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Research Article

Fuzzy Stability of an *n*-Dimensional Quadratic and Additive Functional Equation

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We will investigate a fuzzy version of stability for the functional equation $2f(\sum_{i=1}^{n} x_i)+\sum_{1\le i,j\le n, i\ne j} f(x_i-x_j) = (n+1)\sum_{i=1}^{n} f(x_i)+(n-1)\sum_{i=1}^{n} f(-x_i)$ in the sense of Mirmostafaee and Moslehian.

1. Introduction

In 1940, Ulam [1] gave a wide-ranging talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms.

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, we say that the functional equation for homomorphisms is stable. Hyers [2] was the first mathematician to present the result concerning the stability of functional equations. He answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces. This result of Hyers is stated as follows.

Let $f : E_1 \to E_2$ be a function between Banach spaces such that

$$||f(x+y) - f(x) - f(y)|| \le \delta \tag{1}$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit $A(x) = lim_{n\to\infty} 2^{-n} f(2^n x)$ *exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive function such that $\|f(x) - A(x)\| \le \delta$ for every $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.*

We remark that the additive function A is directly constructed from the given function f, and this method is called the *direct method*. The direct method is a very powerful method for studying the stability problems of various functional equations. Taking this famous result into consideration, the additive Cauchy equation $f(x+y) = f(x)+f(y)$ is said to have the *Hyers-Ulam stability* on (E_1, E_2) if for every function $f : E_1 \to E_2$ satisfying (1) for some $\delta \ge 0$ and for all $x, y \in E_1$, there exists an additive function $A : E_1 \to E_2$ such that $f - A$ is bounded on E_1.

In 1950, Aoki [3] generalized the theorem of Hyers for additive functions, and in the following year, Bourgin [4] extended the theorem without proof. Unfortunately, it seems that their results failed to receive attention from mathematicians at that time. No one has made use of these results for a long time.

In 1978, Rassias [5] addressed the Hyers's stability theorem and attempted to weaken the condition for the bound of the norm of Cauchy difference and generalized the theorem of Hyers for linear functions.

Let $f : E_1 \to E_2$ be a function between Banach spaces. If f satisfies the functional inequality

$$||f(x+y) - f(x) - f(y)|| \le \theta\left(\|x\|^p + ||y||^p\right) \tag{2}$$

for some $\theta \ge 0$, p with $0 \le p < 1$ and for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ such that $\|f(x) - A(x)\| \le (2\theta/(2-2^p))\|x\|^p$ for each $x \in E_1$. If, in

addition, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

This result of Rassias attracted a number of mathematicians who began to be stimulated to investigate the stability problems of functional equations. By regarding a large influence of Ulam, Hyers, and Rassias on the study of stability problems of functional equations, the stability phenomenon proved by Rassias is called the *Hyers-Ulam-Rassias stability*. For the last thirty years many results concerning the Hyers-Ulam-Rassias stability of various functional equations have been obtained (see [6–17]).

In this paper, we investigate a general stability of the *n-dimensional quadratic and additive functional equation*

$$2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i,j\le n, i\ne j} f\left(x_i - x_j\right) = (n+1)\sum_{i=1}^{n} f(x_i) + (n-1)\sum_{i=1}^{n} f(-x_i) \tag{3}$$

in the fuzzy normed space. We will call each solution of (3) a *quadratic-additive function*.

In 2006, Jun and Kim [10] proved the stability of (3) by constructing an additive function A and a quadratic function Q separately and by approximating the given function f with a quadratic-additive function $F = A + Q$. In their approach, A and Q approximate the odd part and the even part of f, respectively. However, their method is not efficient in comparison with our method which approximates the function f with the quadratic-additive function F simultaneously. Indeed, we introduce a Cauchy sequence $\{J^n f(x)\}$ by making use of the given function f, and the sequence $\{J^n f(x)\}$ converges to a quadratic-additive function F which approximates the function f in the fuzzy sense. Our idea seems to be a refinement of previous studies.

2. Preliminaries

In 1984, Katsaras [18] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, mathematicians have introduced several types of fuzzy norm in different points of view. In particular, adhering to the point of view of Cheng and Mordeson, Bag and Samanta suggested an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type (see [19–21]).

We introduce the definition of a fuzzy normed space to establish a reasonable fuzzy version of stability for the n-dimensional quadratic and additive functional equation (3) in the fuzzy normed space (cf. [19]).

Definition 1. Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ is said to be a fuzzy norm on X if the following conditions are true for all $x, y \in X$ and all $s, t \in \mathbb{R}$:

(F_1) $N(x,t) = 0$ *for* $t \le 0$;

(F_2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

(F_3) $N(cx,t) = N(x, t/|c|)$ if $c \ne 0$;

(F_4) $N(x+y, s+t) \ge \min\{N(x,s), N(y,t)\}$;

(F_5) $N(x,\cdot)$ is a nondecreasing function *on* $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (X,N) is called a *fuzzy normed space*. Let (X,N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we write $N\text{-}\lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > 1 - \varepsilon$ for all $n \ge n_0$ and all $p \in \mathbb{N}$. It is known that every convergent sequence in a fuzzy normed space is Cauchy. If every Cauchy sequence in X converges in X, then the fuzzy norm is said to be *complete,* and the fuzzy normed space is called a *fuzzy Banach space*.

In 2008, Mirmostafaee and Moslehian first dealt with the fuzzy stability problems for the additive Cauchy equation, Jensen's functional equation, and the quadratic functional equation (see [22–24]).

3. Fuzzy Stability of (3)

Let (X,N_1) and (Y,N_2) be a fuzzy normed space and a fuzzy Banach space, respectively. Assume that n is a fixed integer greater than 1. For a given function $f : X \to Y$, we use the abbreviation

$$Df(x_1,x_2,\ldots,x_n) := 2f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i,\, j\le n, i\ne j} f\left(x_i - x_j\right) - (n+1)\sum_{i=1}^{n} f(x_i) - (n-1)\sum_{i=1}^{n} f(-x_i) \tag{4}$$

for all $x_1, x_2, \ldots, x_n \in X$. For a given $q > 0$, the function $f : X \to Y$ is called a *fuzzy q-almost quadratic-additive function* if

$$N_2(Df(x_1,\ldots,x_n), t_1 + \cdots + t_n) \ge \min\left\{N_1\left(x_1, t_1^q\right), \ldots, N_1\left(x_n, t_n^q\right)\right\} \tag{5}$$

for all $x_1, x_2, \ldots, x_n \in X$ and all $t_1, t_2, \ldots, t_n \ge 0$.

