Research Article

Lacunary Statistical Limit and Cluster Points of Generalized Difference Sequences of Fuzzy Numbers

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The aim of present work is to introduce and study lacunary statistical limit and lacunary statistical cluster points for generalized difference sequences of fuzzy numbers. Some inclusion relations among the sets of ordinary limit points, statistical limit points, statistical cluster points, lacunary statistical limit points, and lacunary statistical cluster points for these type of sequences are obtained.

1. Introduction

The notion of statistical convergence of sequences of numbers was introduced by Fast [1] and Schoenberg [2] independently and latter discussed in [3–6], and so forth. In 1993, Fridy and Orhan [7] presented an interesting generalization of statistical convergence with the help of a lacunary sequence and called it lacunary statistical convergence or S_θ -convergence. Demirci [8] defined S_θ -limit and cluster points of number sequences and obtained some interesting results analogous to [4]. In past years, statistical convergence has also become an interesting area of research for sequences of fuzzy numbers. The credit goes to Nuray and Savaş [9] who first introduced statistical convergence of sequences of fuzzy numbers. After their pioneer work, many authors have made their contribution to study different generalizations of statistical convergence for sequences of fuzzy numbers (see [10–13], etc.).

Quite recently, statistical convergence of sequences of fuzzy numbers is studied with the help of the difference operator Δ . For instance, Bilgin [14] introduced strongly Δ -summable and Δ -statistical convergence of sequences of fuzzy numbers. Işık [15] studied some notions of generalized difference sequences of numbers. In 2006, Altin et al. [16] united lacunary sequences to introduce the concept of lacunary statistical convergence of generalized difference sequences of fuzzy numbers and obtained some interesting

results. Some more work in this direction can be found in [17–19]. In present work, we continue with this study and introduce the concepts of lacunary statistical limit and cluster points of generalized difference sequences of fuzzy numbers. We obtain some relations among the sets of ordinary limit points, lacunary statistical limit, and cluster points for these type of sequences.

2. Background and Preliminaries

We begin with the following terminology on fuzzy numbers. Given any interval A , we shall denote its end points by $\underline{A}, \bar{A}$ and by D the set of all closed bounded intervals on real line $\mathbb{R}$, that is, $D = \{A \subset \mathbb{R} : A = [\underline{A}, \bar{A}]\}$. For $A, B \in D$ we define $A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $\bar{A} \leq \bar{B}$. Moreover, the distance function d defined by $d(A, B) = \max\{|\underline{A} - \underline{B}|, |\bar{A} - \bar{B}|\}$ is a Hausdorff metric on D and (D, d) is a complete metric space. Also $\leq$ is a partial order on D .

A fuzzy number is a function X from $\mathbb{R}$ to $[0, 1]$ which is satisfying the following conditions: (i) X is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $X(x_0) = 1$; (ii) X is fuzzy convex, that is, for any $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$, $X(\lambda x + (1 - \lambda)y) \geq \min\{X(x), X(y)\}$; (iii) X is upper semicontinuous; and (iv) the closure of the set $\{x \in \mathbb{R} : X(x) > 0\}$ denoted by X^0 is compact.

Properties (i)–(iv) imply that for each $\alpha \in (0, 1]$, the α -level set, $X^\alpha = \{x \in \mathbb{R} : X(x) \geq \alpha\} = [\underline{X}^\alpha, \overline{X}^\alpha]$, is a nonempty compact convex subset of $\mathbb{R}$. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers. The linear structure of $L(\mathbb{R})$ induces an addition $X + Y$ and a scalar multiplication λX in terms of α -level sets by

$$\begin{aligned} [X + Y]^\alpha &= [X]^\alpha + [Y]^\alpha, \\ [\lambda X]^\alpha &= \lambda[X]^\alpha \quad (X, Y \in L(\mathbb{R}), \lambda \in \mathbb{R}) \end{aligned} \quad (1)$$

for each $\alpha \in [0, 1]$. Define a map $\bar{d} : L(\mathbb{R}) \times L(\mathbb{R}) \rightarrow \mathbb{R}$ by

