Research Article

A Note on $(\Phi E_1, \Phi E_2)$ -Convex Fuzzy Processes

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We first define the concept of $(\Phi E_1, \Phi E_2)$ -convex fuzzy processes. Second, we present some basic properties of such processes.

1. Introduction

In recent years, many generalizations of convexity have appeared in the literature aiming at applications to duality theory and optimality conditions. In 1997, Pini and Singh [1] introduced (Φ_1, Φ_2) -convex functions and studied some of their properties. They showed that some of the well-known classes of generalized convex functions (e.g., B-vex functions [2], geodesic convex functions [3], and invex functions [4]) form subclasses of the class of (Φ_1, Φ_2) -convex functions. In 1999, Youness [5] showed that many results for convex sets and convex functions actually hold for a wider class of sets and functions, called E -convex sets and E -convex functions.

Convex processes were studied first by Rockafellar [6] who was interested in extending properties of linear transformations to a large class of maps preserving convexity and which arise naturally in economic theory.

The extension of this notion to the fuzzy framework was done by Matłoka [7, 8] and was investigated by Syau et al. [9] and by Chalco-Cano et al. [10].

In this work, we extend the notion of ΦE -convex set, ΦE -convex fuzzy set, and $(\Phi E_1, \Phi E_2)$ -convex fuzzy process. First, we will present the preliminary definitions and next the main properties of $(\Phi E_1, \Phi E_2)$ -convex fuzzy process.

2. Preliminaries

Let X_i denote a subset of the n_i -dimensional Euclidean space R^{n_i} . Assume that Φ_i is a map satisfying the following assumption:

- (i) $\Phi_i : X_i \times X_i \times [0, 1] \rightarrow R^{n_i}$,
- (ii) $\Phi_i(x, y, 0) = y$, $\Phi_i(x, x, \lambda) = x$, $\forall x, y \in X_i$, $\lambda \in [0, 1]$.

For any subsets A, B , of X_i let us define

$$\Phi_i(A, B, \lambda) = \{\Phi_i(x, y, \lambda) : x \in A, y \in B\}. \quad (1)$$

Definition 1 (see [1]). A set X_i is Φ_i -convex if $\Phi_i(x_1, x_2, \lambda) \in X_i$ for all $x_1, x_2 \in X_i$, $\lambda \in [0, 1]$.

Remark 2. The intersection of Φ_i -convex sets is still Φ_i -convex.

Remark 3. Let X_i be a convex subset of R^{n_i} , and $\Phi_i(x_1, x_2, \lambda) = \lambda x_1 + (1 - \lambda)x_2$. Then a convex set is Φ_i -convex.

Remark 4. If $\eta : R^{n_i} \times R^{n_i} \rightarrow R^{n_i}$, X_i is a preinvex set with respect to η (see [11]), $\Phi_i(x_1, x_2, \lambda) = x_2 + \lambda \cdot \eta(x_1, x_2)$, then X_i is a Φ_i -convex set.

Assume that $E_1 : R^{n_i} \rightarrow R^{n_i}$.

Definition 5 (see [5]). A set $X_i \subset R^{n_i}$ is said to be E_i -convex if $(1 - \lambda)E_i(x_1) + \lambda E_i(x_2) \in X_i$, for each $x_1, x_2 \in X_i$ and $\lambda \in [0, 1]$.

Definition 6. A set $X_i \subset R^{n_i}$ is said to be ΦE_i -convex if $\Phi_i(E_i(x_1), E_i(x_2), \lambda) \in X_i$, for each $x_1, x_2 \in X_i$ and $\lambda \in [0, 1]$.

Remark 7. If $\Phi_i(x_1, x_2, \lambda) = \lambda x_1 + (1 - \lambda)x_2$ and E_i is the identity map then a convex set is ΦE_i -convex.

Definition 8. Let X_i be a ΦE_i -convex set. A function $f : X_i \rightarrow R$ is said to be ΦE_i -quasiconvex on X_i if for any $x_1, x_2 \in X_i$ and $\lambda \in [0, 1]$

$$f(\Phi_i(E_i(x_1), E_i(x_2), \lambda)) \geq \min(f(x_1), f(x_2)). \quad (2)$$

Let $C : R^{n_i} \rightarrow [0, 1]$ denote a fuzzy set in R^{n_i} .