The following theorem gives a fuzzy version of the stability of the n-dimensional quadratic and additive functional equation $Df(x_1,x_2,\ldots,x_n) = 0$.

Theorem 2. *Let q be a positive real number with $q \notin \{1/2, 1\}$. If a function $f : X \to Y$ is a fuzzy q-almost quadratic-additive function from a fuzzy normed space (X,N_1) into a fuzzy Banach space (Y,N_2), then there exists a unique quadratic-additive function $F : X \to Y$ such that*

$$N_2(F(x) - f(x), t) \ge \begin{cases} \sup\limits_{0<t'<t} N_1\left(x, 2^q n^{-q}(n-n^p)^q t'^q\right) & \text{if } q > 1, \\ \sup\limits_{0<t'<t} N_1\left(x, \dfrac{2^q t'^q}{(n/(n^2-n^p)+n/(n^p-n))^q}\right) & \text{if } 1/2 < q < 1, \\ \sup\limits_{0<t'<t} N_1\left(x, 2^q\left(n^p-n^2\right)^q n^{-q} t'^q\right) & \text{if } 0 < q < 1/2 \end{cases} \tag{6}$$

for each $x \in X$ and $t > 0$, where $p = 1/q$.

Proof. It follows from (F_2), (F_3), and (5) that

$$\begin{aligned} N_2(f(0),t) &= N_2(Df(0,\dots,0),(n-1)(n+2)t) \\ &\geq N_1\left(0,\left(\frac{(n-1)(n+2)t}{n}\right)^q\right) \\ &= 1 \end{aligned} \tag{7}$$

for all $t > 0$. Thus, it follows from (F_2) that $f(0) = 0$.

We will split the proof into three parts according to the value of q, namely, $q > 1$, $1/2 < q < 1$, or $0 < q < 1/2$.

Case 1. Assume that $q > 1$ and let $J^m f : X \to Y$ be a function defined by

$$\begin{aligned} J^m f(x) = &\frac{1}{2n^{2m}}(f(n^m x) + f(-n^m x)) \\ &+\frac{1}{2n^m}(f(n^m x) - f(-n^m x)) \end{aligned} \tag{8}$$

for all $x \in X$ and $m \in \mathbb{N}_0$. Then, $J^0 f(x) = f(x)$ and by a long manipulation, we get

$$\begin{aligned} J^i f(x) - J^{i+1} f(x) = &-\frac{(n^{i+1}+1)Df(n^i x,\dots,n^i x)}{4n^{2i+2}} \\ &+\frac{(n^{i+1}-1)Df(-n^i x,\dots,-n^i x)}{4n^{2i+2}} \end{aligned} \tag{9}$$

for all $x \in X$ and for any $i \in \mathbb{N}_0$. Together with (F_3), (F_4), and (5), this equation implies that if $m' + m > m \geq 0$ then

$$\begin{aligned} &N_2\left(J^m f(x) - J^{m'+m} f(x), \sum_{i=m}^{m'+m-1}\left(\frac{n^p}{n}\right)^i \frac{t^p}{2}\right) \\ &= N_2\left(\sum_{i=m}^{m'+m-1}\left(J^i f(x) - J^{i+1} f(x)\right), \sum_{i=m}^{m'+m-1}\left(\frac{n^p}{n}\right)^i \frac{t^p}{2}\right) \\ &\geq \min_{m\leq i<m'+m} N_2\left(J^i f(x) - J^{i+1} f(x), \left(\frac{n^p}{n}\right)^i \frac{t^p}{2}\right) \\ &\geq \min_{m\leq i<m'+m} \min\left\{N_2\left(\frac{-(n^{i+1}+1)Df(n^i x,\dots,n^i x)}{4n^{2i+2}}, \frac{(n^{i+1}+1)n^{ip}t^p}{4n^{2i+1}}\right), \right. \\ &\qquad \left. N_2\left(\frac{(n^{i+1}-1)Df(-n^i x,\dots,-n^i x)}{4n^{2i+2}}, \frac{(n^{i+1}-1)n^{ip}t^p}{4n^{2i+1}}\right)\right\} \\ &\geq \min_{m\leq i<m'+m} N_1\left(n^i x, n^i t\right) \\ &= N_1(x,t) \end{aligned} \tag{10}$$

for all $x \in X$ and for all $t > 0$.

Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty} N_1(x,t) = 1$, there exists an $t_0 > 0$ such that

$$N_1(x,t_0) \geq 1 - \varepsilon. \tag{11}$$

We observe that for some $\tilde{t} > t_0$, the series $\sum_{i=0}^{\infty}(n^p/n)^i(\tilde{t}^p/2)$ converges for $p = 1/q < 1$. Hence, for an arbitrarily given $c > 0$, there exists an $m_0 \geq 0$ such that

$$\sum_{i=m}^{m'+m-1}\left(\frac{n^p}{n}\right)^i \frac{\tilde{t}^p}{2} < c \tag{12}$$

for each $m \geq m_0$ and $m' > 0$. By (F_5) and (10), we have

$$\begin{aligned} &N_2(J^m f(x) - J^{m'+m} f(x), c) \\ &\quad \geq N_2\left(J^m f(x) - J^{m'+m} f(x), \sum_{i=m}^{m'+m-1}\left(\frac{n^p}{n}\right)^i \frac{\tilde{t}^p}{2}\right) \\ &\quad \geq N_1\left(x,\tilde{t}\right) \\ &\quad \geq N_1(x,t_0) \\ &\quad \geq 1-\varepsilon \end{aligned} \tag{13}$$

for all $x \in X$ and for each $c > 0$. Hence, $\{J^m f(x)\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N_2), and, thus, we can define a function $F : X \to Y$ by

$$F(x) := N_2 - \lim_{m\to\infty} J^m f(x) \tag{14}$$

for all $x \in X$. Moreover, if we put $m = 0$ and replace t with $2^q t^q/(\sum_{i=0}^{m'-1}(n^p/n)^i)^q$ in (10), then we have

$$N_2\left(f(x) - J^{m'} f(x), t\right) \geq N_1\left(x, \frac{2^q t^q}{\left(\sum_{i=0}^{m'-1}(n^p/n)^i\right)^q}\right) \tag{15}$$

for all $x \in X$ and for any $t > 0$.

