\eta_1 \subseteq \xi_1$. So $D/C = \bigcup_{E \in \eta_1} E \subseteq \bigcup_{E \in \xi_1} E = D/B.$

(2) Let $\eta = \{E/E \not\subset A \text{ and } EB \subseteq C\}$. Since $B \not\subset A$, we have $AB \not\subset B \subseteq C$, and $A \not\subset A$. Thus $A \in \eta$ and hence $A \subseteq \bigcup_{E \in \eta} E = C/B \subseteq A$, since $C/B$ is an IFI of $A$. Therefore $C/B = A$.

(3) We have $B \subseteq B$. So from (2) $B/B = A$. □

**Corollary 1.** Let $A$ be an IFI of $L$ and $B$, and $C$ be IFI’s of $A$. Then

1. $(B/C)/B = A$,
2. $(B/B)/C = A$,
3. $(B/B \cap C) = A$.

Proof. (1) Since $B \subseteq B/C$, by Theorem 6 (2), $(B/C)/B = A$.

(2) By Theorem 6 (3) $B/B = A$. Since $C \subseteq A = B/B$ by Theorem 6 (2), $(B/B)/C = A$.

(3) Since $B \not\subset A$ and $C \not\subset A$. So $B \cap C \not\subset A$ and $B \cap C \subseteq B$. Hence by Theorem 6 (2), $B/(B \cap C) = A$. □

**Theorem 7.** Let $A$ be an IFI of $L$ and $B_i, i = 1, 2 \ldots, m, C$, are IFI’s of $A$. Then

\[
\left(\bigcap_{i=1}^{m} B_i\right)/C = \bigcap_{i=1}^{m} (B_i/C).
\]

Proof. Since $\bigcap_{i=1}^{m} B_i \subseteq B_i$, by Theorem 6 (1) $(\bigcap_{i=1}^{m} B_i)/C \subseteq B_i/C, \forall i$.

Hence

\[
\left(\bigcap_{i=1}^{m} B_i\right)/C \subseteq \bigcap_{i=1}^{m} (B_i/C).
\]

Let

\[
\eta_1 = \{E/E \not\subset A, \ EC \subseteq B_1\},
\]

\[
\eta_2 = \{E/E \not\subset A, \ EC \subseteq B_2\},
\]

\[
\eta_3 = \{E/E \not\subset A, \ EC \subseteq B_1 \cap B_2\}.
\]

Then $\forall x \in L$

\[
\nu_{B_1/C \cap B_2/C}(x) = \nu_{B_1/C}(x) \lor \nu_{B_2/C}(x)
\]

\[
= \left(\bigvee_{E \in \eta_1} \nu_E(x)\right) \lor \left(\bigvee_{E \in \eta_2} \nu_E(x)\right)
\]

\[
= \bigvee \{\nu_E(x) \lor \nu_E'(x) \mid E \in \eta_1, \ E' \in \eta_2\}.
\]

Similarly

\[
\nu_{B_1/C \cap B_2/C}(x) = \nu_{B_1/C}(x) \lor \nu_{B_2/C}(x)
\]

\[
= \left(\bigwedge_{E \in \eta_3} \nu_E(x)\right) \lor \left(\bigwedge_{E \in \eta_3} \nu_E(x)\right)
\]

\[
= \bigwedge \{\nu_E(x) \lor \nu_E'(x) \mid E \in \eta_3, \ E' \in \eta_3\}.
\]

Thus $C \subseteq B_1 \cap B_2$. Hence

\[
\nu_{B_1/C \cap B_2/C}(x) \leq \bigwedge_{E \in \eta_3} \nu_E(x)
\]

\[
= \bigwedge \{\nu_E(x) \lor \nu_E'(x) \mid E \in \eta_3, \ E' \in \eta_3\}.
\]

Hence

\[
B_1 \cap B_2/C \geq B_1/C \cap B_2/C.
\]

From (43) and (48) $(B_1 \cap B_2)/C = B_1/C \cap B_2/C$. This completes the proof. □

Next, we denote the set of all IFI’s $\{B_i\} i = 1, 2\ldots, m$ of an IFI $A$ that satisfies the property $\mu_{B_0}(0) = \mu_{B_j}(0)$ and $\nu_{B_0}(0) = \nu_{B_j}(0) \forall i, j$ by IFI $(A^*)$. Then we have the following results.

**Lemma 4.** Let $A$ be an IFI of $L$ and $B, C \in IFI (A^*)$. Then

1. $B \subseteq B + C$ and $C \subseteq B + C$.
2. $B/C = B/B + C$.
3. $B + C/B = A$ and $B + C/B \cap C = A$. 

Theorem 8. Then \( \forall x \in L \)

\[
\mu_{C/B_1 \cap C/B_2}(x) = \mu_{C/B_1}(x) \lor \mu_{C/B_2}(x)
\]

\[
= \left( \bigvee_{E \in \eta_1} \mu_E(x) \right) \lor \left( \bigvee_{E \in \eta_2} \mu_E(x) \right)
\]

\[
= \bigvee \{\mu_E(x) \land \mu_F(x)\} / E \in \eta_1, E' \in \eta_2, \}
\] (a.3)

Similarly

\[
\nu_{C/B_1 \cap C/B_2}(x) = \nu_{C/B_1}(x) \lor \nu_{C/B_2}(x)
\]

\[
= \left( \bigwedge_{E \in \eta_1} \nu_E(x) \right) \lor \left( \bigwedge_{E \in \eta_2} \nu_E(x) \right)
\]

\[
= \bigwedge \{\nu_E(x) \lor \nu_F(x)\} / E \in \eta_1, E' \in \eta_2.
\] (b.3)

Now let \( E \in \eta_1 \) and \( E' \in \eta_2 \). Then \( EB_1 \subseteq C \) and \( E'B_2 \subseteq C \). Also \( E \lor E' \) IFI of \( A \), so that

\[
(E \lor E')(B_1 + B_2) \subseteq (E \lor E')B_1 + (E \lor E')B_2 \subseteq EB_1 + E'B_2 \subseteq C + C = C.
\] (54)

So \( E \lor E' \subseteq \eta_3 \). Hence \( \eta_1 \cap \eta_2 \subseteq \eta_3 \). Thus \( C/B_1 + B_2 = \bigcup_{E \in \eta_1} E \bigcup_{E \in \eta_2} (E \lor E') \). So

\[
\mu_{C/B_1 + B_2}(x) \geq \bigvee \mu_{E \lor E'}(x)
\]

\[
= \bigvee \{\mu_E(x) \land \mu_F(x)\}
\]

\[
= \mu_{C/B_1 \cap C/B_2}(x)
\] from (a.3),

\[
\nu_{C/B_1 + B_2}(x) \leq \bigwedge \nu_{E \lor E'}(x)
\]

\[
= \nu_{C/B_1 \cap C/B_2}(x)
\] from (b.3).

Therefore

\[
C/B_1 + B_2 \supseteq C/B_1 \cap C/B_2.
\] (56)

From (52) and (56) \( C/B_1 + B_2 = C/B_1 \cap C/B_2 \), hence the result.

References