Definition 9. A fuzzy set C is called ΦE_i -convex if and only if

$$C(\Phi_i(E_i(x_1), E_i(x_2), \lambda)) \geq \min(C(x_1), C(x_2)), \quad (3)$$

for all $x_1, x_2 \in R^n$.

Definition 10. An α -cut of a fuzzy set C is defined as follows:

$$C^\alpha = \begin{cases} \{x : C(x) \geq \alpha\} & \text{for } \alpha \in (0, 1] \\ \{x : C(x) > \alpha\} & \text{for } \alpha = 0. \end{cases} \quad (4)$$

Proposition 11. If C is ΦE_i -convex fuzzy set then C^α is ΦE_i -convex (crisp) set.

Proof. We have to prove that if $x_1, x_2 \in C^\alpha$ then for any $\lambda \in [0, 1]$, $\Phi_i(E_i(x_1), E_i(x_2), \lambda) \in C^\alpha$. So, taking into account the above definitions, we observe that if $x_1, x_2 \in C^\alpha$ then $C(x_1) \geq \alpha$ and $C(x_2) \geq \alpha$ and $\min(C(x_1), C(x_2)) \geq \alpha$. So, $C(\Phi_i(E_i(x_1), E_i(x_2), \lambda)) \geq \min(C(x_1), C(x_2)) \geq \alpha$. This means that $\Phi_i(E_i(x_1), E_i(x_2), \lambda) \in C^\alpha$. $\square$

3. Main Results

In this section, we present the definition and some properties of the $(\Phi E_1, \Phi E_2)$ -convex fuzzy processes.

Let X_i denote ΦE_i -convex set and $F(X_i)$ the set of all non-void fuzzy sets in X_i ($i = 1, 2, 3$).

Definition 12. A mapping A from X_1 to $F(X_2)$ is called $(\Phi E_1, \Phi E_2)$ -convex fuzzy process if and only if for any $x_1, x_2 \in X_1$ and $\lambda \in [0, 1]$ and $y \in X_2$

$$\begin{aligned} & A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))(y) \\ & \geq \sup_{\substack{y_1, y_2 \in X_2: \\ y = \Phi_2(E_2(y_1), E_2(y_2), \lambda)}} \min(A(x_1)(y_1), A(x_2)(y_2)). \end{aligned} \quad (5)$$

Example 13. Let $X_1 = R$, $X_2 = R - (-1/2, 1/2)$ and let $E_1(x) = |x|$, $E_2(y) = y$, $\Phi_1(x_1, x_2, \lambda) = \lambda x_1 + (1 - \lambda)x_2$, and

$$\Phi_2(y_1, y_2, \lambda) = \begin{cases} \lambda y_1 + (1 - \lambda)y_2 & \text{if } y_1 y_2 > 0 \\ (1 + \lambda)y_2 - \lambda y_1 & \text{if } y_1 y_2 < 0. \end{cases} \quad (6)$$

Consider $A : X_1 \rightarrow F(X_2)$ defined by

$$A(X) := \chi_{(f(x), \infty)}, \quad (7)$$

where $f(x) = k|x|$, $k > 0$, and χ_C denotes the characteristics function of C .

Then a mapping A is $(\Phi E_1, \Phi E_2)$ -convex fuzzy process.

Now, let us consider $A : X_1 \rightarrow F(X_2)$ defined by

$$A(x)(y) = \begin{cases} \frac{y}{f(x)} & \text{if } y \in \langle 0, f(x) \rangle, \\ 1 & \text{if } y \in \langle f(x), \infty \rangle, \\ 0 & \text{if } y \in \langle -\infty, 0 \rangle, \end{cases} \quad (8)$$

for all $x \neq 0$ and

$$A(x)(y) = \begin{cases} 1 & \text{if } y \in \langle 0, \infty \rangle, \\ 0 & \text{if } y \in \langle -\infty, 0 \rangle, \end{cases} \quad (9)$$

for $x = 0$, where $f(x) = k|x|$, $k > 0$.

The above mapping is $(\Phi E_1, \Phi E_2)$ -convex fuzzy mapping too.

