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Research Article

$\mathcal{H}_\infty$ Stability Conditions for Fuzzy Neural Networks

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This paper presents a novel approach to assess the stability of fuzzy neural networks. First, we propose a new condition for the $\mathcal{H}_\infty$ stability of fuzzy neural networks. Second, a new $\mathcal{H}_\infty$ stability condition based on linear matrix inequality (LMI) is presented for fuzzy neural networks. These conditions also ensure asymptotic stability without external input.

1. Introduction

In this paper, we consider the following Takagi-Sugeno (T-S) fuzzy Hopfield neural network:

$$\begin{aligned}&\textbf{Fuzzy Rule } i\text{:}\\&\textbf{IF } \omega_1 \text{ is } \mu_{i1} \text{ and} \ldots \omega_s \text{ is } \mu_{is} \textbf{ THEN}\\&\dot{x}(t) = A_i x(t) + W_i \phi(x(t)) + J(t),\end{aligned} \tag{1}$$

where $x(t) = [x_1(t) \cdots x_n(t)]^T \in R^n$ is the state vector, $A_i = \mathrm{diag}\{-a_{(i,1)}, \ldots, -a_{(i,n)}\} \in R^{n\times n}$ ($a_{(i,k)} > 0$, $k = 1, \ldots, n$) is the self-feedback matrix, $W_i \in R^{n\times n}$ is the connection weight matrix, $\phi(x(t)) = [\phi_1(x(t)) \cdots \phi_n(x(t))]^T : R^n \to R^n$ is the nonlinear function vector satisfying the global Lipschitz condition with Lipschitz constant $L_\phi > 0$, $J(t) \in R^n$ is an external input vector, ω_j ($j = 1, \ldots, s$) is the premise variable, μ_{ij} ($i = 1, \ldots, r$, $j = 1, \ldots, s$) is the fuzzy set that is characterized by membership function, r is the number of the IF-THEN rules, and s is the number of the premise variables. Using a singleton fuzzifier, product fuzzy inference, and weighted average defuzzifier, the system (1) is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(\omega)\left[A_i x(t) + W_i \phi(x(t)) + J(t)\right], \tag{2}$$

where $\omega = [\omega_1, \ldots, \omega_s]$, $h_i(\omega) = w_i(\omega)/\sum_{j=1}^{r} w_j(\omega)$, and $w_i : R^s \to [0,1]$ ($i = 1, \ldots, r$) is the membership function of the system with respect to the fuzzy rule i. h_i can be regarded as the normalized weight of each IF-THEN rule, and it satisfies $h_i(\omega) \geq 0$, $\sum_{i=1}^{r} h_i(\omega) = 1$. The T-S fuzzy models have been of great importance in academic research and industrial applications. Essentially, these models are based on the use of a set of fuzzy rules to describe nonlinear systems in terms of a set of local linear models that are smoothly connected by fuzzy membership functions [1]. The T-S fuzzy models can be also used to represent several complex nonlinear systems by having a set of neural networks as its consequent parts. Some stability problems for neural networks based on T-S fuzzy models have been investigated [2–8]. In [2, 3], some linear matrix inequality (LMI) conditions for the stochastic stability of fuzzy neural networks were derived. The authors in [4] investigated the robust stability of uncertain fuzzy neural networks and proposed a delay-dependent condition such that these neural networks are asymptotically stable. In [5], Ahn presented a new delay-dependent state estimation method for fuzzy neural networks. In [6], Mathiyalagan et al. further dealt with the stability problem of fuzzy neural networks with various activation functions. Recently, some conditions for the passivity analysis of fuzzy neural networks were proposed in [7, 8]. Despite these advances in the stability analysis of fuzzy neural networks, most research results were restricted to fuzzy neural networks without external disturbance. With the existing results, it is difficult to guarantee the stability of fuzzy neural networks with external disturbance. Thus, it is desirable to investigate some stability conditions of fuzzy neural networks with external disturbance. This situation motivates our investigation.