$$\bar{d}(X, Y) = \sup_{\alpha \in [0, 1]} d(X^\alpha, Y^\alpha). \quad (2)$$

Puri and Ralescu [20] proved that $(L(\mathbb{R}), \bar{d})$ is a complete metric space. Also the ordered structure on $L(\mathbb{R})$ is defined as follows. For $X, Y \in L(\mathbb{R})$, we define $X \leq Y$ if and only if $\underline{X}^\alpha \leq \underline{Y}^\alpha$ and $\overline{X}^\alpha \leq \overline{Y}^\alpha$ for each $\alpha \in [0, 1]$. We say that $X < Y$ if $X \leq Y$ and there exist $\alpha_0 \in [0, 1]$ such that $\underline{X}^{\alpha_0} < \underline{Y}^{\alpha_0}$ or $\overline{X}^{\alpha_0} < \overline{Y}^{\alpha_0}$. The fuzzy numbers X and Y are said to be incomparable if neither $X \leq Y$ nor $Y \leq X$.

We next recall some definitions and results which form the base for present study. For any set $K \subseteq \mathbb{N}$, let K_n denote the set $\{k \in K : k \leq n\}$ and $|K_n|$ denote the number of elements in K_n . The natural density δ of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$. The natural density may not exist for each set K . But the upper density $\bar{\delta}$ defined by $\bar{\delta}(K) = \limsup_n n^{-1}|K_n|$ always exists for each set K . Moreover, $\delta(K)$ different from zero means $\bar{\delta}(K) > 0$. Besides that, $\delta(K^C) = 1 - \delta(K)$ and if $A \subseteq B$, then $\bar{\delta}(A) \leq \bar{\delta}(B)$.

For any sequence $X = (X_k)$ of fuzzy numbers, we write $\{X_k : k \in \mathbb{N}\}$ to denote the range of X . If $(X_{k(j)})$ is a subsequence of X and $K = \{k(j) : j \in \mathbb{N}\}$, then we abbreviate $(X_{k(j)})$ by $(X)_K$. If $\delta(K) = 0$, $(X)_K$ is called a thin subsequence, otherwise if $\delta(K) \neq 0$, $(X)_K$ is called nonthin subsequence of X .

For w , the set of all sequences of fuzzy numbers, the operator $\Delta^m : w \rightarrow w$ is defined by

$$\begin{aligned} \Delta^0 X_k &= X_k, \\ \Delta^1 X_k &= X_k - X_{k+1}, \\ &\vdots \\ \Delta^m X_k &= \Delta^1(\Delta^{m-1} X_k). \end{aligned} \quad (3)$$

Definition 1. A sequence $X = (X_k)$ of fuzzy numbers is said to be Δ^m -statistically convergent to a fuzzy number X_0 , in symbol: $S(\Delta^m) - \lim_k X_k = X_0$, if for each $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \in \mathbb{N} : \bar{d}(\Delta^m X_k, X_0) \geq \epsilon \right\} \right| = 0. \quad (4)$$

Let $S(\Delta^m(X))$ denote the set of all Δ^m -statistically convergent sequences of fuzzy numbers.

Definition 2. Let $X = (X_k)$ be a sequence of fuzzy numbers. A fuzzy number X_0 is said to be a statistical limit point (s.l.p) of

the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers provided that there is a nonthin subsequence of X that is Δ^m -convergent to X_0 .

Let $\Lambda_S(\Delta^m(X))$ denote the set of all s.l.p. of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers.

Definition 3. Let $X = (X_k)$ be a sequence of fuzzy numbers. A fuzzy number Y_0 is said to be a statistical cluster point (s.c.p) of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers provided that, for each $\epsilon > 0$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \left| \left\{ k \in \mathbb{N} : \bar{d}(\Delta^m X_k, Y_0) < \epsilon \right\} \right| > 0. \quad (5)$$

Let $\Gamma_S(\Delta^m(X))$ denote the set of all s.c.p of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers.

By a lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of positive integers such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by $\theta = (k_r)$ will be denoted by $I_r = (k_{r-1}, k_r]$ whereas the ratio k_r/k_{r-1} is denoted by q_r . Further, a lacunary sequence $\theta' = (k'_r)$ is called a lacunary refinement of the lacunary sequence $\theta = (k_r)$ if $\{k_r\} \subseteq \{k'_r\}$.

Definition 4 (see [21]). Let $\theta = (k_r)$ be a lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary statistical convergent to a fuzzy number X_0 provided that for each $\epsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \bar{d}(X_k, Y_0) \geq \epsilon \right\} \right| = 0. \quad (6)$$

Let S_θ denote the set of all lacunary statistically convergent sequences of fuzzy numbers.

Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ a sequence of fuzzy numbers. If $(X)_K$ where $K = \{k(j) : j \in \mathbb{N}\}$ is a subsequence of $X = (X_k)$ such that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k(j) \in I_r : j \in \mathbb{N} \right\} \right| = 0, \quad (7)$$

we call $(X)_K$ a θ -thin subsequence. On the other hand, $(X)_K$ is a θ -nonthin subsequence of X provided that

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k(j) \in I_r : j \in \mathbb{N} \right\} \right| > 0. \quad (8)$$

Definition 5. Let $\theta = (k_r)$ be a lacunary sequence. A sequence $X = (X_k)$ of fuzzy numbers is said to be lacunary Δ^m -statistically convergent to a fuzzy number X_0 , in symbol: $S_\theta(\Delta^m) - \lim_k X_k = X_0$, if for each $\epsilon > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \bar{d}(\Delta^m X_k, X_0) \geq \epsilon \right\} \right| = 0. \quad (9)$$

Let $S_\theta(\Delta^m(X))$ denote the set of all lacunary Δ^m -statistically convergent sequences of fuzzy numbers.

We now consider the natural definitions of statistical limit and cluster points for generalized difference sequences of fuzzy numbers with respect to lacunary sequences.

3. Main Results

Definition 6. Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ a sequence of fuzzy numbers. A fuzzy number X_0 is said to be a lacunary statistical limit point (l.s.l.p) of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers provided that there is a θ -nonthin subsequence of X that is Δ^m -convergent to X_0 .

Let $\Lambda_{S_\theta}(\Delta^m(X))$ denote the set of all l.s.l.p. of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers.

Definition 7. Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ a sequence of fuzzy numbers. A fuzzy number Y_0 is said to be a lacunary statistical cluster point (l.s.c.p) of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers provided that, for each $\epsilon > 0$,

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \bar{d}(\Delta^m X_k, Y_0) < \epsilon \right\} \right| > 0. \quad (10)$$

Let $\Gamma_{S_\theta}(\Delta^m(X))$ denote the set of all l.s.c.p of the generalized difference sequence $(\Delta^m X_k)$ of fuzzy numbers.

Example 8. Let $\theta = (k_r)$ be a lacunary sequence. We define a sequence of fuzzy numbers $X = (X_k)$ as follows. For $x \in \mathbb{R}$, define

$$X_k(x) = \begin{cases} \begin{cases} x - k + 1, & \text{if } k - 1 \leq x \leq k \\ -x + k + 1, & \text{if } k < x \leq k + 1 \\ 0, & \text{otherwise} \end{cases} & \text{if } k_r - \left[\sqrt{h_r} \right] + 1 \leq k \leq k_r; r \in \mathbb{N} \\ \begin{cases} x - 5, & \text{if } 5 \leq x \leq 6 \\ 7 - x, & \text{if } 6 < x \leq 7 \\ 0, & \text{otherwise} \end{cases} & \text{otherwise.} \end{cases} \quad (11)$$

Then, we obtain

$$[X_k]^\alpha = \begin{cases} [k - 1 + \alpha, k + 1 - \alpha], & \text{if } k \in [k_r - \left[\sqrt{h_r} \right] + 1, k_r] \\ [5 + \alpha, 7 - \alpha] & \text{otherwise,} \end{cases}$$

$$[\Delta X_k]^\alpha = \begin{cases} \begin{cases} [-3 + \alpha, 1 - 2\alpha], & \text{if both } k, k + 1 \in [k_r - \left[\sqrt{h_r} \right] + 1, k_r] \\ [k - 8 + 2\alpha, k - 4 - 2\alpha], & \text{if } k \in [k_r - \left[\sqrt{h_r} \right] + 1, k_r] \text{ but not } k + 1, \\ [-k + 3 + 2\alpha, -k + 7 - 2\alpha], & \text{if } k + 1 \in [k_r - \left[\sqrt{h_r} \right] + 1, k_r] \text{ but not } k, \\ [-2 + 2\alpha, 2 - 2\alpha], & \text{otherwise.} \end{cases} \end{cases} \quad (12)$$

Thus, for $m = 1$, it is clear that the sequence ΔX_k has two different subsequences which converge to μ_1 and μ_2 , respectively, where $[\mu_1]^\alpha = [-3 + \alpha, 1 - 2\alpha]$ and $[\mu_2]^\alpha = [-2 + 2\alpha, 2 - 2\alpha]$. Hence, if $L(\Delta(X))$ denotes the set of ordinary limit points of (ΔX_k) , then $L(\Delta(X)) = \{\mu_1, \mu_2\}$; however, $\Gamma_{S_\theta}(\Delta(X)) = \{\mu_2\}$.

