Research Article

A Note on \((\Phi E_1, \Phi E_2)\)-Convex Fuzzy Processes

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1. Introduction

In recent years, many generalizations of convexity have appeared in the literature aiming at applications to duality theory and optimality conditions. In 1997, Pini and Singh [1] introduced \((\Phi_1, \Phi_2)\)-convex functions and studied some of their properties. They showed that some of the well-known classes of generalized convex functions (e.g., B-vex functions [2], geodesic convex functions [3], and invex functions [4]) form subclasses of the class of \((\Phi_1, \Phi_2)\)-convex functions. In 1999, Youness [5] showed that many results for convex sets and convex functions actually hold for a wider class of sets and functions, called \(E\)-convex sets and \(E\)-convex functions.

Convex processes were studied first by Rockafellar [6] who was interested in extending properties of linear transformations to a large class of maps preserving convexity and which arise naturally in economic theory.

The extension of this notion to the fuzzy framework was done by Matłoka [7, 8] and was investigated by Syau et al. [9] and by Chalco-Cano et al. [10].

In this work, we extend the notion of \(\Phi E\)-convex set, \(\Phi E\)-convex fuzzy set, and \((\Phi E_1, \Phi E_2)\)-convex fuzzy process. First, we will present the preliminary definitions and next the main properties of \((\Phi E_1, \Phi E_2)\)-convex fuzzy process.

2. Preliminaries

Let \(X_i\) denote a subset of the \(n_i\)-dimensional Euclidean space \(R^{n_i}\). Assume that \(\Phi_i\) is a map satisfying the following assumption:

\[(i)\quad \Phi_i : X_i \times X_i \times [0, 1] \to R^{n_i},\]
\[(ii)\quad \Phi_i(x, y, 0) = y, \quad \Phi_i(x, x, \lambda) = x, \quad \forall x, y \in X_i, \lambda \in [0, 1].\]

For any subsets \(A, B\) of \(X_i\) let us define
\[
\Phi_i(A, B, \lambda) = \{ \Phi_i(x, y, \lambda) : x \in A, y \in B \}.
\]

Definition 1 (see [1]). A set \(X_i\) is \(\Phi_1\)-convex if \(\Phi_i(x_1, x_2, \lambda) \in X_i\) for all \(x_1, x_2 \in X_i, \lambda \in [0, 1]\).

Remark 2. The intersection of \(\Phi_1\)-convex sets is still \(\Phi_1\)-convex.

Remark 3. Let \(X_i\) be a convex subset of \(R^{n_i}\), and \(\Phi_i(x_1, x_2, \lambda) = \lambda x_1 + (1 - \lambda)x_2\). Then a convex set is \(\Phi_1\)-convex.

Remark 4. If \(\eta : R^{n_i} \times R^{n_i} \to R^{n_i}\), \(X_i\) is a preinvex set with respect to \(\eta\) (see [11]), \(\Phi_i(x_1, x_2, \lambda) = x_2 + \lambda \cdot \eta(x_1, x_2)\), then \(X_i\) is a \(\Phi_1\)-convex set.

Assume that \(E_i : R^{n_i} \to R^{n_i}\).

Definition 5 (see [5]). A set \(X_i \subset R^{n_i}\) is said to be \(E_i\)-convex if \((1 - \lambda)E(x_1) + \lambda E(x_2) \in X_i\), for each \(x_1, x_2 \in X_i\) and \(\lambda \in [0, 1]\).

Definition 6. A set \(X_i \subset R^{n_i}\) is said to be \(\Phi E_i\)-convex if \(\Phi_i(E_i(x_1), E_i(x_2), \lambda) \in X_i\), for each \(x_1, x_2 \in X_i\) and \(\lambda \in [0, 1]\).

Remark 7. If \(\Phi_i(x_1, x_2, \lambda) = \lambda x_1 + (1 - \lambda)x_2\) and \(E_i\) is the identity map then a convex set is \(\Phi E_i\)-convex.

