Research Article
Common Coupled Fixed-Point Theorems in Generalized Fuzzy Metric Spaces

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We prove two unique common coupled fixed-point theorems for self maps in symmetric G-fuzzy metric spaces.

1. Introduction and Preliminaries

Mustafa and Sims [1–3] and Naidu et al. [4] demonstrated that most of the claims concerning the fundamental topological structure of G-metric introduced by Dhage [5–8] and hence all theorems are incorrect. Alternatively, Mustafa and Sims [1, 2] introduced a G-metric space and obtained some fixed-point theorems in it. Some interesting references in G-metric spaces are [3, 9–15]. In this paper, we prove two unique common coupled fixed-point theorems for Jungck type and for three mappings in symmetric G-fuzzy metric spaces.

Before giving our main results, we recall some of the basic concepts and results in G-metric spaces and G-fuzzy metric spaces.

Definition 1 (see [2]). Let X be a nonempty set and let \( G : X \times X \times X \rightarrow [0, \infty) \) be a function satisfying the following properties:

\( (G_1) \ G(x, y, z) = 0 \) if \( x = y = z \),
\( (G_2) \ 0 < G(x, x, y) \) for all \( x, y \in X \) with \( x \neq y \),
\( (G_3) \ G(x, x, y) \leq G(x, y, z) \) for all \( x, y, z \in X \) with \( y \neq z \),
\( (G_4) \ G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots \), symmetry in all three variables,
\( (G_5) \ G(x, y, z) \leq G(x, a, a) + G(a, y, z) \) for all \( x, y, z, a \in X \).

Then, the function \( G \) is called a generalized metric or a G-metric on \( X \) and the pair \( (X, G) \) is called a G-metric space.

Definition 2 (see [2]). The G-metric space \( (X, G) \) is called symmetric if \( G(x, x, y) = G(x, y, y) \) for all \( x, y \in X \).

Definition 3 (see [2]). Let \( (X, G) \) be a G-metric space and let \( \{x_n\} \) be a sequence in \( X \). A point \( x \in X \) is said to be a limit of \( \{x_n\} \) if and only if \( \lim_{n,m \to \infty} G(x, x_n, x_m) = 0 \). In this case, the sequence \( \{x_n\} \) is said to be G-convergent to \( x \).

Definition 4 (see [2]). Let \( (X, G) \) be a G-metric space and let \( \{x_n\} \) be a sequence in \( X \). \( \{x_n\} \) is called G-Cauchy if and only if \( \lim_{n,m \to \infty} G(x, x_n, x_m) = 0 \). \( (X, G) \) is called G-complete if every G-Cauchy sequence in \( (X, G) \) is G-convergent in \( (X, G) \).

Proposition 5 (see [2]). In a G-metric space \( (X, G) \), the following are equivalent.

(i) The sequence \( \{x_n\} \) is G-Cauchy.
(ii) For every \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( G(x_n, x_m, x_m) < \epsilon \), for all \( n, m \geq N \).

Proposition 6 (see [2]). Let \( (X, G) \) be a G-metric space. Then, the function \( G(x, y, z) \) is jointly continuous in all three of its variables.

Proposition 7 (see [2]). Let \( (X, G) \) be a G-metric space. Then, for any \( x, y, z, a \in X \), it follows that

(i) if \( G(x, y, z) = 0 \), then \( x = y = z \),
(ii) \( G(x, y, z) \leq G(x, x, y) + G(x, x, z) \),
Definition 9 (see [16]). A 3-tuple \((X, G, \ast)\) is called a \(G\)-fuzzy metric space if \(X\) is an arbitrary nonempty set, \(\ast\) is a continuous \(t\)-norm, and \(G\) is a fuzzy set on \(X^3 \times (0, \infty)\) satisfying the following conditions for each \(t, s > 0\):

(i) \(G(x, y, t) > 0\) for all \(x, y \in X\) with \(x \neq y\),
(ii) \(G(x, y, t) \geq G(x, y, z, t)\) for all \(x, y, z \in X\) with \(y \neq z\),
(iii) \(G(x, y, z) = 1\) if and only if \(x = y = z\),
(iv) \(G(x, y, z, t + s) \geq G(a, y, z, s) \ast G(x, a, s)\) for all \(x, y, z, a \in X\),
(v) \(G(x, y, z, t) : (0, \infty) \to [0, 1]\) is continuous.