On the other hand, model uncertainties always exist, along with a lack of statistical information on the signals in real physical systems. This has led in recent years to an interest in an $\mathcal{H}_\infty$ approach [9]. Analysis and synthesis in an $\mathcal{H}_\infty$ framework have advantages such as effective disturbance attenuation, less sensitivity to uncertainties, and many practical applications. This paper provides an answer to the question of whether an $\mathcal{H}_\infty$ stability condition can be obtained for T-S fuzzy neural networks. To the best of our knowledge, the $\mathcal{H}_\infty$ analysis of T-S fuzzy neural networks has not yet been reported in the literature.

In this paper, we present new $\mathcal{H}_\infty$ stability conditions for T-S fuzzy Hopfield neural networks. The conditions proposed in this paper are a new contribution to the stability analysis of fuzzy neural networks. In contrast to the existing stability conditions for fuzzy neural networks, an advantage of the proposed conditions is to attenuate the effect of external disturbance to a prescribed level. This paper is organized as follows. In Section 2, new $\mathcal{H}_\infty$ stability conditions are derived. In Section 3, a numerical example is given, and finally, conclusions are presented in Section 4.

2. New $\mathcal{H}_\infty$ Stability Conditions

Given a prescribed level of noise attenuation $\gamma > 0$, the purpose of this paper is to find conditions such that the fuzzy neural network (2) with $J(t) = 0$ is asymptotically stable ($\lim_{t\to\infty} x(t) = 0$) and

$$\int_0^\infty x^T(t)x(t)dt < \gamma^2 \int_0^\infty J^T(t)J(t)dt, \tag{3}$$

under zero-initial conditions for all nonzero $J(t) \in L_2[0,\infty)$, where $L_2[0,\infty)$ is the space of square integrable vector functions over $[0,\infty)$.

Now we derive an $\mathcal{H}_\infty$ stability condition of the T-S fuzzy Hopfiel neural network (2) in the following theorem.

Theorem 1. *For a given level $\gamma > 0$, the T-S fuzzy Hopfiel neural network (2) is $\mathcal{H}_\infty$ stable if*

$$\begin{gathered} \|W_i\| < \frac{1}{L_\phi}\sqrt{k_i - 1 - \left(1 + \frac{1}{\gamma^2}\right)\|P\|^2}, \\ \|P\| < \sqrt{\frac{k_i - 1}{1 + 1/\gamma^2}}, \quad k_i > 1, \ P = P^T > 0, \end{gathered} \tag{4}$$

where P satisfies the Lyapunov inequality $A_i^T P + PA_i < -k_i I$ for $i = 1, \ldots, r$.

Proof. We consider the function $V(t) = x^T(t)Px(t)$. Its time derivative along the trajectory of (2) satisfies

$$\begin{aligned} \dot V(t) < \sum_{i=1}^r h_i(\omega)\Big\{&-k_i x^T(t)x(t) + 2x^T(t)PW_i\phi(x(t)) \\ &+2x^T(t)PJ(t)\Big\}. \end{aligned} \tag{5}$$

By Young's inequality [10], we have

$$\begin{aligned} 2x^T(t)PW_i\phi(x(t)) &\le x^T(t)PP^Tx(t) \\ &\quad + \phi^T(x(t))W_i^TW_i\phi(x(t)) \\ &\le \|P\|^2\|x(t)\|^2 + \|W_i\|^2\|\phi(x(t))\|^2 \\ &\le \|P\|^2\|x(t)\|^2 + L_\phi^2\|W_i\|^2\|x(t)\|^2, \\ 2x^T(t)PJ(t) &\le \frac{1}{\gamma^2}x^T(t)PP^Tx(t) + \gamma^2 J^T(t)J(t) \\ &\le \frac{1}{\gamma^2}\|P\|^2\|x(t)\|^2 + \gamma^2\|J(t)\|^2. \end{aligned} \tag{6}$$

Substituting (6) into (5), we finally obtain

$$\begin{aligned} \dot V(t) &< \sum_{i=1}^r h_i(\omega) \\ &\quad \times \left\{-\left(k_i - \left(1 + \frac{1}{\gamma^2}\right)\|P\|^2 - L_\phi^2\|W_i\|^2\right)\right. \\ &\qquad \left.\times\|x(t)\|^2 + \gamma^2\|J(t)\|^2\right\} \\ &= -\sum_{i=1}^r h_i(\omega)\left(k_i - 1 - \left(1 + \frac{1}{\gamma^2}\right)\|P\|^2 - L_\phi^2\|W_i\|^2\right) \\ &\quad \times \|x(t)\|^2 + \sum_{i=1}^r h_i(\omega)\left\{-\|x(t)\|^2 + \gamma^2\|J(t)\|^2\right\}. \end{aligned} \tag{7}$$