Next, we will show that F is a quadratic-additive function. Using (F_4) and (4), we have

$$\begin{aligned} &N_2(DF(x_1,\dots,x_n),t), \\ &\quad \geq \min\left\{N_2\left(2(F - J^m f)\left(\sum_{i=1}^n x_i\right), \frac{t}{5}\right), \right. \\ &\qquad \min_{1\leq i,\, j\leq n,\, i\neq j} N_2\left((F - J^m f)\left(x_i - x_j\right), \frac{t}{5n(n-1)}\right), \\ &\qquad \min_{1\leq i\leq n} N_2\left((n+1)(J^m f - F)(x_i), \frac{t}{5n}\right), \\ &\qquad \min_{1\leq i\leq n} N_2\left((n-1)(J^m f - F)(-x_i), \frac{t}{5n}\right), \\ &\qquad \left. N_2\left(DJ^m f(x_1,x_2,\dots,x_n), \frac{t}{5}\right)\right\} \end{aligned} \tag{16}$$

for all $x_1,\dots,x_n \in X$, $t > 0$, and $m \in \mathbb{N}$. In view of the definition of F and (F_2), the first four terms on the right hand

side of (16) tend to 1 as $m \to \infty$. It follows from (4) and (8) that

$$\begin{aligned} DJ^m f(x_1, x_2, \ldots, x_n) &= \frac{1}{2n^{2m}} Df(n^m x_1, \ldots, n^m x_n) \\ &\quad + \frac{1}{2n^{2m}} Df(-n^m x_1, \ldots, -n^m x_n) \\ &\quad + \frac{1}{2n^m} Df(n^m x_1, \ldots, n^m x_n) \\ &\quad - \frac{1}{2n^m} Df(-n^m x_1, \ldots, -n^m x_n) \end{aligned} \tag{17}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $m \in \mathbb{N}_0$. Hence, by (F_3) and (F_4), we have

$$\begin{aligned} &N_2\left(DJ^m f(x_1, x_2, \ldots, x_n), \frac{t}{5}\right) \\ &\quad \geq \min\left\{N_2\left(\frac{Df(n^m x_1, \ldots, n^m x_n)}{2n^{2m}}, \frac{t}{20}\right), \right. \\ &\qquad N_2\left(\frac{Df(-n^m x_1, \ldots, -n^m x_n)}{2n^{2m}}, \frac{t}{20}\right), \\ &\qquad N_2\left(\frac{Df(n^m x_1, \ldots, n^m x_n)}{2n^m}, \frac{t}{20}\right), \\ &\qquad \left. N_2\left(\frac{Df(-n^m x_1, \ldots, -n^m x_n)}{2n^m}, \frac{t}{20}\right)\right\} \end{aligned} \tag{18}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$.

By (F_3) and (5), we obtain

$$\begin{aligned} &N_2\left(\frac{Df(\pm n^m x_1, \ldots, \pm n^m x_n)}{2n^{2m}}, \frac{t}{20}\right) \\ &\quad \geq \min\left\{N_1\left(x_1, 10^{-q} n^{(2q-1)m-q} t^q\right), \ldots, \right. \\ &\qquad \left. N_1\left(x_n, 10^{-q} n^{(2q-1)m-q} t^q\right)\right\}, \\ &N_2\left(\frac{Df(\pm n^m x_1, \ldots, \pm n^m x_n)}{2n^m}, \frac{t}{20}\right) \\ &\quad \geq \min\left\{N_1\left(x_1, 10^{-q} n^{(q-1)m-q} t^q\right), \ldots, \right. \\ &\qquad \left. N_1\left(x_n, 10^{-q} n^{(q-1)m-q} t^q\right)\right\} \end{aligned} \tag{19}$$

for all $x_1, \ldots, x_n \in X$, $t > 0$, and $m \in \mathbb{N}$. Since $q > 1$, it follows from (F_5) that the last term in (16) also goes to 1 as $m \to \infty$. In view of (16), we conclude that

$$N_2(DF(x_1, x_2, \ldots, x_n), t) = 1 \tag{20}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$. By (F_2), this implies that

$$DF(x_1, x_2, \ldots, x_n) = 0 \tag{21}$$

for all $x_1, x_2, \ldots, x_n \in X$.

We will now prove the first inequality in (6). Let $x \in X$, let $t > t' > 0$, and let $0 < \varepsilon < 1$ be given. Since F is the limit of the sequence $\{J^m f(x)\}$, there exists an $m \in \mathbb{N}$ such that

$$N_2(F(x) - J^m f(x), t - t') \geq 1 - \varepsilon. \tag{22}$$

By (F_4) and (15), we have

$$\begin{aligned} &N_2(F(x) - f(x), t) \\ &\quad \geq \min\{N_2(F(x) - J^m f(x), t - t'), \\ &\qquad N_2(J^m f(x) - f(x), t')\} \\ &\quad \geq \min\left\{1 - \varepsilon, N_1\left(x, \frac{2^q t'^q}{\left(\sum_{i=0}^{m-1} (n^p/n)^i\right)^q}\right)\right\} \\ &\quad \geq \min\{1 - \varepsilon, N_1(x, 2^q n^{-q} (n - n^p)^q t'^q)\}. \end{aligned} \tag{23}$$

Because $0 < \varepsilon < 1$ is arbitrary, we get the first inequality in (6) for the case of $q > 1$.