Definition 8. Let \(X_i\) be a \(\Phi E_i\)-convex set. A function \(f : X_i \to R\) is said to be \(\Phi E_i\)-quasiconcave on \(X_i\) if for any \(x_1, x_2 \in X_i\) and \(\lambda \in [0, 1]\)
\[
f(\Phi_i(E_i(x_1), E_i(x_2), \lambda)) \geq \min\{f(x_1), f(x_2)\}.
\]

Let \(C : R^{n_i} \to [0, 1]\) denote a fuzzy set in \(R^{n_i}\).
Definition 9. A fuzzy set $C$ is called $\Phi E_i$-convex if and only if
\[
C(\Phi E_i(x_1), E(x_2), \lambda) \geq \min(C(x_1), C(x_2)),
\]
for all $x_1, x_2 \in \mathbb{R}^n$.

Definition 10. An $\alpha$-cut of a fuzzy set $C$ is defined as follows:
\[
C^\alpha = \begin{cases} 
\{x : C(x) \geq \alpha\} & \text{for } \alpha \in (0, 1] \\
\{x : C(x) > \alpha\} & \text{for } \alpha = 0.
\end{cases}
\]

Proposition 11. If $C$ is $\Phi E_i$-convex fuzzy set then $C^\alpha$ is $\Phi E_i$-convex (crisp) set.

Proof. We have to prove that if $x_1, x_2 \in C^\alpha$ then for any $\lambda \in [0, 1]$, $\Phi E_i(x_1), E(x_2), E(\lambda) \in C^\alpha$. So, taking into account the above definitions, we observe that if $x_1, x_2 \in C^\alpha$ then $C(x_1) \geq \alpha$ and $C(x_2) \geq \alpha$ and $\min(C(x_1), C(x_2)) \geq \alpha$. So, $C(\Phi E_i(x_1), E(x_2), \lambda) \geq \min(C(x_1), C(x_2)) \geq \alpha$. This means that $\Phi E_i(x_1), E(x_2), \lambda) \in C^\alpha$.

3. Main Results

In this section, we present the definition and some properties of the $(\Phi E_1, \Phi E_2)$-convex fuzzy processes.

Let $X_i$ denote $\Phi E_i$-convex set and $F(X_i)$ the set of all non-void fuzzy sets in $X_i$ $(i = 1, 2, 3)$.

Definition 12. A mapping $A$ from $X_1$ to $F(X_2)$ is called $(\Phi E_1, \Phi E_2)$-convex fuzzy process if and only if for any $x_1, x_2 \in X_1$ and $\lambda \in [0, 1]$ and $y \in X_2$
\[
A(\Phi E_1(x_1), E(x_2), \lambda)) \geq \min(A(x_1)(y_1), A(x_2)(y_2)).
\]

Example 13. Let $X_1 = \mathbb{R}, X_2 = \mathbb{R} - (-1/2, 1/2)$ and let $E_1(x) = |x|$, $E_2(y) = y$, $\Phi E_1(x_1, x_2, \lambda) = \lambda x_1 + (1 - \lambda)x_2$, and
\[
\Phi E_2(y_1, y_2, \lambda) = \begin{cases} 
\lambda y_1 + (1 - \lambda)y_2 & \text{if } y_1 y_2 > 0 \\
(1 + \lambda)y_2 - \lambda y_1 & \text{if } y_1 y_2 < 0.
\end{cases}
\]

Consider $A : X_1 \rightarrow F(X_2)$ defined by
\[
A(x)(y) = \chi_{(f(x)), \alpha},
\]
where $f(x) = k|x|$, $k > 0$, and $\chi_C$ denotes the characteristics function of $C$.

Then a mapping $A$ is $(\Phi E_1, \Phi E_2)$-convex fuzzy process. Now, let us consider $A : X_1 \rightarrow F(X_2)$ defined by
\[
A(x)(y) = \begin{cases} 
y / f(x) & \text{if } y \in (0, f(x)), \\
1 & \text{if } y \in \{f(x), \alpha\}, \\
0 & \text{if } y \in (-\alpha, 0),
\end{cases}
\]

for all $x \neq 0$ and
\[
A(x)(y) = \begin{cases} 
1 & \text{if } y \in \{0, \alpha\}, \\
0 & \text{if } y \in (-\alpha, 0),
\end{cases}
\]

for $x = 0$, where $f(x) = k|x|$, $k > 0$.