If the following condition is satisfied

$$\left(k_i - 1 - \left(1 + \frac{1}{\gamma^2}\right)\|P\|^2 - L_\phi^2\|W_i\|^2\right) > 0, \tag{8}$$

we have

$$\dot V(t) < -\|x(t)\|^2 + \gamma^2\|J(t)\|^2. \tag{9}$$

Integrating both sides of (9) from 0 to ∞ gives

$$V(\infty) - V(0) < -\int_0^\infty x^T(t)x(t)dt + \gamma^2\int_0^\infty J^T(t)J(t)dt. \tag{10}$$

Since $V(\infty) \ge 0$ and $V(0) = 0$, we have the relation (3). The condition (8) implies

$$\begin{gathered} \|W_i\|^2 < \frac{1}{L_\phi^2}\left(k_i - 1 - \left(1 + \frac{1}{\gamma^2}\right)\|P\|^2\right), \\ \|P\|^2 < \frac{k_i - 1}{1 + 1/\gamma^2}, \end{gathered} \tag{11}$$

for $i = 1, \ldots, r$. This completes the proof. □

Corollary 2. *When $J(t) = 0$, the condition (4) ensures that the fuzzy neural network (2) is asymptotically stable.*

Proof. When $J(t) = 0$, from (9), we have

$$\begin{aligned} \dot V(t) &< -\|x(t)\|^2 \\ &< 0, \quad \forall x(t) \neq 0. \end{aligned} \tag{12}$$

This relation ensures that the fuzzy neural network (2) is asymptotically stable from Lyapunov stability theory. This completes the proof. □

Next, we propose a new LMI-based condition for the $\mathcal{H}_\infty$ stability of the fuzzy neural network (2). This LMI condition can be facilitated readily via standard numerical algorithms [11, 12].

Theorem 3. *For a given level $\gamma > 0$, the T-S fuzzy Hopfiel neural network (2) is $\mathcal{H}_\infty$ stable if there exist a positive symmetric matrix P and a positive scalar ϵ such that*

$$\begin{bmatrix} A_i^T P + PA_i + \left(\epsilon L_\phi^2 + 1\right) I & PW_i & P \\ W_i^T P & -\epsilon I & 0 \\ P & 0 & -\gamma^2 I \end{bmatrix} < 0, \qquad (13)$$

for $i = 1, \ldots, r$.

Proof. Consider the function $V(t) = x^T(t)Px(t)$. By Young's inequality [10], it is clear that the following relation is satisfied:

$$\epsilon\left[L_\phi^2 x^T(t)x(t) - \phi^T(x(t))\phi(x(t))\right] \geq 0. \qquad (14)$$

By using (14), the time derivative of $V(t)$ along the trajectory of (2) is

$$\begin{aligned}
&\dot{V}(t) \\
&= \sum_{i=1}^{r} h_i(\omega)\Big\{x^T(t)\left[A_i^T P + PA_i\right]x(t) + 2x^T(t)PW_i\phi(x(t)) \\
&\qquad +2x^T(t)PJ(t)\Big\} \\
&\leq \sum_{i=1}^{r} h_i(\omega)\Big\{x^T(t)\left[A_i^T P + PA_i\right]x(t) \\
&\qquad + 2x^T(t)PW_i\phi(x(t)) + 2x^T(t)PJ(t) \\
&\qquad +\epsilon\left[L_\phi^2 x^T(t)x(t) - \phi^T(x(t))\phi(x(t))\right]\Big\} \\
&= \sum_{i=1}^{r} h_i(\omega)\begin{bmatrix} x(t) \\ \phi(x(t)) \\ J(t) \end{bmatrix}^T \\
&\quad \times \begin{bmatrix} A_i^T P + PA_i + \left(\epsilon L_\phi^2 + 1\right) I & PW_i & P \\ W_i^T P & -\epsilon I & 0 \\ P & 0 & -\gamma^2 I \end{bmatrix}\begin{bmatrix} x(t) \\ \phi(x(t)) \\ J(t) \end{bmatrix} \\
&\quad + \sum_{i=1}^{r} h_i(\omega)\left\{-x^T(t)x(t) + \gamma^2 J^T(t)J(t)\right\}.
\end{aligned} \qquad (15)$$