Definition 10 (see [16]). A \(G\)-fuzzy metric space \((X, G, \ast)\) is said to be symmetric if \(G(x, y, t) = G(x, y, t)\) for all \(x, y \in X\) and for each \(t > 0\).

Example 11. Let \(X\) be a nonempty set and let \(G\) be a \(G\)-metric on \(X\). Denote \(a \ast b = ab\) for all \(a, b \in [0, 1]\). For each \(t > 0\), \(G(x, y, t) = t/(t + G(x, y))\) is a \(G\)-fuzzy metric on \(X\).

Let \((X, G, \ast)\) be a \(G\)-fuzzy metric space. For \(t > 0\), \(0 < r < 1\), and \(x \in X\), the set \(B_G(x, r, t) = \{y \in X : G(x, y, t) > 1 - r\}\) is called an open ball with center \(x\) and radius \(r\).

A subset \(A\) of \(X\) is called an open set if for each \(x \in X\), there exist \(t > 0\) and \(0 < r < 1\) such that \(B_G(x, r, t) \subseteq A\).

A sequence \(\{x_n\}\) in \(G\)-fuzzy metric space \(X\) is said to be \(G\)-convergent to \(x \in X\) if \(G(x_n, x, t) \to 1\) as \(n \to \infty\) for each \(t > 0\). It is called a \(G\)-Cauchy sequence if \(G(x_n, x_m, t) \to 1\) as \(n, m \to \infty\) for each \(t > 0\). \(X\) is called \(G\)-complete if every \(G\)-Cauchy sequence in \(X\) is \(G\)-convergent in \(X\).

Lemma 12 (see [16]). Let \((X, G, \ast)\) be a \(G\)-fuzzy metric space. Then, \(G(x, y, z, t)\) is nondecreasing with respect to \(t\) for all \(x, y, z \in X\).

Lemma 13 (see [16]). Let \((X, G, \ast)\) be a \(G\)-fuzzy metric space. Then, \(G\) is a continuous function on \(X^3 \times (0, \infty)\).

Now onwards, we assume the following condition:

\[
\lim_{t \to \infty} G(x, y, z, t) = 1 \quad \forall x, y, z \in X. \tag{P}
\]

Using (P), one can prove the following lemma.

Lemma 14. Let \((X, G, \ast)\) be a \(G\)-fuzzy metric space. If there exists \(k \in (0, 1)\) such that

\[
\min\{G(x, y, z, kt), G(u, v, w, kt)\} \geq \min\{G(x, y, z, t), G(u, v, w, t)\}
\]

for all \(x, y, z, u, v, w \in X\) and \(t > 0\), then \(x = y = z\) and \(u = v = w\).

Definition 15 (see [17]). Let \(X\) be a nonempty set. An element \((x, y) \in X \times X\) is called a coupled fixed point of the mapping \(F : X \times X \to X\) if \(F(x, y) = x\) and \(y = F(y, x)\).

Definition 16 (see [18]). Let \(X\) be a nonempty set. An element \((x, y) \in X \times X\) is called

(i) a coupled coincidence point of \(F : X \times X \to X\) and \(g : X \to X\) if \(gx = F(x, y)\) and \(gy = F(y, x)\),
(ii) a common coupled fixed point of \(F : X \times X \to X\) and \(g : X \to X\) if \(gx = F(x, y)\) and \(gy = F(y, x)\).