The above mapping is $(\Phi E_1, \Phi E_2)$-convex fuzzy mapping too.

\[\text{Theorem 14. If } A \text{ is a } (\Phi E_1, \Phi E_2)\text{-convex fuzzy process from } X_1 \text{ to } F(X_2) \text{ and } E_i \text{ is an identity mapping then for any } x \in X_1 A(x) \text{ is a } \Phi E_2\text{-convex fuzzy set in } X_2.\]

Proof. We will prove that if $y_1, y_2 \in X_2, \lambda \in [0, 1]$ then for any $x \in X_1$
\[
A(x)(\Phi E_2(E_1(y_1), E_2(y_2), \lambda)) \geq \min(A(x)(y_1), A(x)(y_2)).
\]

Let us note that for any $x \in X_1 E_1(x) = x = \Phi_1(x, x, \lambda)$.

So, using the definition of $(\Phi E_1, \Phi E_2)$-convex fuzzy processes, we have
\[
A(x)(\Phi E_2(E_1(y_1), E_2(y_2), \lambda))
\]
\[
= A(\Phi_1(x, x, \lambda))(\Phi E_2(E_1(y), E_2(y), \lambda))
\]
\[
\geq \sup_{y', y'' \in X_2: \Phi E_1(x, y', E_1(y''), \lambda)} \min(A(x)(y'), A(x)(y''))
\]
\[
\geq \min(A(x)(y_1), A(x)(y_2)).
\]

\[\text{Definition 15. The graph of a } (\Phi E_1, \Phi E_2)\text{-convex fuzzy process } A \text{ from } X_1 \text{ to } F(X_2) \text{ denoted } G_A, \text{ is a fuzzy set in } X_1 \times X_2 \text{ such that for any } (x_1, y_1) \in X_1 \times X_2
\]
\[
G_A(x, y) = A(x)(y).
\]

In the analogous way as in the Definition 9, we can define the $(\Phi E_1, \Phi E_2)$-convex fuzzy subset $B$ of $X_1 \times X_2$, that is,
\[
B(\Phi E_1(x_1), E(x_2), \lambda) \geq \min(B(x_1, y_1), B(x_2, y_2)),
\]

for $x_1, x_2 \in X_1, y_1, y_2 \in X_2, \lambda \in \{0, 1\}$.

\[\text{Theorem 16. The graph of a } (\Phi E_1, \Phi E_2)\text{-convex fuzzy process } A \text{ from } X_1 \text{ to } F(X_2) \text{ is a } (\Phi E_1, \Phi E_2)\text{-convex fuzzy subset of } X_1 \times X_2.\]

Proof. Taking into account the definitions of the graph and $(\Phi E_1, \Phi E_2)$-convex fuzzy process, we observe that for any $(x_1, y_1), (x_2, y_2) \in X_1 \times X_2$ and $\lambda \in \{0, 1\}$
\[
G_A(\Phi E_1(x_1), E(x_2), \lambda) \geq \min(G_A(x_1, y_1), G_A(x_2, y_2)),
\]
\[
\sup_{y', y'' \in X_2: \Phi E_2(E_1(y', y''), \lambda)} \min(G_A(x_1)(y_1), G_A(x_2)(y_2))
\]
\[
= \min(G_A(x_1, y_1), G_A(x_2, y_2)).
\]
Theorem 18. If $\Phi$ is a $(\Phi E_1, \Phi E_2)$-convex fuzzy process from $X_1$ to $F(X_2)$ and $B$ is a $(\Phi E_1, \Phi E_3)$-convex fuzzy process from $X_3$ to $F(X_3)$, then $B \circ A$ is a $(\Phi E_1, \Phi E_3)$-convex fuzzy process from $X_1$ to $F(X_3)$.

Proof. Let $x_1, x_2 \in X_1$ and $\lambda \in [0, 1]$. Then for any $z \in X_3$, we have
\[
(B \circ A)(\Phi_1(E_1(x_1), E_1(x_2), \lambda))(z) = \sup_{y \in X_2} \min(A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))(y), B(y)(z)) 
\]
for any $x \in X_1$.