Finally, it remains to prove the uniqueness of F. Let $G : X \to Y$ be another quadratic-additive function satisfying the first inequality in (6). Then, by (9) and (21), we get

$$\begin{aligned} F(x) - J^m F(x) &= \sum_{i=0}^{m-1} \left(J^i F(x) - J^{i+1} F(x)\right) = 0, \\ G(x) - J^m G(x) &= \sum_{i=0}^{m-1} \left(J^i G(x) - J^{i+1} G(x)\right) = 0 \end{aligned} \tag{24}$$

for all $x \in X$ and $m \in \mathbb{N}$. Together with (F_3), (F_4), (F_5), (6), and (8), this implies that

$$\begin{aligned} &N_2(F(x) - G(x), t) \\ &\quad = N_2(J^m F(x) - J^m G(x), t) \\ &\quad \geq \min\left\{N_2\left((J^m F - J^m f)(x), \frac{t}{2}\right), \right. \\ &\qquad \left. N_2\left((J^m f - J^m G)(x), \frac{t}{2}\right)\right\} \\ &\quad \geq \min\left\{N_2\left(\frac{(F - f)(n^m x)}{2n^{2m}}, \frac{t}{8}\right), \right. \\ &\qquad N_2\left(\frac{(f - G)(n^m x)}{2n^{2m}}, \frac{t}{8}\right), \\ &\qquad N_2\left(\frac{(F - f)(-n^m x)}{2n^{2m}}, \frac{t}{8}\right), \\ &\qquad N_2\left(\frac{(f - G)(-n^m x)}{2n^{2m}}, \frac{t}{8}\right), \\ &\qquad N_2\left(\frac{(F - f)(n^m x)}{2n^m}, \frac{t}{8}\right), \\ &\qquad N_2\left(\frac{(f - G)(n^m x)}{2n^m}, \frac{t}{8}\right), \\ &\qquad N_2\left(\frac{(F - f)(-n^m x)}{2n^m}, \frac{t}{8}\right), \\ &\qquad \left. N_2\left(\frac{(f - G)(-n^m x)}{2n^m}, \frac{t}{8}\right)\right\} \\ &\quad \geq \sup_{0<t'<t} N_1\left(x, n^{(q-1)m-q} 2^{-q} (n - n^p)^q t'^q\right) \\ &\quad \longrightarrow 1 \text{ as } m \longrightarrow \infty \end{aligned} \tag{25}$$

for all $x \in X$ and $t > 0$. This implies that $N_2(F(x) - G(x), t) = 1$ for all $x \in X$ and $t > 0$. Thus, it follows from (F_2) that

$$F(x) = G(x) \tag{26}$$

for all $x \in X$.

Case 2. Assume that $1/2 < q < 1$. Let us define a function $J^m f : X \to Y$ by

$$J^m f(x) := \frac{1}{2n^{2m}}\left(f(n^m x) + f(-n^m x)\right) + \frac{n^m}{2}\left(f\left(\frac{x}{n^m}\right) - f\left(-\frac{x}{n^m}\right)\right) \tag{27}$$

for all $x \in X$ and $m \in \mathbb{N}_0$. Then, we have $J^0 f(x) = f(x)$, and it follows from (4) and (27) that

$$\begin{aligned} J^i f(x) - J^{i+1} f(x) &= -\frac{Df(n^i x, \ldots, n^i x)}{4n^{2i+2}} - \frac{Df(-n^i x, \ldots, -n^i x)}{4n^{2i+2}} \\ &\quad + \frac{n^i}{4} Df\left(\frac{x}{n^{i+1}}, \ldots, \frac{x}{n^{i+1}}\right) \\ &\quad - \frac{n^i}{4} Df\left(-\frac{x}{n^{i+1}}, \ldots, -\frac{x}{n^{i+1}}\right) \end{aligned} \tag{28}$$

for all $x \in X$ and $i \in \mathbb{N}_0$. If $m' + m > m \geq 0$, then it follows from (F_3), (F_4), and (5) that

$$\begin{aligned} &N_2\left(J^m f(x) - J^{m'+m} f(x), \sum_{i=m}^{m'+m-1}\left(\frac{1}{2n}\left(\frac{n^p}{n^2}\right)^i + \frac{n}{2n^p}\left(\frac{n}{n^p}\right)^i\right)t^p\right) \\ &\quad = N_2\left(\sum_{i=m}^{m'+m-1}\left(J^i f(x) - J^{i+1} f(x)\right), \sum_{i=m}^{m'+m-1}\left(\frac{1}{2n}\left(\frac{n^p}{n^2}\right)^i + \frac{n}{2n^p}\left(\frac{n}{n^p}\right)^i\right)t^p\right) \\ &\quad \geq \min_{m \leq i < m'+m} \min\left\{N_2\left(-\frac{Df(n^i x, \ldots, n^i x)}{4n^{2i+2}}, \frac{n^{ip}t^p}{4n^{2i+1}}\right), N_2\left(-\frac{Df(-n^i x, \ldots, -n^i x)}{4n^{2i+2}}, \frac{n^{ip}t^p}{4n^{2i+1}}\right), \right. \\ &\qquad N_2\left(\frac{n^i}{4} Df\left(\frac{x}{n^{i+1}}, \ldots, \frac{x}{n^{i+1}}\right), \frac{n^{i+1}t^p}{4n^{(i+1)p}}\right), \\ &\qquad \left. N_2\left(-\frac{n^i}{4} Df\left(-\frac{x}{n^{i+1}}, \ldots, -\frac{x}{n^{i+1}}\right), \frac{n^{i+1}t^p}{4n^{(i+1)p}}\right)\right\} \\ &\quad \geq \min_{m \leq i < m'+m} \min\left\{N_1\left(n^i x, n^i t\right), N_1\left(\frac{x}{n^{i+1}}, \frac{t}{n^{i+1}}\right)\right\} \\ &\quad = N_1(x, t) \end{aligned} \tag{29}$$

for all $x \in X$ and $t > 0$.

By a similar argument of the preceding part, we can define the limit $F(x) := N_2\text{-}\lim_{m\to\infty} J^m f(x)$ of the Cauchy sequence $\{J^m f(x)\}$ in the fuzzy Banach space Y. Moreover, putting $m = 0$ and replacing t with $t^q/(\sum_{i=0}^{m'-1}((1/2n)(n^p/n^2)^i + (n/2n^p)(n/n^p)^i))^q$ in the last inequality yield

$$N_2(f(x) - J^{m'} f(x), t) \geq N_1\left(x, \frac{t^q}{\left(\sum_{i=0}^{m'-1}\left((1/2n)(n^p/n^2)^i + (n/2n^p)(n/n^p)^i\right)\right)^q}\right) \tag{30}$$

for each $x \in X$ and for each $m' \in \mathbb{N}$.