Theorem 9. Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ a sequence of fuzzy numbers. Then, one has $\Lambda_{S_\theta}(\Delta^m(X)) \subseteq \Gamma_{S_\theta}(\Delta^m(X))$.

Proof. Suppose $X_0 \in \Lambda_{S_\theta}(\Delta^m(X))$. By definition, there is a θ -nonthin subsequence $(X_{k(j)})$ of $X = (X_k)$ which is Δ^m -convergent to X_0 , and therefore we have

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k(j) \in I_r : j \in \mathbb{N} \right\} \right| = d > 0. \quad (13)$$

Since, for every $\epsilon > 0$,

$$\left\{ k \in I_r : \bar{d}(\Delta^m X_k, X_0) < \epsilon \right\} \supseteq \left\{ k(j) \in I_r : \bar{d}(\Delta^m X_{k(j)}, X_0) < \epsilon \right\}, \quad (14)$$

so we have the containment

$$\begin{aligned} & \left\{ k \in I_r : \bar{d}(\Delta^m X_k, X_0) < \epsilon \right\} \\ & \supset \left\{ k(j) \in I_r : j \in \mathbb{N} \right\} \\ & \quad - \left\{ k(j) \in I_r : \bar{d}(\Delta^m X_{k(j)}, X_0) \geq \epsilon \right\}. \end{aligned} \quad (15)$$

Now, $(X_{k(j)})$ is Δ^m -convergent to X_0 , which implies that, for every $\epsilon > 0$, $\{k(j) \in I_r : \bar{d}(\Delta^m X_{k(j)}, X_0) \geq \epsilon\}$ is finite for which we have

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k(j) \in I_r : \bar{d}(\Delta^m X_{k(j)}, X_0) \geq \epsilon \right\} \right| = 0. \quad (16)$$

Thus from (15), we obtain

$$\begin{aligned} & \limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \bar{d}(\Delta^m X_k, X_0) < \epsilon \right\} \right| \\ & \geq \limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k(j) \in I_r : j \in \mathbb{N} \right\} \right| \\ & \quad - \limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k(j) \in I_r : \bar{d}(\Delta^m X_{k(j)}, X_0) \geq \epsilon \right\} \right| \\ & \geq \limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k(j) \in I_r : j \in \mathbb{N} \right\} \right| = d > 0, \end{aligned} \quad (17)$$

using (13) and (16). This shows that $X_0 \in \Gamma_{S_\theta}(\Delta^m(X))$ and therefore the result is proved. $\square$

Theorem 10. Let $\theta = (k_r)$ be a lacunary sequence. Then, for any sequence $X = (X_k)$ of fuzzy numbers, one has $\Gamma_{S_\theta}(\Delta^m(X)) \subseteq L(\Delta^m(X))$.

Proof. Assume $Y_0 \in \Gamma_{S_\theta}(\Delta^m(X))$. By definition, for each $\epsilon > 0$ we have

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \bar{d}(\Delta^m X_k, Y_0) < \epsilon \right\} \right| > 0. \quad (18)$$

We set $(X)_K$ a θ -nonthin subsequence of $X = (X_k)$ such that $K = \{k(j) \in I_r : \bar{d}(\Delta^m X_k, Y_0) < \epsilon\}$ for $\epsilon > 0$. Since $\limsup_{r \rightarrow \infty} (1/h_r) |K| > 0$, it follows that K is an infinite set. Thus we have a subsequence $(X)_K$ of X that is Δ^m -convergent to Y_0 . This shows that $Y_0 \in L(\Delta^m(X))$. Hence $\Gamma_{S_\theta}(\Delta^m(X)) \subseteq L(\Delta^m(X))$. $\square$

Theorem 11. Let $\theta = (k_r)$ be a lacunary sequence. If $X = (X_k)$ and $Y = (Y_k)$ are two sequences of fuzzy numbers such that $\lim_{r \rightarrow \infty} (1/h_r) |\{k \in I_r : X_k \neq Y_k\}| = 0$, then $\Lambda_{S_\theta}(\Delta^m(X)) = \Lambda_{S_\theta}(\Delta^m(Y))$ and $\Gamma_{S_\theta}(\Delta^m(X)) = \Gamma_{S_\theta}(\Delta^m(Y))$.