If the LMI (13) is satisfied, we have

$$\dot{V}(t) < -x^T(t)x(t) + \gamma^2 J^T(t)J(t). \qquad (16)$$

Integrating both sides of (16) from 0 to ∞ gives

$$V(\infty) - V(0) < -\int_0^\infty x^T(t)x(t)dt + \gamma^2\int_0^\infty J^T(t)J(t)dt. \qquad (17)$$

Since $V(\infty) \geq 0$ and $V(0) = 0$, we have the relation (3). This completes the proof. □

Corollary 4. *When $J(t) = 0$, the LMI condition (13) ensures that the fuzzy neural network (2) is asymptotically stable.*

Proof. When $J(t) = 0$, from (16), we have

$$\begin{aligned} \dot{V}(t) &< -x^T(t)x(t) \\ &< 0, \quad \forall x(t) \neq 0. \end{aligned} \qquad (18)$$

This relation ensures that the fuzzy neural network (2) is asymptotically stable from Lyapunov stability theory. This completes the proof. □

Remark 5. The proposed scheme can be applied to several real-world problems. For example, in image-based visual servo control of an unknown aerial robot vehicle [13, 14], a possible application of the proposed scheme is to use fuzzy neural networks to build a mathematical model from experimental data and then check the $\mathcal{H}_\infty$ stability of this model. If the $\mathcal{H}_\infty$ stability of this model is guaranteed by Theorem 1 or Theorem 3, we can use this model in the implementation of the nonlinear controller. Learning algorithms already exist, which ensures error convergence for fuzzy neural networks [15]. Even in presence of model mismatching, the identification error remains bounded. Therefore, from the point of view of control, the $\mathcal{H}_\infty$ stability analysis for fuzzy neural networks is a prerequisite for successful applications of the networks.

Remark 6. The condition (4) in Theorem 1 is a matrix norm-based criterion for checking the $\mathcal{H}_\infty$ stability of fuzzy neural networks. But the condition (13) in Theorem 3 is an LMI based criterion. This condition can be facilitated readily via several numerical algorithms [11, 12]. Hence, this condition is computationally attractive.

3. Numerical Example

Consider the following T-S fuzzy Hopfield neural network:

Fuzzy Rule 1:
IF ω_1 is μ_{11} and... ω_s is μ_{1s} **THEN** (19)
$\dot{x}(t) = A_1 x(t) + W_1\phi(x(t)) + J(t)$,

Fuzzy Rule 2:
IF ω_1 is μ_{21} and... ω_s is μ_{2s} **THEN** (20)
$\dot{x}(t) = A_2 x(t) + W_2\phi(x(t)) + J(t)$,

where

$$\begin{gathered}
x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad J(t) = \begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix}, \\
\phi(x(t)) = \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix}, \\
A_1 = \begin{bmatrix} -2.8 & 0 \\ 0 & -4.1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -3.9 & 0 \\ 0 & -2.8 \end{bmatrix}, \\
W_1 = \begin{bmatrix} -1 & 0.4 \\ 0 & -0.1 \end{bmatrix}, \qquad W_2 = \begin{bmatrix} 0.2 & -0.8 \\ 0.4 & 0.5 \end{bmatrix}.
\end{gathered} \qquad (21)$$

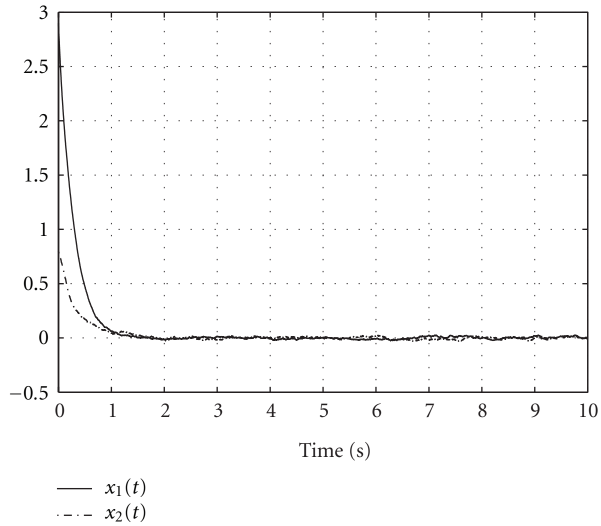

Figure 1: Responses of the state vector $x(t)$.