To prove that F is a quadratic-additive function, it suffices to show that the last term of (16) tends to 1 as $m \to \infty$. By (4) and (27), we obtain

$$\begin{aligned} DJ^m f(x_1, x_2, \ldots, x_n) &= \frac{1}{2n^{2m}} Df(n^m x_1, \ldots, n^m x_n) \\ &\quad + \frac{1}{2n^{2m}} Df(-n^m x_1, \ldots, -n^m x_n) \\ &\quad + \frac{n^m}{2} Df\left(\frac{x_1}{n^m}, \ldots, \frac{x_n}{n^m}\right) \\ &\quad - \frac{n^m}{2} Df\left(-\frac{x_1}{n^m}, \ldots, -\frac{x_n}{n^m}\right) \end{aligned} \tag{31}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $m \in \mathbb{N}_0$. Hence, it follows from (F_3), (F_4), (F_5), and (5) that

$$\begin{aligned} &N_2\left(DJ^m f(x_1, x_2, \ldots, x_n), \frac{t}{5}\right) \\ &\quad \geq \min\left\{N_2\left(\frac{1}{2n^{2m}} Df(n^m x_1, \ldots, n^m x_n), \frac{t}{20}\right), \right. \\ &\qquad N_2\left(\frac{1}{2n^{2m}} Df(-n^m x_1, \ldots, -n^m x_n), \frac{t}{20}\right), \\ &\qquad N_2\left(\frac{n^m}{2} Df\left(\frac{x_1}{n^m}, \ldots, \frac{x_n}{n^m}\right), \frac{t}{20}\right), \\ &\qquad \left. N_2\left(-\frac{n^m}{2} Df\left(-\frac{x_1}{n^m}, \ldots, -\frac{x_n}{n^m}\right), \frac{t}{20}\right)\right\} \\ &\quad \geq \min\left\{N_1\left(x_1, 10^{-q} n^{(2q-1)m-q} t^q\right), \ldots, \right. \\ &\qquad N_1\left(x_n, 10^{-q} n^{(2q-1)m-q} t^q\right), \\ &\qquad N_1\left(x_1, 10^{-q} n^{(1-q)m-q} t^q\right), \ldots, \\ &\qquad \left. N_1\left(x_n, 10^{-q} n^{(1-q)m-q} t^q\right)\right\} \\ &\quad \longrightarrow 1 \text{ as } m \longrightarrow \infty \end{aligned} \tag{32}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$, since $1/2 < q < 1$. Hence, as we did for the preceding case of $q > 1$, it follows from (16) that $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in X$.

By a similar way as the first inequality in (6) follows from (15), we see that the second inequality of (6) follows from (30).

We will prove the uniqueness of F. Let G be another quadratic-additive function satisfying the second inequality

in (6). Notice that (24) also holds true for the case of $1/2 < q < 1$. Then, by (F_3), (F_4), (F_5), (6), (24), and (27), we have

$$\begin{aligned}
&N_2(F(x)-G(x),t)\\
&\quad= N_2(J^mF(x)-J^mG(x),t)\\
&\quad\geq \min\Big\{N_2\Big((J^mF-J^mf)(x),\frac{t}{2}\Big),\\
&\qquad N_2\Big((J^mf-J^mG)(x),\frac{t}{2}\Big)\Big\}\\
&\quad\geq \min\Big\{N_2\Big(\frac{(F-f)(n^mx)}{2n^{2m}},\frac{t}{8}\Big),\\
&\qquad N_2\Big(\frac{(f-G)(n^mx)}{2n^{2m}},\frac{t}{8}\Big),\\
&\qquad N_2\Big(\frac{(F-f)(-n^mx)}{2n^{2m}},\frac{t}{8}\Big),\\
&\qquad N_2\Big(\frac{(f-G)(-n^mx)}{2n^{2m}},\frac{t}{8}\Big),\\
&\qquad N_2\Big(\frac{n^m}{2}(F-f)\Big(\frac{x}{n^m}\Big),\frac{t}{8}\Big),\\
&\qquad N_2\Big(\frac{n^m}{2}(f-G)\Big(\frac{x}{n^m}\Big),\frac{t}{8}\Big),\\
&\qquad N_2\Big(-\frac{n^m}{2}(F-f)\Big(-\frac{x}{n^m}\Big),\frac{t}{8}\Big),\\
&\qquad N_2\Big(-\frac{n^m}{2}(f-G)\Big(-\frac{x}{n^m}\Big),\frac{t}{8}\Big)\Big\}\\
&\quad\geq \min\Big\{\sup_{0<t'<t} N_1\Big(x,\frac{2^{-q}n^{(2q-1)m}t'^q}{(n/(n^2-n^p)+n/(n^p-n))^q}\Big),\\
&\qquad \sup_{0<t'<t} N_1\Big(x,\frac{2^{-q}n^{(1-q)m}t'^q}{(n/(n^2-n^p)+n/(n^p-n))^q}\Big)\Big\}\\
&\quad\longrightarrow 1 \text{ as } m\longrightarrow\infty
\end{aligned}\tag{33}$$

for all $x \in X$ and $t > 0$, since $\lim_{m\to\infty} n^{(2q-1)m} = \lim_{m\to\infty} n^{(1-q)m} = \infty$ in this case. This implies that $N_2(F(x)-G(x),t) = 1$ for all $x \in X$ and $t > 0$, and hence $F(x) = G(x)$ for all $x \in X$ by (F_2).

Case 3. Finally, we consider the case of $0 < q < 1/2$. Let us define a function $J^mf : X \to Y$ by

$$\begin{aligned}
J^mf(x) := {}&\frac{n^{2m}}{2}\Big(f\Big(\frac{x}{n^m}\Big)+f\Big(-\frac{x}{n^m}\Big)\Big)\\
&+\frac{n^m}{2}\Big(f\Big(\frac{x}{n^m}\Big)-f\Big(-\frac{x}{n^m}\Big)\Big)
\end{aligned}\tag{34}$$

for all $x \in X$ and $m \in \mathbb{N}_0$. Then, we have $J^0f(x) = f(x)$, and a somewhat tedious manipulation yields

$$\begin{aligned}
&J^if(x)-J^{i+1}f(x)\\
&\quad= \frac{n^i(n^i+1)}{4}Df\Big(\frac{x}{n^{i+1}},\dots,\frac{x}{n^{i+1}}\Big)\\
&\qquad+\frac{n^i(n^i-1)}{4}Df\Big(-\frac{x}{n^{i+1}},\dots,-\frac{x}{n^{i+1}}\Big)
\end{aligned}\tag{35}$$

for all $x \in X$ and $i \in \mathbb{N}_0$.