Definition 17 (see [18]). Let \(X\) be a nonempty set. The mappings \(F : X \times X \to X\) and \(g : X \to X\) are called \(W\)-compatible if \(g(F(x, y)) = F(gx, gy)\) and \(g(F(y, x)) = F(gy, gx)\) whenever \(gx = F(x, y)\) and \(gy = F(y, x)\) for some \((x, y) \in X \times X\).

Now, we give our main results.

2. Main Results

Theorem 18. Let \((X, G, \ast)\) be a \(G\)-fuzzy metric space with \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0, 1]\) and \(S : X \times X \to X\) and let \(f : X \to X\) be mappings satisfying

\[
G(S(x, y), S(u, v), S(u, v), kt) \geq \min\{G(fx, fu, fu, t), G(fy, fv, fv, t)\}
\]

for all \(x, y, u, v \in X\), where \(0 \leq k < 1\),

\[
S(X \times X) \subseteq f(X) \text{ and } f(X) \text{ is a complete subspace of } X, \quad \text{the pair } (f, S) \text{ is } W\text{-compatible.}
\]

Then \(S\) and \(f\) have a unique common coupled fixed point of the form \((a, a)\) in \(X \times X\).

Proof. Let \(x_0, y_0 \in X\) and denote \(z_n = S(x_n, y_n) = fx_{n+1}, p_n = S(y_n, x_n) = fy_{n+1}, n = 0, 1, 2, \ldots\). Let \(d_n(t) =\)
\(G(z_n, z_{n+1}, z_{n+1}, t), c_n(t) = G(p_n, p_{n+1}, p_{n+1}, t). \) From (2), we have
\[
d_{n+1}(kt) = G(z_{n+1}, z_{n+2}, z_{n+2}, kt)
= G(S(x_{n+1}, y_{n+1}), S(x_{n+2}, y_{n+2}), S(x_{n+2}, y_{n+2}), kt)
\geq \min \{G(z_n, z_{n+1}, z_{n+1}, t), G(p_n, p_{n+1}, p_{n+1}, t)\}
\geq \min \{d_n(t), c_n(t)\}.
\]

Thus, \(\min \{d_{n+1}(kt), c_{n+1}(kt)\} \geq \min \{d_n(t), c_n(t)\}. \) Hence, \(\min \{d_n(t), c_n(t)\} \)
\[
\geq \min \left\{d_{n-1}\left(\frac{t}{k}\right), c_{n-1}\left(\frac{t}{k}\right)\right\}
\geq \min \left\{d_{n-2}\left(\frac{t}{k^2}\right), c_{n-2}\left(\frac{t}{k^2}\right)\right\}
\vdots
\geq \min \left\{d_0\left(\frac{t}{k^n}\right), c_0\left(\frac{t}{k^n}\right)\right\}
= \min \left\{G(z_0, z_1, z_1, t), G(p_0, p_1, p_1, t)\right\}.
\]

For any positive integer \(n\) and fixed positive integer \(p\), we have
\[
G(z_n, z_{n+p}, z_{n+p}, t)
\geq G(z_{n+p-1}, z_{n+p}, z_{n+p}, \frac{t}{p}) * G(z_{n+p-2}, z_{n+p-1}, z_{n+p-1}, \frac{t}{p})
* \cdots * G(z_0, z_1, z_1, \frac{t}{p})
\geq \min \left\{G(z_0, z_1, z_1, \frac{t}{pk^n}), G(p_0, p_1, p_1, \frac{t}{pk^n}) \right\}
* \min \left\{G(z_0, z_1, z_1, \frac{t}{pk^n-2}), G(p_0, p_1, p_1, \frac{t}{pk^n-2}) \right\}
* \cdots * \min \left\{G(z_0, z_1, z_1, \frac{t}{pk^n}), G(p_0, p_1, p_1, \frac{t}{pk^n}) \right\}.
\]