Theorem 14. If A is a $(\Phi E_1, \Phi E_2)$ -convex fuzzy process from X_1 to $F(X_2)$ and E_1 is an identity mapping then for any $x \in X_1$ $A(x)$ is a ΦE_2 -convex fuzzy set in X_2 .

Proof. We will prove that if $y_1, y_2 \in X_2$, $\lambda \in [0, 1]$ then for any $x \in X_1$

$$A(x)(\Phi_2(E_2(y_1), E_2(y_2), \lambda)) \geq \min(A(x)(y_1), A(x)(y_2)). \quad (10)$$

Let us note that for any $x \in X_1$ $E_1(x) = x = \Phi_1(x, x, \lambda)$.

So, using the definition of $(\Phi E_1, \Phi E_2)$ -convex fuzzy processes, we have

$$\begin{aligned} & A(x)(\Phi_2(E_2(y_1), E_2(y_2), \lambda)) \\ & = A(\Phi_1(x, x, \lambda))(\Phi_2(E_2(y_1), E_2(y_2), \lambda)) \\ & \geq \sup_{\substack{y', y'' \in X_2: \\ \Phi_2(E_2(y_1), E_2(y_2), \lambda) \\ = \Phi_2(E_2(y'), E_2(y''), \lambda)}} \min(A(x)(y'), A(x)(y'')) \quad (11) \\ & \geq \min(A(x)(y_1), A(x)(y_2)). \end{aligned}$$

$\square$

Definition 15. The graph of a $(\Phi E_1, \Phi E_2)$ -convex fuzzy process A from X_1 to $F(X_2)$, denoted G_A , is a fuzzy set in $X_1 \times X_2$ such that for any $(x, y) \in X_1 \times X_2$

$$G_A(x, y) = A(x)(y). \quad (12)$$

In the analogous way as in the Definition 9, we can define the $(\Phi E_1, \Phi E_2)$ -convex fuzzy subset B of $X_1 \times X_2$, that is,

$$\begin{aligned} & B(\Phi_1(E_1(x_1), E_1(x_2), \lambda), \Phi_2(E_2(y_1), E_2(y_2), \lambda)) \\ & \geq \min(B(x_1, y_1), B(x_2, y_2)), \end{aligned} \quad (13)$$

for $x_1, x_2 \in X_1$, $y_1, y_2 \in X_2$, $\lambda \in [0, 1]$.

Theorem 16. The graph of a $(\Phi E_1, \Phi E_2)$ -convex fuzzy process A from X_1 to $F(X_2)$ is a $(\Phi E_1, \Phi E_2)$ -convex fuzzy subset of $X_1 \times X_2$.

Proof. Taking into account the definitions of the graph and $(\Phi E_1, \Phi E_2)$ -convex fuzzy process, we observe that for any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$ and $\lambda \in [0, 1]$

$$\begin{aligned} & G_A(\Phi_1(E_1(x_1), E_1(x_2), \lambda), \Phi_2(E_2(y_1), E_2(y_2), \lambda)) \\ & = A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))(\Phi_2(E_2(y_1), E_2(y_2), \lambda)) \\ & \geq \sup_{\substack{y', y'' \in X_2: \\ \Phi_2(E_2(y_1), E_2(y_2), \lambda) \\ = \Phi_2(E_2(y'), E_2(y''), \lambda)}} \min(A(x_1)(y'), A(x_2)(y'')) \quad (14) \\ & \geq \min(A(x_1)(y_1), A(x_2)(y_2)) \\ & = \min(G_A(x_1, y_1), G_A(x_2, y_2)). \end{aligned}$$

$\square$

Definition 17. A composition of $(\Phi E_1, \Phi E_2)$ -convex fuzzy process A from X_1 to $F(X_2)$ and $(\Phi E_2, \Phi E_3)$ -convex fuzzy process B from X_2 to $F(X_3)$ is mapping $B \circ A$ from X_1 to $F(X_3)$ such that

$$(B \circ A)(x)(z) = \sup_{y \in X_2} \min(A(x)(y), B(y)(z)) \quad (15)$$

for any $(x, z) \in X_1 \times X_3$.