Theorem 19. If $A$ is a $(\Phi E_1, \Phi E_2)$-convex fuzzy process from $X_1$ to $F(X_2)$ and $C$ is a $(\Phi E_1, \Phi E_3)$-convex fuzzy process from $X_3$ to $F(X_3)$, then $C \circ A$ is a $(\Phi E_1, \Phi E_3)$-convex fuzzy process from $X_1$ to $F(X_3)$.

Proof. Let $y_1, y_2 \in X_2$ and $\lambda \in [0, 1]$. Then we have
\[
A(C)(\Phi_2(E_2(y_1), E_2(y_2), \lambda)) = \sup_{x \in C} A(x)(\Phi_2(E_2(y_1), E_2(y_2), \lambda)) 
\]
for any $x \in X_1$.

Theorem 20. If $A$ is a $(\Phi E_1, \Phi E_2)$-convex fuzzy process from $X_1$ to $F(X_2)$ then for any $x \in [0, 1]$
\[
[A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))^*] \supset \Phi_2(E_2([A(x_1)]^+, E_2([A(x_2)]^+, \lambda), X_2)
\]
for any $x_1, x_2 \in X_1$, $\lambda \in [0, 1]$. 

Proof. According to the definition of $\alpha$-cut, we have
\[
[A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))]^* = \{ y \in X_2 : A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))(y) \geq \alpha \},
\]
\[
E_2([A(x_1)]^+) = \{ E_2(y_1) \in X_2 : x_1(y_1) \geq \alpha \},
\]
\[
E_2([A(x_2)]^+) = \{ E_2(y_2) \in X_2 : x_2(y_2) \geq \alpha \}.
\]

Moreover,
\[
\Phi_2(E_2([A(x_1)]^+, E_2([A(x_2)]^+, \lambda), X_2)
\]
\[
A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))(y) \supset \Phi_2(E_2([A(x_1)]^+, E_2([A(x_2)]^+, \lambda).
\]

This means that if $y \in \Phi_2(E_2([A(x_1)]^+, E_2([A(x_2)]^+, \lambda)$ then $y \in [A(\Phi_1(E_1(x_1), E_1(x_2), \lambda)]^*$, that is,
\[
[A(\Phi_1(E_1(x_1), E_1(x_2), \lambda))]^*
\]
\[
[\Phi_2(E_2([A(x_1)]^*, E_2([A(x_2)]^*, \lambda).
\]

Now, for any $\alpha \in [0, 1]$ and any $\Phi E_2$-quasiconvex function $g : X_2 \rightarrow R$, let us define a function
\[
g^*(x) = \max_{y \in [A(x)]^*} g(y) \text{ for } x \in X_1.
\]
Theorem 21. If $A$ is a $(\Phi E_1, \Phi E_2)$-convex fuzzy process from $X_1$ to $F(X_2)$, then the function $q^a \Phi$ is $\Phi E_1$-quasiconcave.

Proof. Let $x_1, x_2 \in X_1, \lambda \in [0, 1]$. Then we have

\[
q^a \Phi(x_1, x_2, \lambda) = \max_{y \in [A(\Phi E_1, E_1(x_1), E_1(x_2), \lambda)]} g(y)
\]

\[
\geq \max_{y \in \Phi E_2(\Phi E_1, E_1(x_1), E_1(x_2), \lambda)} g(y)
\]

\[
= \max \{ g(y) : y = \Phi_2(E_2(y_1), E_2(y_2), \lambda),
\]

\[y_1 \in [A(x_1)]^a, y_2 \in [A(x_2)]^a \}
\]

\[
= \max \{ \min (g(y_1), g(y_2)) : y_1 \in [A(x_1)]^a, y_2 \in [A(x_2)]^a \}
\]

\[
\geq \min \left( \max_{y_1 \in [A(x_1)]^a} g(y_1), \max_{y_2 \in [A(x_2)]^a} g(y_2) \right)
\]

\[
= \min (q^a \Phi(x_1), q^a \Phi(x_2)).
\]

References