Letting \(n \to \infty\) and using (P), we get
\[
\lim_{n \to \infty} G(z_n, z_{n+p}, z_{n+p}, t) \geq 1 * 1 * \cdots * 1 = 1.
\]

Hence, \(\lim_{n \to \infty} G(z_n, z_{n+p}, z_{n+p}, t) = 1.\) Thus, \(\{z_n\}\) is \(G\)-Cauchy in \(X\). Similarly, we can show that \(\{p_n\}\) is \(G\)-Cauchy in \(X\). Since \(f(X)\) is \(G\)-complete, \(\{z_n\}\) and \(\{p_n\}\) converge to some \(\alpha\) and \(\beta\) in \(f(X)\), respectively. Hence, there exist \(x\) and \(y\) in \(X\) such that \(\alpha = fx, \beta = fy:\)
\[
G(z_n, S(x, y), S(x, y), kt)
= G(S(x_n, y_n), S(x, y), S(x, y), kt)
\geq \min \{G(z_{n-1}, fx, fx, t), G(p_{n-1}, fy, fy, t)\}.
\]

Letting \(n \to \infty\), we get
\[
G(fx, S(x, y), S(x, y), kt) \geq \min \{1, 1\} = 1.
\]

Hence, \(S(x, y) = fx\). Similarly, it can be shown that \(S(y, x) = fy\). Since \((f, S)\) is \(W\)-compatible, we have
\[
f \alpha = ffx = f(S(x, y)) = S(fx, fy) = S(\alpha, \beta),
\]
\[
f \beta = ffy = f(S(y, x)) = S(fy, fx) = S(\beta, \alpha).
\]

Thus, \(\min \{G(z_{n-1}, f \alpha, f \alpha, t), G(p_{n-1}, f \beta, f \beta, t)\} \geq \min \{G(z_{n-1}, f \alpha, f \alpha, t), G(p_{n-1}, f \beta, f \beta, t)\}. \)

Similarly, we can show that
\[
G(\beta, f \beta, f \beta, kt) \geq \min \{G(\alpha, f \alpha, f \alpha, t), G(\beta, f \beta, f \beta, t)\}. \]

Thus,
\[
\min \{G(z_{n-1}, f \alpha, f \alpha, t), G(p_{n-1}, f \beta, f \beta, t)\} \geq \min \{G(z_{n-1}, f \alpha, f \alpha, t), G(p_{n-1}, f \beta, f \beta, t)\}. \]

From Lemma 14, we have \(f \alpha = \alpha\) and \(f \beta = \beta\). Thus, \(\alpha = f \alpha = S(\alpha, \beta)\) and \(\beta = f \beta = S(\beta, \alpha)\). Hence, \((\alpha, \beta)\) is a common coupled fixed point of \(S\) and \(f\).

Suppose \((\alpha', \beta')\) is another common coupled fixed point of \(S\) and \(f\):
\[
G(\alpha, \alpha', \alpha', kt) = G(S(\alpha, \beta), S(\alpha', \beta'), S(\alpha', \beta'), kt)
\geq \min \{G(\alpha, \alpha', \alpha', t), G(\beta, \beta', \beta', t)\}. \]