Theorem 18. If A is $(\Phi E_1, \Phi E_2)$ -convex fuzzy process from X_1 to $F(X_2)$ and B is $(\Phi E_2, \Phi E_3)$ -convex fuzzy process from X_2 to $F(X_3)$, then $B \circ A$ is a $(\Phi E_1, \Phi E_3)$ -convex fuzzy process from X_1 to $F(X_3)$.

Proof. Let $x_1, x_2 \in X_1$ and $\lambda \in [0, 1]$. Then for any $z \in X_3$ we have

$$\begin{aligned} & (B \circ A)(\Phi_1(E_1(x_1), E_1(x_2), \lambda))(z) \\ &= \sup_{y \in X_2} \min(A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))(y), B(y)(z)) \\ &= \sup_{\substack{y_1, y_2 \in X_2: \\ y = \Phi_2(E_2(y_1), E_2(y_2), \lambda)}} \min(A(\Phi_1(E_1(x_1), E_1(x_2), \lambda)), \\ & \quad (\Phi_2(E_2(y_1), E_2(y_2), \lambda))), \\ & B(\Phi_2(E_2(y_1), E_2(y_2), \lambda))(z)) \\ &\geq \sup_{\substack{y_1, y_2 \in X_2: \\ y = \Phi_2(E_2(y_1), E_2(y_2), \lambda)}} \min(A(x_1)(y_1), A(x_2)(y_2), \\ & \quad \sup_{\substack{z_1, z_2 \in X_3: \\ z = \Phi_3(E_3(z_1), E_3(z_2), \lambda)}} \min(B(y_1)(z_1), B(y_2)(z_2))) \\ &\geq \sup_{\substack{z_1, z_2 \in X_3: \\ z = \Phi_3(E_3(z_1), E_3(z_2), \lambda)}} \sup_{y_1, y_2 \in X_2} \min(A(x_1)(y_1), A(x_2)(y_2), \\ & \quad B(y_1)(z_1), B(y_2)(z_2))) \\ &= \sup_{\substack{z_1, z_2 \in X_3: \\ z = \Phi_3(E_3(z_1), E_3(z_2), \lambda)}} \min \left(\sup_{y_1 \in X_2} \min(A(x_1)(y_1), B(y_1)(z_1)), \right. \\ & \quad \left. \sup_{y_2 \in X_2} \min(A(x_2)(y_2), B(y_2)(z_2)) \right) \\ &= \sup_{\substack{z_1, z_2 \in X_3: \\ z = \Phi_3(E_3(z_1), E_3(z_2), \lambda)}} \min((B \circ A)(x_1)(z_1), (B \circ A)(x_2)(z_2)). \end{aligned} \quad (16)$$

So, $B \circ A$ is a $(\Phi E_1, \Phi E_3)$ -convex fuzzy process.

For any set $C \subset X_1$, we put

$$A(C)(y) = \sup_{x \in C} A(x)(y) \quad (17)$$

for any $y \in X_2$. $\square$

Theorem 19. If A is $(\Phi E_1, \Phi E_2)$ -convex fuzzy process from X_1 to $F(X_2)$ and C is a ΦE_1 -convex subset of X_1 and E_1 is an identity mapping then $A(C)$ is ΦE_2 -convex subset of X_2 .

Proof. Let $y_1, y_2 \in X_2$ and $\lambda \in [0, 1]$. Then we have

$$\begin{aligned} & A(C)(\Phi_2(E_2(y_1), E_2(y_2), \lambda)) \\ &= \sup_{x \in C} A(x)(\Phi_2(E_2(y_1), E_2(y_2), \lambda)) \\ &= \sup_{x \in C} A(\Phi_1(x, x, \lambda))(\Phi_2(E_2(y_1), E_2(y_2), \lambda)) \\ &\geq \sup_{x \in C} \sup_{\substack{y', y'' \in X_2: \\ \Phi_2(E_2(y'), E_2(y''), \lambda) \\ = \Phi_2(E_2(y_1), E_2(y_2), \lambda)}} \min(A(x)(y'), A(x)(y'')) \\ &= \sup_{x \in C} \min(A(x)(y_1), A(x)(y_2)) \\ &= \min \left(\sup_{x \in C} A(x)(y_1), \sup_{x \in C} A(x)(y_2) \right) \\ &= \min(A(C)(y_1), A(C)(y_2)). \end{aligned} \quad (18)$$