If m and m' are nonnegative integers with $m' + m > m$, then it follows from (F_3), (F_4), and (5) that

$$\begin{aligned}
&N_2\Big(J^mf(x)-J^{m'+m}f(x),\sum_{i=m}^{m'+m-1}\Big(\frac{n^2}{n^p}\Big)^{i+1}\frac{t^p}{2n}\Big)\\
&\quad= N_2\Big(\sum_{i=m}^{m'+m-1}\big(J^if(x)-J^{i+1}f(x)\big),\sum_{i=m}^{m'+m-1}\Big(\frac{n^2}{n^p}\Big)^{i+1}\frac{t^p}{2n}\Big)\\
&\quad\geq \min_{m\leq i<m'+m}\min\Big\{N_2\Big(\frac{n^i(n^i+1)}{4}Df\Big(\frac{x}{n^{i+1}},\dots,\frac{x}{n^{i+1}}\Big),\\
&\qquad \frac{n^{i+1}(n^i+1)t^p}{4n^{(i+1)p}}\Big),\\
&\qquad N_2\Big(\frac{n^i(n^i-1)}{4}Df\Big(-\frac{x}{n^{i+1}},\dots,-\frac{x}{n^{i+1}}\Big),\\
&\qquad \frac{n^{i+1}(n^i-1)t^p}{4n^{(i+1)p}}\Big)\Big\}\\
&\quad\geq N_1\Big(\frac{x}{n^{i+1}},\frac{t}{n^{i+1}}\Big)\\
&\quad= N_1(x,t)
\end{aligned}\tag{36}$$

for all $x \in X$ and $t > 0$.

Similarly to the preceding cases, let us define a function $F : X \to Y$ by $F(x) := N_2\text{-}\lim_{m\to\infty} J^mf(x)$. Putting $m = 0$ and replacing t with $2^qn^qt^q/(\sum_{i=0}^{m'-1}(n^2/n^p)^{i+1})^q$ in the last inequality yield

$$N_2\big(f(x)-J^{m'}f(x),t\big)\geq N_1\Big(x,\frac{2^qn^qt^q}{\big(\sum_{i=0}^{m'-1}(n^2/n^p)^{i+1}\big)^q}\Big)\tag{37}$$

for all $x \in X$ and $t > 0$. Notice that

$$\begin{aligned}
&DJ^mf(x_1,x_2,\dots,x_n)\\
&\quad= \frac{n^{2m}}{2}Df\Big(\frac{x_1}{n^m},\dots,\frac{x_n}{n^m}\Big)+\frac{n^{2m}}{2}Df\Big(-\frac{x_1}{n^m},\dots,-\frac{x_n}{n^m}\Big)\\
&\qquad+\frac{n^m}{2}Df\Big(\frac{x_1}{n^m},\dots,\frac{x_n}{n^m}\Big)-\frac{n^m}{2}Df\Big(-\frac{x_1}{n^m},\dots,-\frac{x_n}{n^m}\Big)
\end{aligned}\tag{38}$$

for all $x_1, x_2, \dots, x_n \in X$. Hence, it follows from (F_3), (F_4) and (5) that

$$\begin{aligned}
&N_2\Big(DJ^mf(x_1,x_2,\dots,x_n),\frac{t}{5}\Big)\\
&\quad\geq \min\Big\{N_2\Big(\frac{n^{2m}}{2}Df\Big(\frac{x_1}{n^m},\dots,\frac{x_n}{n^m}\Big),\frac{t}{20}\Big),\\
&\qquad N_2\Big(\frac{n^{2m}}{2}Df\Big(-\frac{x_1}{n^m},\dots,-\frac{x_n}{n^m}\Big),\frac{t}{20}\Big),\\
&\qquad N_2\Big(\frac{n^m}{2}Df\Big(\frac{x_1}{n^m},\dots,\frac{x_n}{n^m}\Big),\frac{t}{20}\Big),\\
&\qquad N_2\Big(-\frac{n^m}{2}Df\Big(-\frac{x_1}{n^m},\dots,-\frac{x_n}{n^m}\Big),\frac{t}{20}\Big)\Big\}
\end{aligned}$$

$$\begin{aligned}
&\geq \min\Big\{N_1\left(x_1, 10^{-q}n^{(1-2q)m-q}t^q\right),\ldots,\\
&\qquad N_1\left(x_n, 10^{-q}n^{(1-2q)m-q}t^q\right),\\
&\qquad N_1\left(x_1, 10^{-q}n^{(1-q)m-q}t^q\right),\ldots,\\
&\qquad N_1\left(x_n, 10^{-q}n^{(1-q)\ m-q}t^q\right)\Big\}\\
&\longrightarrow 1 \text{ as } m \longrightarrow \infty
\end{aligned} \tag{39}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $t > 0$, since $0 < q < 1/2$. Thus, all terms on the right hand side of (16) tend to 1 as $m \to \infty$. Therefore, it follows from (F_2) and (16) that $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in X$. Moreover, using the same argument as in the first case, the third inequality in (6) follows from (37) for the case of $0 < q < 1/2$.