Similarly,
\[
G(\beta, \beta', \beta', kt) = G(S(\beta, \alpha), S(\beta', \alpha'), S(\beta', \alpha'), kt)
\geq \min \{G(\alpha, \alpha', \alpha', t), G(\beta, \beta', \beta', t)\}. \]
Thus,
\[
\min \{ G(a, a^1, a^1, kt), G(\beta, \beta^1, \beta^1, kt) \} 
\geq \min \{ G(a, a^1, a^1, t), G(\beta, \beta^1, \beta^1, t) \}.
\]
(17)
From Lemma 14, \( a^1 = a \) and \( \beta^1 = \beta \). Thus, \((a, \beta)\) is the unique common coupled fixed point of \( S \) and \( f \). Now, we will show that \( a = \beta \):
\[
G(a, a, \beta, kt) = G(S(\alpha, \beta), S(\alpha, \beta), S(\beta, \alpha), kt) 
\geq \min \{ G(a, a, \beta, t), G(\beta, \alpha, t) \},
\]
(18)
\[
G(\alpha, \beta, \beta, kt) = G(S(\alpha, \beta), S(\beta, \alpha), S(\beta, \alpha), kt) 
\geq \min \{ G(a, a, \beta, t), G(\beta, a, t) \}.
\]
Thus,
\[
\min \{ G(a, a, \beta, kt), G(\alpha, \beta, \beta, kt) \} 
\geq \min \{ G(a, a, \beta, t), G(\alpha, \beta, \beta, t) \}.
\]
(19)
From Lemma 14, we have \( a = \beta \). Thus, \( a \) is a common fixed point of \( S \) and \( f \), that is, \( a = f^a = S(a, a) \). Suppose \( a^1 \) is another common fixed point of \( S \) and \( f \):
\[
G(a^1, a, a, t) = G(S(a^1, a^1), S(a, a), S(a, a), t) 
\geq \min \{ G(a^1, a, a, \frac{t}{K}), G(a^1, a, a, \frac{t}{K^2}) \} 
\geq \min \{ G(\alpha, \alpha, a, \frac{t}{K}), G(\alpha, a, a, \frac{t}{K}) \} 
\geq G(\alpha, a, a, \frac{t}{K^n}) \rightarrow 1 \quad \text{as } n \rightarrow \infty.
\]
Hence, \( a^1 = a \). Thus, \( S \) and \( f \) have a unique common coupled fixed point of the form \((a, a)\).

Finally, we prove a common coupled fixed-point theorem for three mappings in symmetric \( G \)-fuzzy metric spaces.

**Theorem 19.** Let \((X, G, *)\) be a symmetric \( G \)-complete fuzzy metric space with \( a \neq b \in [0, 1] \) and let \( S, T, R : X \times X \rightarrow X \) be mappings satisfying
\[
G(S(x, y), T(u, v), R(p, q), kt) 
\geq \min \{ G(x, u, p, t), G(y, v, q, t), G(x, S(x, y), t), G(u, u, T(u, v), t), G(p, p, R(p, q), t) \}
\]
(21)
for all \( x, y, u, v, p, q \in X \), where \( 0 \leq k < 1 \). Then, there exists \((x, y) \in X \times X \) such that
\[
x = S(x, y) = T(x, y) = R(x, y),
\]
(22)
y = S(y, x) = T(y, x) = R(y, x).
(23)

Or
\[
S, T, \text{ and } R \text{ have a unique common coupled fixed point of the form } (x, x) \text{ in } X \times X.
\]
(24)