Proof. We prove the theorem into two parts. In the first part we prove that $\Lambda_{S_\theta}(\Delta^m(X)) = \Lambda_{S_\theta}(\Delta^m(Y))$; however, in the second part we shall prove $\Gamma_{S_\theta}(\Delta^m(X)) = \Gamma_{S_\theta}(\Delta^m(Y))$.

Part (i). Let $X_0 \in \Lambda_{S_\theta}(\Delta^m(Y))$. By definition, there is a θ -nonthin subsequence $(Y)_K$ of $Y = (Y_k)$ that is Δ^m -convergent to X_0 . Since $\lim_{r \rightarrow \infty} (1/h_r) |\{k \in I_r : k \in K \text{ and } X_k \neq Y_k\}| = 0$, it follows that $\lim_{r \rightarrow \infty} (1/h_r) |\{k \in I_r : k \in K \text{ and } X_k = Y_k\}| > 0$. Therefore, from the later set, we can yield a θ -nonthin subsequence $(X)_{K'}$ of $X = (X_k)$ that is Δ^m -convergent to X_0 . Hence, $X_0 \in \Lambda_{S_\theta}(\Delta^m(X))$, and therefore we have $\Lambda_{S_\theta}(\Delta^m(Y)) \subset \Lambda_{S_\theta}(\Delta^m(X))$. Also by symmetry one get $\Lambda_{S_\theta}(\Delta^m(X)) \subset \Lambda_{S_\theta}(\Delta^m(Y))$. On combining we have $\Lambda_{S_\theta}(\Delta^m(X)) = \Lambda_{S_\theta}(\Delta^m(Y))$.

Part (ii). Let $Y_0 \in \Gamma_{S_\theta}(\Delta^m(X))$. By definition, for each $\epsilon > 0$,

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : \bar{d}(\Delta^m X_k, Y_0) < \epsilon\} \right| > 0. \quad (19)$$

Since $X_k = Y_k$ for all most all k , it follows that, for each $\epsilon > 0$,

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : \bar{d}(\Delta^m Y_k, Y_0) < \epsilon\} \right| > 0. \quad (20)$$

This shows that $Y_0 \in \Gamma_{S_\theta}(\Delta^m(Y))$ and therefore $\Gamma_{S_\theta}(\Delta^m(X)) \subset \Gamma_{S_\theta}(\Delta^m(Y))$. By symmetry, we see that $\Gamma_{S_\theta}(\Delta^m(Y)) \subset \Gamma_{S_\theta}(\Delta^m(X))$, whence $\Gamma_{S_\theta}(\Delta^m(X)) = \Gamma_{S_\theta}(\Delta^m(Y))$. $\square$

Theorem 12. Let $\theta = (k_r)$ be a lacunary sequence. If $X = (X_k)$ is a sequence of fuzzy numbers such that $S_\theta(\Delta^m) - \lim_k X_k = X_0$, then $\Lambda_{S_\theta}(\Delta^m(X)) = \Gamma_{S_\theta}(\Delta^m(X)) = \{X_0\}$.

Proof. We prove the theorem in two parts. In the first part, we prove that $\Lambda_{S_\theta}(\Delta^m(X)) = \{X_0\}$ whereas in the second part we obtain $\Gamma_{S_\theta}(\Delta^m(X)) = \{X_0\}$.