We will prove the uniqueness of F. Let $G : X \to Y$ be another quadratic-additive function satisfying the third inequality in (6). Then, (F_3), (F_4), (F_5), (6), (24), and by (34), we get

$$\begin{aligned}
&N_2(F(x) - G(x), t)\\
&\quad\geq \min\Big\{N_2\left((J^mF - J^mf)(x), \frac{t}{2}\right),\\
&\qquad N_2\left((J^mf - J^mG)(x), \frac{t}{2}\right)\Big\}\\
&\quad\geq \min\Big\{N_2\left(\frac{n^{2m}}{2}(F-f)\left(\frac{x}{n^m}\right), \frac{t}{8}\right),\\
&\qquad N_2\left(\frac{n^{2m}}{2}(f-G)\left(\frac{x}{n^m}\right), \frac{t}{8}\right),\\
&\qquad N_2\left(\frac{n^{2m}}{2}(F-f)\left(-\frac{x}{n^m}\right), \frac{t}{8}\right),\\
&\qquad N_2\left(\frac{n^{2m}}{2}(f-G)\left(-\frac{x}{n^m}\right), \frac{t}{8}\right),\\
&\qquad N_2\left(\frac{n^{m}}{2}(F-f)\left(\frac{x}{n^m}\right), \frac{t}{8}\right),\\
&\qquad N_2\left(\frac{n^{m}}{2}(f-G)\left(\frac{x}{n^m}\right), \frac{t}{8}\right),\\
&\qquad N_2\left(-\frac{n^{m}}{2}(F-f)\left(-\frac{x}{n^m}\right), \frac{t}{8}\right),\\
&\qquad N_2\left(-\frac{n^{m}}{2}(f-G)\left(-\frac{x}{n^m}\right), \frac{t}{8}\right)\Big\}\\
&\quad\geq \sup_{0<t'<t} N_1\left(x,\ 2^{-q}\left(n^p - n^2\right)^q n^{(1-2q)m-q}t'^q\right)\\
&\quad\longrightarrow 1 \text{ as } m \longrightarrow \infty
\end{aligned} \tag{40}$$

for all $x \in X$ and $t > 0$. This implies that $N_2(F(x) - G(x), t) = 1$ for all $x \in X$ and $t > 0$, and hence $F(x) = G(x)$ for all $x \in X$ by (F_2). □

4. Applications

Corollary 3. *If an even function $f : X \to Y$ satisfies all of the conditions of Theorem 2, then there exists a unique quadratic function $F : X \to Y$ such that*

$$N_2(F(x) - f(x), t) \geq \sup_{0<t'<t} N_1\left(x, 2^q\left|n^p - n^2\right|^q n^{-q}t'^q\right) \tag{41}$$

for all $x \in X$ and $t > 0$, where $q \notin \{1/2, 1\}$.

Proof. Let $J^mf : X \to Y$ be defined as in Theorem 2. Since f is an even function, it follows from (8), (27), and (34) that

$$J^mf(x) = \begin{cases} \dfrac{1}{2n^{2m}}\left(f(n^mx) + f(-n^mx)\right) & \text{if } q > 1/2,\ q \neq 1,\\[2ex] \dfrac{n^{2m}}{2}\left(f\left(\dfrac{x}{n^m}\right) + f\left(-\dfrac{x}{n^m}\right)\right) & \text{if } 0 < q < 1/2 \end{cases} \tag{42}$$

for all $x \in X$. Notice that $J^0f(x) = f(x)$ and

$$\begin{aligned}
&J^if(x) - J^{i+1}f(x)\\
&\quad= \begin{cases} -\dfrac{Df(n^ix, \ldots, n^ix)}{4n^{2i+2}} - \dfrac{Df(-n^ix, \ldots, -n^ix)}{4n^{2i+2}} \\ \qquad\qquad \text{if } q > 1/2,\ q \neq 1,\\[2ex] \dfrac{n^{2i}}{4}Df\left(\dfrac{x}{n^{i+1}}, \ldots, \dfrac{x}{n^{i+1}}\right) + \dfrac{n^{2i}}{4}Df\left(-\dfrac{x}{n^{i+1}}, \ldots, -\dfrac{x}{n^{i+1}}\right) \\ \qquad\qquad \text{if } 0 < q < 1/2 \end{cases}
\end{aligned} \tag{43}$$

for all $x \in X$ and $i \in \mathbb{N}_0$. From these equalities, using the similar method as presented in Theorem 2, we obtain the quadratic-additive function F satisfying (41). Notice that $F(x) := N_2\text{-}\lim_{n\to\infty} J^nf(x)$ for all $x \in X$, F is even, and $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in X$. Hence, we get

$$\begin{aligned}
&F(x+y) + F(x-y) - 2F(x) - 2F(y)\\
&\qquad = \frac{1}{2}DF(x, y, 0, \ldots, 0) = 0
\end{aligned} \tag{44}$$

for all x, $y \in X$. This implies that F is a quadratic function. □

Corollary 4. *If an odd function $f : X \to Y$ satisfies all of the conditions of Theorem 2, then there exists a unique additive function $F : X \to Y$ such that*

$$N_2(F(x) - f(x), t) \geq \sup_{0<t'<t} N_1\left(x, 2^qn^{-q}\left|n - n^p\right|^q t'^q\right) \tag{45}$$

for all $x \in X$ and $t > 0$, where $p = 1/q$ and $q \notin \{1/2, 1\}$.

Proof. Let $J^mf : X \to Y$ be defined as in Theorem 2. Since f is an odd function, we obtain

$$J^mf(x) = \begin{cases} \dfrac{1}{2n^m}\left(f(n^mx) - f(-n^mx)\right) & \text{if } q > 1,\\[2ex] \dfrac{n^m}{2}\left(f\left(\dfrac{x}{n^m}\right) - f\left(-\dfrac{x}{n^m}\right)\right) \\ \qquad \text{if } 0 < q < 1\ q \neq 1/2 \end{cases} \tag{46}$$

for all $x \in X$. Notice that $J^0 f(x) = f(x)$ and

$$J^i f(x) - J^{i+1} f(x) = \begin{cases} -\dfrac{1}{4n^{i+1}} Df(n^i x, \ldots, n^i x) + \dfrac{1}{4n^{i+1}} Df(-n^i x, \ldots, -n^i x) & \text{if } q > 1, \\ \dfrac{n^i}{4} Df\left(\dfrac{x}{n^{i+1}}, \ldots, \dfrac{x}{n^{i+1}}\right) - \dfrac{n^i}{4} Df\left(-\dfrac{x}{n^{i+1}}, \ldots, -\dfrac{x}{n^{i+1}}\right) & \text{if } 0 < q < 1, \quad q \neq 1/2 \end{cases} \tag{47}$$