**Proof.** Let \( x_0, y_0 \in X \). Define the sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) as follows: \( x_{n+1} = S(x_n, y_n), y_{n+1} = S(y_n, x_n); x_{3n+2} = T(x_{3n+1}, y_{3n+1}), y_{3n+2} = T(y_{3n+1}, x_{3n+1}); x_{3n+3} = R(x_{3n+2}, y_{3n+2}), y_{3n+3} = R(y_{3n+2}, x_{3n+2}), n = 0, 1, 2, \ldots \). Suppose \( x_{3n+1} = x_{3n} \) for some \( n \). Then, \( S(x, y) = x \), where \( x = x_{3n}, y = y_{3n} \). Suppose \( T(x, y) \neq R(x, y) \). Then,
\[
G(x, T(x, y), R(x, y), kt) 
= G(S(x, y), T(x, y), R(x, y), kt) 
\geq \min \{ 1, 1, 1, G(x, x, T(x, y), t), G(x, x, R(x, y), t) \} 
\geq G(x, T(x, y), R(x, y), t).
\]
(25)
It is a contradiction. Hence, \( T(x, y) = R(x, y) \). From (25) and since \( X \) is symmetric,
\[
G(x, T(x, y), T(x, y), kt) \geq G(x, x, T(x, y), t) = G(x, T(x, y), T(x, y), t).
\]
(26)
From Lemma 14, we have \( T(x, y) = x \). Thus, \( S(x, y) = T(x, y) = R(x, y) = x \). Similarly, if \( x_{3n+1} = x_{3n+2} \) or \( x_{3n+2} = x_{3n+3} \), then also we can show that \( S(x, y) = T(x, y) = R(x, y) = x \) for some \( y \) in \( X \). Similarly, it can be shown that if \( y_{3n} = y_{3n+1} \) or \( y_{3n+1} = y_{3n+2} \) or \( y_{3n+2} = y_{3n+3} \) then there exists \((x, y) \in X \times X \) such that
\[
S(y, x) = T(y, x) = R(y, x) = y.
\]
(27)
Now, assume that \( x_n \neq x_{n+1} \) and \( y_n \neq y_{n+1} \) for all \( n \). Write \( d_{3n}(t) = G(x_n, x_{n+1}, x_{n+2}, t) \) and \( e_{3n}(t) = G(y_n, y_{n+1}, y_{n+2}, t) \):
\[
d_{3n}(kt) 
= G(x_{3n}, x_{3n+1}, x_{3n+2}, kt) 
= G(S(x_{3n}, y_{3n}), T(x_{3n+1}, y_{3n+1}), R(x_{3n+1}, y_{3n+1}), kt) 
\geq \min \{ d_{3n-1}(t), e_{3n-1}(t), G(x_{3n}, x_{3n+1}, x_{3n+1}, t), G(x_{3n+1}, x_{3n+1}, x_{3n+2}, t), G(x_{3n+1}, x_{3n+1}, x_{3n+2}, t) \} 
\geq \min \{ d_{3n-1}(t), e_{3n-1}(t), d_{3n-1}(t), d_{3n-1}(t), d_{3n-1}(t) \}.
\]
(28)
Thus, \( d_{3n}(kt) \geq \min \{ d_{3n-1}(t), e_{3n-1}(t) \} \). Similarly, we have \( e_{3n}(kt) \geq \min \{ d_{3n-1}(t), e_{3n-1}(t) \} \).
Thus, \( \min \{ d_{3n}(kt), e_{3n}(kt) \} \geq \min \{ d_{3n-1}(t), e_{3n-1}(t) \} \).
(29)
Similarly, we can show that
\[
\min \{ d_{3n+1}(kt), e_{3n+1}(kt) \} \geq \min \{ d_{3n}(t), e_{3n}(t) \},
\]
\[
\min \{ d_{3n+2}(kt), e_{3n+2}(kt) \} \geq \min \{ d_{3n+1}(t), e_{3n+1}(t) \}.
\]
(30)
Thus, 
\begin{align*}
\min\{d_{n+1}(kt), e_{n+1}(kt)\} &\geq \min\{d_n(t), e_n(t)\}. 
\end{align*}
\tag{31}

Hence
\begin{align*}
\min\{d_n(t), e_n(t)\} \\
&\geq \min\left\{d_n\left(\frac{t}{k^n}\right), e_n\left(\frac{t}{k^n}\right)\right\} \\
&\geq \min\left\{d_0\left(\frac{t}{k^n}\right), e_0\left(\frac{t}{k^n}\right)\right\} \\
&\geq \min\left\{d_0\left(\frac{t}{k^n}\right), e_0\left(\frac{t}{k^n}\right)\right\} \\
&= \min\left\{G(x_0, x_1, x_2, \frac{t}{k^n}), G(y_0, y_1, y_2, \frac{t}{k^n})\right\}.
\end{align*}
\tag{32}