Part (i). Suppose that $\Lambda_{S_\theta}(\Delta^m(X)) = \{X_0, Y_0\}$, where $X_0 \neq Y_0$, that is, Y_0 is a l.s.l.p. of the generalized difference sequence $(\Delta^m X_k)$ different from X_0 . Choose $\epsilon > 0$ such that $0 < \epsilon < \bar{d}(X_0, Y_0)/2$. By definition there exist two θ -nonthin subsequences $(X_{k(j)})$ and $(X_{l(i)})$ of the sequence $X = (X_k)$ which are Δ^m -convergent to X_0 and Y_0 , respectively. Since $(X_{l(i)})$ is Δ^m -convergent to Y_0 , so for each $\epsilon > 0$, $\{l(i) \in I_n : \bar{d}(\Delta^m X_{l(i)}, Y_0) \geq \epsilon\}$ is a finite set for which

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{l(i) \in I_r : \bar{d}(\Delta^m X_{l(i)}, Y_0) \geq \epsilon\} \right| = 0. \quad (21)$$

Further, we can write

$$\begin{aligned} \{l(i) \in I_r : i \in \mathbb{N}\} &= \{l(i) \in I_r : \bar{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\} \\ &\cup \{l(i) \in I_r : \bar{d}(\Delta^m X_{l(i)}, Y_0) \geq \epsilon\}, \end{aligned} \quad (22)$$

for which we have

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{1}{h_r} |\{l(i) \in I_r : i \in \mathbb{N}\}| \\ = \limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{l(i) \in I_r : \bar{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\} \right| \\ + \limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{l(i) \in I_r : \bar{d}(\Delta^m X_{l(i)}, Y_0) \geq \epsilon\} \right|. \end{aligned} \quad (23)$$

Since $(X_{l(i)})$ is nonthin, so, by use of (21), we have

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{l(i) \in I_r : \bar{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\} \right| > 0. \quad (24)$$

Since $S_\theta(\Delta^m) - \lim_k X_k = X_0$, so for each $\epsilon > 0$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : \bar{d}(\Delta^m X_k, X_0) \geq \epsilon\} \right| = 0, \quad (25)$$

and therefore we can write

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : \bar{d}(\Delta^m X_k, X_0) < \epsilon\} \right| > 0. \quad (26)$$

Furthermore for $0 < 2\epsilon < \bar{d}(X_0, Y_0)$,

$$\begin{aligned} \{l(i) \in I_r : \bar{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\} \\ \cap \{k \in I_r : \bar{d}(\Delta^m X_k, X_0) < \epsilon\} = \emptyset, \end{aligned} \quad (27)$$

which immediately gives the containment

$$\{l(i) \in I_r : \bar{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\} \subset \{k \in I_r : \bar{d}(\Delta^m X_k, X_0) \geq \epsilon\} \quad (28)$$

for which we have

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{l(i) \in I_r : \bar{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\} \right| \\ \leq \limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : \bar{d}(\Delta^m X_k, X_0) \geq \epsilon\} \right| = 0. \end{aligned} \quad (29)$$

As left side of (29) cannot be negative, so we must have

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{l(i) \in I_r : \bar{d}(\Delta^m X_{l(i)}, Y_0) < \epsilon\} \right| = 0. \quad (30)$$

This contradicts (24). Hence, $\Lambda_{S_\theta}(\Delta^m(X)) = \{X_0\}$.

Part (ii). Let Z_0 be a l.s.c.p. of the generalized difference sequence $(\Delta^m X_k)$ different from X_0 , that is, $\Gamma_{S_\theta}(\Delta^m(X)) = \{X_0, Z_0\}$, where $X_0 \neq Z_0$. Choose ϵ such that $0 < \epsilon < \bar{d}(X_0, Z_0)/2$. Since Z_0 is a l.s.c.p. of $(\Delta^m X_k)$, so for each $\epsilon > 0$ we have

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : \bar{d}(\Delta^m X_k, Z_0) < \epsilon\} \right| > 0. \quad (31)$$

Since $\{k \in I_r : \bar{d}(\Delta^m X_k, X_0) < \epsilon\} \cap \{k \in I_r : \bar{d}(\Delta^m X_k, Z_0) < \epsilon\} = \emptyset$ for every $0 < \epsilon < \bar{d}(X_0, Z_0)/2$, it follows that

$\{k \in I_r : \bar{d}(\Delta^m X_k, X_0) \geq \epsilon\} \supseteq \{k \in I_r : \bar{d}(\Delta^m X_k, Z_0) < \epsilon\}$ for which we have

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : \bar{d}(\Delta^m X_k, X_0) \geq \epsilon\} \right| \\ \geq \limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : \bar{d}(\Delta^m X_k, Z_0) < \epsilon\} \right| > 0 \end{aligned} \quad (32)$$

by (31), which is impossible as by (25) $\limsup_{r \rightarrow \infty} (1/h_r) |\{k \in I_r : \bar{d}(\Delta^m X_k, X_0) \geq \epsilon\}| = 0$. In this way we obtained a contradiction. Hence, $\Gamma_{S_\theta}(\Delta^m(X)) = \{X_0\}$. $\square$