for all $x \in X$ and $i \in \mathbb{N}_0$. From these equalities, by using a similar method in Theorem 2, we obtain the quadratic-additive function F satisfying (45). Notice that $F(x) := N_2\text{-}\lim_{n\to\infty} J^n f(x)$ for all $x \in X$, F is odd, and $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in X$. Hence, we get

$$F(x+y) - F(x) - F(y) = \frac{1}{2} DF(x, -y, 0, \ldots, 0) = 0 \tag{48}$$

for all $x, y \in X$. This implies that F is an additive function. □

We can use Theorem 2 to prove a classical result in the framework of normed spaces. Let $(X, \|\cdot\|)$ be a normed space and let $(Y, \|\cdot\|)$ be a Banach space. Then we can define fuzzy norms by

$$N_X(x,t) = \begin{cases} 0 & \text{if } t \le \|x\|, \\ 1 & \text{if } t > \|x\|, \end{cases} \qquad N_Y(y,t) = \begin{cases} 0 & \text{if } t \le \|y\|, \\ 1 & \text{if } t > \|y\|, \end{cases} \tag{49}$$

where $x \in X$, $y \in Y$ and $t \in \mathbb{R}$ (see [22]). Suppose a function $f : X \to Y$ satisfies

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p \tag{50}$$

for all $x_1, x_2, \ldots, x_n \in X$, where $p > 0$ and $p \notin \{1, 2\}$. Let N_Y be a fuzzy norm on Y. Then we get

$$N_Y(Df(x_1, \ldots, x_n),\ t_1 + \cdots + t_n) = \begin{cases} 0 & \text{if } t_1 + \cdots + t_n \le \|Df(x_1, \ldots, x_n)\|, \\ & \text{if } t_1 + \cdots + t_n > \|Df(x_1, \ldots, x_n)\| \end{cases} \tag{51}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $t_1, \ldots, t_n \in \mathbb{R}$.

Consider the case

$$N_Y(Df(x_1, \ldots, x_n), t_1 + \cdots + t_n) = 0. \tag{52}$$

This implies that

$$\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p \ge \|Df(x_1, x_2, \ldots, x_n)\| \ge t_1 + \cdots + t_n. \tag{53}$$

Hence, there exists an $i \in \{1, \ldots, n\}$ such that $\|x_i\|^p \ge t_i$. Thus, with $q = 1/p$, we have

$$\min\left\{N_X\left(x_1, t_1^q\right), \ldots, N_X\left(x_n, t_n^q\right)\right\} = 0 \tag{54}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $t_1, \ldots, t_n > 0$. Therefore, the inequality

$$N_Y(Df(x_1, \ldots, x_n), t_1 + \cdots + t_n) \ge \min\left\{N_X\left(x_1, t_1^q\right), \ldots, N_X\left(x_n, t_n^q\right)\right\} \tag{55}$$

holds true. That is, f is a fuzzy q-almost quadratic-additive function, and by Theorem 2, we obtain the following stability result.

Corollary 5. *Let $(X, \|\cdot\|)$ be a normed linear space and let $(Y, \|\cdot\|)$ be a Banach space. Assume that n is an integer greater than 1 and p is a positive real number with $p \notin \{1, 2\}$. If a function $f : X \to Y$ satisfies*

$$\|Df(x_1, x_2, \ldots, x_n)\| \le \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p \tag{56}$$

for all $x_1, x_2, \ldots, x_n \in X$, then there exists a unique quadratic-additive function $F : X \to Y$ such that

$$\|F(x) - f(x)\| = \begin{cases} \dfrac{n}{2|n^p - n^2|} \|x\|^p & \text{if } p > 2, \\ \left(\dfrac{n}{2|n^p - n|} + \dfrac{n}{2|n^2 - n^p|}\right) \|x\|^p & \text{if } 1 < p < 2, \\ \dfrac{n\|x\|^p}{2(n - n^p)} & \text{if } p < 1 \end{cases} \tag{57}$$

for all $x \in X$.

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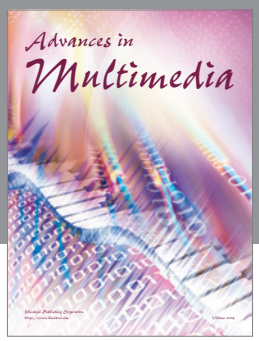

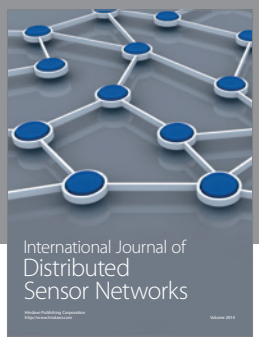

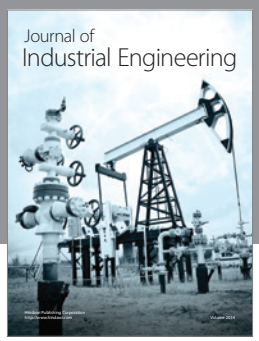

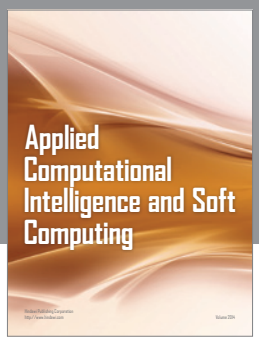

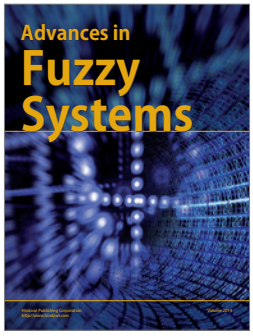

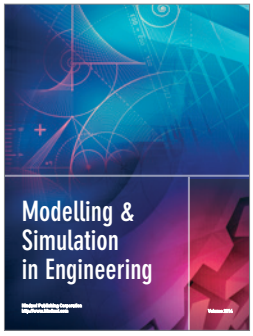

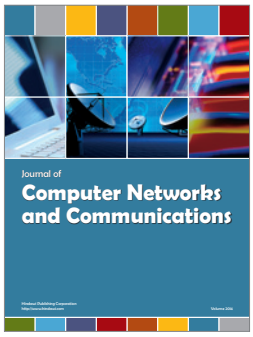

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