The fuzzy membership functions are taken as $h_1(\omega) = \sin^2(x_1(t))$ and $h_2(\omega) = \cos^2(x_1(t))$. By applying Theorem 3 via the Matlab LMI Control Toolbox [12], we have the following feasible solution:

$$P = \begin{bmatrix} 0.6925 & 0.0101 \\ 0.0101 & 0.9734 \end{bmatrix}, \qquad \epsilon = 0.8563, \tag{22}$$

with the $\mathcal{H}_\infty$ performance index $\gamma = 0.6$. When the initial condition is given by $x(0) = [2.9 \;\; 0.8]^T$ and the external disturbance $J_i(t)$ $(i = 1, 2)$ is given by a Gaussian noise with mean 0 and variance 1, Figure 1 shows the trajectories of state vector $x(t)$. This simulation result confirms that the proposed condition guarantees to reduce the effect of the external disturbance $J(t)$ on the state vector $x(t)$.

4. Conclusion

In this paper, as our main contribution, we establish new conditions for the weights of the interconnection matrix of fuzzy neural networks, in order to ensure $\mathcal{H}_\infty$ stability. These conditions also guarantee asymptotic stability for external input identically equal to zero.

References

[1] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 15, no. 1, pp. 116–132, 1985.

[2] H. Huang, D. W. C. Ho, and J. Lam, "Stochastic stability analysis of fuzzy Hopfield neural networks with time-varying delays," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 52, no. 5, pp. 251–255, 2005.

[3] H. Li, B. Chen, Q. Zhou, and W. Qian, "Robust stability for uncertain delayed fuzzy Hopfield neural networks with Markovian jumping parameters," *IEEE Transactions on Systems, Man, and Cybernetics, Part B*, vol. 39, no. 1, pp. 94–102, 2009.

[4] M. S. Ali and P. Balasubramaniam, "Stability analysis of uncertain fuzzy Hopfield neural networks with time delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 6, pp. 2776–2783, 2009.

[5] C. K. Ahn, "Delay-dependent state estimation for T-S fuzzy delayed Hopfield neural networks," *Nonlinear Dynamics*, vol. 61, no. 3, pp. 483–489, 2010.

[6] K. Mathiyalagan, R. Sakthivel, and S. Marshal Anthoni, "New stability and stabilization criteria for fuzzy neural networks with various activation functions," *Physica Scripta*, vol. 84, no. 1, Article ID 015007, 2011.

[7] R. Sakthivel, K. Mathiyalagan, and S. M. Anthoni, "Design of a passification controller for uncertain fuzzy Hopfield neural networks with time-varying delays," *Physica Scripta*, vol. 84, Article ID 045024, 2011.

[8] K. Mathiyalagan, R. Sakthivel, and S. M. Anthoni, "New robust passivity criteria for stochastic fuzzy BAM neural networks with time-varying delays," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 3, pp. 1392–1407, 2012.

[9] A. Stoorvogel, *The $\mathcal{H}_\infty$ Control Problem: A State-Space Approach*, Prentice Hall, UK, 1992.

[10] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, vol. 60 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1989.

[11] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *SIAM Studies in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, Pa, USA, 1994.

[12] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox*, The Mathworks Inc., 1995.

[13] E. B. Kosmatopoulos, "Control of unknown nonlinear systems with efficient transient performance using concurrent exploitation and exploration," *IEEE Transactions on Neural Networks*, vol. 21, no. 8, Article ID 5510184, pp. 1245–1261, 2010.

[14] T. Hamel and R. Mahony, "Image based visual servo control for a class of aerial robotic systems," *Automatica*, vol. 43, no. 11, pp. 1975–1983, 2007.

[15] C. K. Ahn, "Takagi-Sugeno fuzzy hopfield neural networks for $\mathcal{H}_\infty$ nonlinear system identification," *Neural Processing Letters*, vol. 34, no. 1, pp. 59–70, 2011.