Theorem 13. Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ a sequence of fuzzy numbers. Then one has the following:

- (i) if $\liminf q_r > 1$, then $\Lambda_{S_\theta}(\Delta^m(X)) \subset \Lambda_S(\Delta^m(X))$,
- (ii) if $\limsup q_r < \infty$, then $\Lambda_S(\Delta^m(X)) \subset \Lambda_{S_\theta}(\Delta^m(X))$,
- (iii) if $1 < \liminf q_r \leq \limsup q_r < \infty$, then $\Lambda_S(\Delta^m(X)) = \Lambda_{S_\theta}(\Delta^m(X))$.

Proof. (i) Suppose $\liminf q_r > 1$; there exists a $\delta > 0$ such that $q_r > 1 + \delta$ for sufficient large r , which implies that $h_r/k_r \geq \delta/(1 + \delta)$. Assume that $X_0 \in \Lambda_{S_\theta}(\Delta^m(X))$, then there is θ -nonthin subsequence $(X_{k(j)})$ of (X_k) that is Δ^m -convergent to X_0 and

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| = d > 0. \quad (33)$$

Since

$$\begin{aligned} \frac{1}{k_r} |\{k(j) \leq k_r : j \in \mathbb{N}\}| \\ \geq \frac{1}{k_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| \\ = \frac{h_r}{k_r} \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}| \\ \geq \left(\frac{\delta}{1 + \delta} \right) \frac{1}{h_r} |\{k(j) \in I_r : j \in \mathbb{N}\}|, \end{aligned} \quad (34)$$

it follows by (33) that $\limsup_{r \rightarrow \infty} (1/k_r) |\{k(j) \leq k_r : j \in \mathbb{N}\}| > 0$. Since $(X_{k(j)})$ is already Δ^m -convergent to X_0 , so we have $X_0 \in \Lambda_S(\Delta^m(X))$. Hence $\Lambda_{S_\theta}(\Delta^m(X)) \subset \Lambda_S(\Delta^m(X))$.

(ii) If $\limsup q_r < \infty$, then there exists a real number H such that $q_r < H$ for all r . Without loss of generality, we can assume $H > 1$ (as otherwise $k_r < k_{r-1}$). Now for all r , $(h_r/k_{r-1}) = (k_r - k_{r-1})/k_{r-1} = q_r - 1 \leq H - 1$. Let $X_0 \in \Lambda_S(\Delta^m(X))$, then there is a set $K = \{k(j) : j \in \mathbb{N}\}$ with $\delta(K) \neq 0$ and $\lim_j \Delta^m X_{k(j)} = X_0$.

Let $N_r = |\{k \in I_r : k \in K\}| = |K \cap I_r|$ and $t_r = N_r/h_r$. For any integer n satisfying $k_{r-1} < n \leq k_r$, we can write

$$\begin{aligned} \frac{1}{n} |\{k \leq n : k \in K\}| &\leq \frac{1}{k_{r-1}} |\{k \leq k_r : k \in K\}| \\ &= \frac{1}{k_{r-1}} \{N_1 + N_2 + \dots + N_r\} \\ &= \frac{1}{k_{r-1}} \{t_1 h_1 + t_2 h_2 + \dots + t_r h_r\} \\ &= \frac{1}{\sum_{i=1}^{r-1} h_i} \sum_{i=1}^{r-1} h_i t_i + \frac{h_r}{k_{r-1}} t_r \\ &\leq \frac{1}{\sum_{i=1}^{r-1} h_i} \sum_{i=1}^{r-1} h_i t_i + (H - 1) t_r. \end{aligned} \quad (35)$$

Suppose $t_r \rightarrow 0$ as $r \rightarrow \infty$. Since θ is a lacunary sequence and the first part on the right side of above expression is a regular weighted mean transform of the sequence $t = (t_r)$, therefore it too tends to zero as $r \rightarrow \infty$. Since $n \rightarrow \infty$ as $r \rightarrow \infty$, it follows that $\delta(K) = 0$ which is a contradiction as $\delta(K) \neq 0$. Thus $\lim_{r \rightarrow \infty} t_r \neq 0$ and therefore $X_0 \in \Lambda_{S_\theta}(\Delta^m(X))$. Hence $\Lambda_S(\Delta^m(X)) \subset \Lambda_{S_\theta}(\Delta^m(X))$.