So, $A(C)$ is a ΦE_2 -convex fuzzy subset of X_2 . $\square$

Theorem 20. If A is $(\Phi E_1, \Phi E_2)$ -convex fuzzy process from X_1 to $F(X_2)$ then for any $\alpha \in [0, 1]$

$$\begin{aligned} & [A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))]^\alpha \\ & \supset \Phi_2(E_2([A(x_1)]^\alpha), E_2([A(x_2)]^\alpha), \lambda) \end{aligned} \quad (19)$$

for any $x_1, x_2 \in X_1$, $\lambda \in [0, 1]$.

Proof. According to the definition of α -cut, we have

$$\begin{aligned} & [A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))]^\alpha \\ &= \{y \in X_2 : A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))(y) \geq \alpha\}, \\ & E_2([A(x_1)]^\alpha) = \{E_2(y_1) \in X_2 : A(x_1)(y_1) \geq \alpha\}, \\ & E_2([A(x_2)]^\alpha) = \{E_2(y_2) \in X_2 : A(x_2)(y_2) \geq \alpha\}. \end{aligned} \quad (20)$$

Moreover,

$$\begin{aligned} & \Phi_2(E_2([A(x_1)]^\alpha), E_2([A(x_2)]^\alpha), \lambda) \\ &= \{y = \Phi_2(E_2(y_1), E_2(y_2), \lambda) : \\ & \quad y_1 \in [A(x_1)]^\alpha, y_2 \in [A(x_2)]^\alpha\}, \\ & A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))(y) \\ &\geq \sup_{\substack{y_1, y_2 \in X_2: \\ y = \Phi_2(E_2(y_1), E_2(y_2), \lambda)}} \min(A(x_1)(y_1), A(x_2)(y_2)). \end{aligned} \quad (21)$$

This means that if $y \in \Phi_2(E_2([A(x_1)]^\alpha), E_2([A(x_2)]^\alpha), \lambda)$ then $y \in [A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))]^\alpha$, that is,

$$\begin{aligned} & [A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))]^\alpha \\ & \supset \Phi_2(E_2([A(x_1)]^\alpha), E_2([A(x_2)]^\alpha), \lambda). \end{aligned} \quad (22)$$

Now, for any $\alpha \in [0, 1]$ and any ΦE_2 -quasiconcave function $g : X_2 \rightarrow R$, let us define a function

$$g_g^\alpha(x) = \max_{y \in [A(x)]^\alpha} g(y) \text{ for } x \in X_1. \quad (23)$$

$\square$

Theorem 21. If A is a $(\Phi E_1, \Phi E_2)$ -convex fuzzy process from X_1 to $F(X_2)$, then the function q_g^α is ΦE_1 -quasiconcave.

Proof. Let $x_1, x_2 \in X_1, \lambda \in [0, 1]$. Then we have

$$\begin{aligned}
 & q_g^\alpha(\Phi_1(E_1(x_1), E_1(x_2), \lambda)) \\
 &= \max_{y \in [A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))]^\alpha} g(y) \\
 &\geq \max_{y \in \Phi_2(E_2([A(x_1)]^\alpha), E_2([A(x_2)]^\alpha), \lambda)} g(y) \\
 &= \max\{g(y) : y = \Phi_2(E_2(y_1), E_2(y_2), \lambda), \\
 &\quad y_1 \in [A(x_1)]^\alpha, y_2 \in [A(x_2)]^\alpha\} \\
 &= \max\{g(\Phi_2(E_2(y_1), E_2(y_2), \lambda) : \quad (24) \\
 &\quad y_1 \in [A(x_1)]^\alpha, y_2 \in [A(x_2)]^\alpha\} \\
 &\geq \max\{\min(g(y_1), g(y_2)) : \\
 &\quad y_1 \in [A(x_1)]^\alpha, y_2 \in [A(x_2)]^\alpha\} \\
 &\geq \min\left(\max_{y_1 \in [A(x_1)]^\alpha} g(y_1), \max_{y_2 \in [A(x_2)]^\alpha} g(y_2)\right) \\
 &= \min(q_g^\alpha(x_1), q_g^\alpha(x_2)).
 \end{aligned}$$

□

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