Thus,
\begin{align*}
G(x_n, x_{n+1}, x_{n+2}, t) \\
&\geq \min\left\{G(x_0, x_1, x_2, \frac{t}{k^n}), G(y_0, y_1, y_2, \frac{t}{k^n})\right\}. 
\end{align*}
\tag{33}

From (G3), we have
\begin{align*}
G(x_n, x_{n+1}, t) \\
&\geq G(x_n, x_{n+1}, x_{n+2}, t) \\
&\geq \min\left\{G(x_0, x_1, x_2, \frac{t}{k^n}), G(y_0, y_1, y_2, \frac{t}{k^n})\right\}. 
\end{align*}
\tag{34}

As in Theorem 18, we can show that \{x_n\} and \{y_n\} are G-Cauchy sequences in X. Since X is G-complete, there exist \(x, y \in X\) such that \(x_n \to x\) and \(y_n \to y\):
\begin{align*}
G(S(x, y), x_{3n+1}, x_{3n+3}, kt) \\
&= G(S(x, y), T(x_{3n+1}, y_{3n+1}), R(x_{3n+2}, y_{3n+2}), kt) \\
&\geq \min\{G(x, x_{3n+1}, x_{3n+2}, t), G(y, y_{3n+1}, y_{3n+2}, t), \\
&G(x, x, S(x, y), t), G(x_{3n+1}, x_{3n+1}, x_{3n+2}, t), \\
&G(x_{3n+1}, x_{3n+1}, x_{3n+2}, t)\}. 
\end{align*}
\tag{35}

Letting \(n \to \infty\),
\begin{align*}
G(S(x, y), x, x, kt) &\geq \min\{1, 1, G(x, x, S(x, y), t), 1, 1\} \\
&= G(x, x, S(x, y), t). 
\end{align*}
\tag{36}

From this, we have \(S(x, y) = x\). As in the first part of proof, we can show that \(S(x, y) = T(x, y) = R(x, y) = x\). Similarly, it can be shown that \(S(y, x) = T(y, x) = R(y, x) = y\). Thus, \((x, y)\) is a common coupled fixed point of \(S, T\), and

R. Suppose \((x^1, y^1)\) is another common coupled fixed point of \(S, T, \) and \(R\). Consider
\begin{align*}
G(x, x, x^1, kt) &= G(S(x, y), T(x, y), R(x^1, y^1), kt) \\
&\geq \min\{G(x, x, x^1, t), G(y, y, y^1, t), 1, 1, 1\} \\
&= \min\{G(x, x, x^1, t), G(y, y, y^1, t)\}. 
\end{align*}
\tag{37}

Also,
\begin{align*}
G(y, y, y^1, kt) &= G(S(x, y), T(y, x), R(y^1, x^1), kt) \\
&\geq \min\{G(x, x, x^1, t), G(y, y, y^1, t), 1, 1, 1\} \\
&= \min\{G(x, x, x^1, t), G(y, y, y^1, t)\}. 
\end{align*}
\tag{38}

Thus,
\begin{align*}
\min\{G(x, x, x^1, kt), G(y, y, y^1, kt)\} \\
&\geq \min\{G(x, x, x^1, t), G(y, y, y^1, t)\}. 
\end{align*}
\tag{39}

From Lemma 14, we have \(x^1 = x\) and \(y^1 = y\). Thus, \((x, y)\) is the unique common coupled fixed point of \(S, T, \) and \(R\). Now, we will show that \(x = y\). Consider
\begin{align*}
G(x, x, y, kt) &= G(S(x, y), T(x, y), R(x, y), kt) \\
&\geq \min\{G(x, x, y, t)G(y, x, y, t), 1, 1, 1\} \\
&= G(x, x, y, t). 
\end{align*}
\tag{40}

Hence, \(x = y\). Thus, \(S, T, \) and \(R\) have a unique common coupled fixed point of the form \((x, x)\). \(\Box\)

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\textbf{References}