(iii) This is an immediate consequence of (i) and (ii). $\square$

Theorem 14. Let $\theta = (k_r)$ be a lacunary sequence and $X = (X_k)$ a sequence of fuzzy numbers. Then one has the following:

- (i) if $\liminf q_r > 1$, then $\Gamma_{S_\theta}(\Delta^m(X)) \subset \Gamma_S(\Delta^m(X))$,

- (ii) if $\limsup q_r < \infty$, then $\Gamma_S(\Delta^m(X)) \subset \Gamma_{S_\theta}(\Delta^m(X))$,

- (iii) if $1 < \liminf q_r \leq \limsup q_r < \infty$, then $\Gamma_S(\Delta^m(X)) = \Gamma_{S_\theta}(\Delta^m(X))$.

Proof. The proof of the theorem can be obtain on the similar lines as that of the above theorem and therefore is omitted here. $\square$

Theorem 15. For any lacunary refinement θ' of a lacunary sequence $\theta = (k_r)$, $\Gamma_{S_\theta}(\Delta^m(X)) \subseteq \Gamma_{S_{\theta'}}(\Delta^m(X))$ and $\Lambda_{S_\theta}(\Delta^m(X)) \subseteq \Lambda_{S_{\theta'}}(\Delta^m(X))$.

Proof. Suppose each I_r of θ contains the points $\{k'_{r,i}\}_{i=1}^{v(r)}$ of θ' so that $k_{r-1} < k'_{r,1} < k'_{r,2} < \dots < k'_{r,v(r)} = k_r$, where $I'_r = (k'_{r,i-1}, k'_{r,i}]$. Note that for all r , $v(r) \geq 1$. Let $(I_j^*)_{j=1}^\infty$ be the sequence of abutting intervals $I'_{r,i}$ ordered by increasing right end points. Let $Y_0 \in \Gamma_{S_\theta}(\Delta^m(X))$, then for each $\epsilon > 0$,

$$\limsup_{r \rightarrow \infty} \frac{1}{h_r} \left| \{k \in I_r : \bar{d}(\Delta^m X_k, Y_0) < \epsilon\} \right| > 0. \quad (36)$$

As before, write $h'_{r,i} = k'_{r,i} - k'_{r,i-1}$ and $h'_{r,1} = k'_{r,1} - k'_{r-1}$. Now for each $\epsilon > 0$, we can write

$$\begin{aligned} & \frac{1}{h_r} \left| \left\{ k \in I_r : \bar{d}(\Delta^m X_k, Y_0) < \epsilon \right\} \right| \\ &= \frac{1}{h_r} \sum_{I_j^* \subseteq I_r} h_j^* \frac{1}{h_j^*} \left| \left\{ k \in I_j^* : \bar{d}(\Delta^m X_k, Y_0) < \epsilon \right\} \right| \quad (37) \\ &= \frac{1}{h_r} \sum_{I_j^* \subseteq I_r} h_j^* (C_{\theta'} \chi_K)_j, \end{aligned}$$

where χ_K is the characteristics function of the set $K = \{k \in I_j^* : \bar{d}(\Delta^m X_k, Y_0) < \epsilon\}$ and $(C_{\theta'} \chi_K)_j = |K \cap I_j^*|/h_j^*$. Suppose $\lim_{j \rightarrow \infty} (C_{\theta'} \chi_K)_j = 0$. Then the right side of above expression is a regular weighted mean transform of $(C_{\theta'} \chi_K)_j$ and therefore tends to zero as $j \rightarrow \infty$ which contradicts (36). Thus $\lim_{j \rightarrow \infty} (C_{\theta'} \chi_K)_j \neq 0$, which shows that $Y_0 \in \Gamma_{S_{\theta'}}(\Delta^m(X))$. Hence $\Gamma_{S_{\theta'}}(\Delta^m(X)) \subseteq \Lambda_{S_{\theta'}}(\Delta^m(X))$.

Similarly, we can prove $\Lambda_{S_{\theta}}(\Delta^m(X)) \subseteq \Gamma_{S_{\theta}}(\Delta^m(X))$. $\square$

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