Research Article

On α -Layer Ideal Topologies

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The purpose of this paper is to study theory of two different kinds of α -layer order-preserving operator space, namely, ω_α -*opos* and $\omega_\alpha^*(\mathfrak{I})$ -*opos*. The former kind of space is formed by α -layer function in L -fuzzy order-preserving operator space. The later kind of space is derived by local α -remote neighborhood function, which is related with ω_α -*opos* and α -ideal. We study characteristic properties of the two kinds of spaces, respectively, and give some applications to show the intimate relations under two different $\omega_\alpha^*(\mathfrak{I})$ -*oposs*.

1. Introduction

In general topology, Vaidyanathaswamy firstly defined concepts of local function and its derived ideal topology from initial topology and ideal [1]. Some interesting extensive works were done by Janković and Hamlett [2]. After M. E. Abd El-Monsef introduced the concept of I -open set, many researchers were devoted to research on local semitopology. There are many local semiopen sets, such as β - I -open set, strong β - I -open set, and δ - I -open set [3–5]. All of these local semiopen sets are given by comparing interior and closure operators in the initial topology and its ideal derived topology.

By utilizing q -neighborhood which is mentioned in [6], Sarkar generalized the concepts of local function and derived topology into fuzzy topology in 1997 [7].

As there is a layer structure in L -fuzzy topology, fuzzy local functions and their derived fuzzy ideal topology must be more complex. Hence, in this paper, we will analyze the ideal topological properties in terms of layer structure of L -fuzzy topology and reveal the inner relations between α -layer ideal space and fuzzy ideal topological space.

In the first part of the paper, we establish the theory of ω_α -*opos*. We introduce the concept of α -layer function in L -fuzzy order-preserving operator space. Then, based on its basic properties, we form ω_α -*opos*. We also prove it preserves

many good properties. In the second part of the paper, we establish the theory of $\omega_\alpha^*(\mathfrak{I})$ -*opos*. It is a α -layer space with local topological properties. We introduce the concept of local α -remote neighborhood function via an ω_α -*opos* and an α -ideal. On the basis of its basic properties, we form $\omega_\alpha^*(\mathfrak{I})$ -*opos*. It is finer than the old one. We observe the structures of local α -remote neighborhood functions under different α -ideals and ω_α -*opos* as well as the relations of the correspondent ω_α^* -*opos*. We also obtain some equivalent conditions of ω_α^* -*opos* under compatibility of α -ideal and ω_α -*opos*. Finally, as an application, we define four kinds of connectivity and reveal their inner relations.

2. Preliminaries

In a general topological space (X, τ) , $\mathfrak{I} \subset X$ is an ideal, $U \subset X$. The concept of local function of U with respect to τ and $\mathfrak{I}$ is given by $U^* = \{x \in X : U \cap V \notin \mathfrak{I}, V \in \mathcal{N}(x)\}$, in which $\mathcal{N}(x)$ is the neighborhood system of x [1].

In this paper, a lattice L is called a completely distributive lattice with an order preserving involution'. X, Y will always denote nonempty crisp sets, A mapping $A : X \rightarrow L$ is called an L -fuzzy set. L^X is the set of all L -fuzzy sets on X . An element $e \in L$ is called an irreducible element in L , if $p \vee q = e$ implies $p = e$ or $q = e$, where $p, q \in L$. The set of

all nonzero irreducible elements in L will be denoted by $M(L)$ (see [1]). If $x \in X$, $\alpha \in M(L)$, then x_α is called a molecule in L^X . The set of all molecules in L^X is denoted by $M^*(L^X)$. If $A \in L^X$, $\alpha \in M(L)$, take $A_{[\alpha]} = \{x \in X : A(x) \geq \alpha\}$. If $E \subset X$, the complement of E , denoted by E' , and $E' = X - E = \{y \in X : y \notin E\}$ [8].

An L -fuzzy order-preserving operator space and some related conceptions are given in the following.

Let X be a nonempty set. An operator $\omega : L^X \rightarrow L^X$ is called a L -fuzzy order preserving operator in L^X , if it satisfies (1) $\omega(1_X) = 1_X$, (2) for all $A, B \in L^X$ and $A \leq B$ implies $\omega(A) \leq \omega(B)$. A set $A \in L^X$ is called an ω -set, if $\omega(A) = A$. The set of all ω -sets in L^X is denoted by Ω . And (L^X, Ω) is called an order-preserving operator space (briefly, L -fopos). A molecule $x_\alpha \in M^*(L^X)$, $P \in \Omega$, P is called an ω -remote neighborhood of x_α , if $x_\alpha \notin P$. The set of all ω -remote neighborhood of x_α is denoted by $\omega\eta(x_\alpha)$. Let $x_\alpha \in M^*(L^X)$, $A \in L^X$, x_α is called an ω -adherent point of A , if for all $P \in \omega\eta(x_\alpha)$, $A \not\subset P$. The union of all ω -adherent points of A is called the ω -closure of A , denoted by A_ω^- . A set $A \in L^X$ is called ω -closed, if $A_\omega^- = A$. The set of all ω -closed sets in L^X is denoted by Ω_ω^- . Ω_ω^- is finite union and infinite intersection preserving [9].

Similar concepts in a general topology are defined as follows.

Let X be a nonempty set, and let $\mathcal{P}(X)$ be the family of all subsets of X . An operator $\sigma : X \rightarrow X$ is called an order-preserving operator in X , if it satisfies (1) $\sigma(X) = X$, (2) for all $A \subset B \subset X$ implies $\sigma(A) \subset \sigma(B)$. A set $A \subset X$ is called a σ -set, if $\sigma(A) = A$. The set of all σ -sets in X is denoted by Δ . (X, Δ) is called an order-preserving operator space on X (briefly, opos). Let $x \in X$, $P \in \Delta$, P is called a σ -remote neighborhood of x , if there is $Q \subset X$, such that $x \notin Q$, $P \subset Q$. The set of all σ -remote neighborhood of x is denoted by $\sigma\eta(x)$. Let $x \in X$, $A \subset X$, x is called a σ -adherent point of A , if for all $P \in \sigma\eta(x)$, $A \not\subset P$. The union of all σ -adherent points of A is called the σ -closure of A , denoted by A_σ^- . A set $A \subset X$ is called σ -closed, if $A_\sigma^- = A$. The set of all σ -closed sets in X is denoted by Δ_σ^- . Δ_σ^- is finite union and infinite intersection preserving.

Let (X, Δ) be an opos, and let L be a fuzzy lattice, for all $\alpha \in L$. A fuzzy set $A : L \rightarrow X$ is called a lower continuous function, if $\{x \in X : A(x) \leq \alpha\} \in \Delta_\sigma^-$. Then, the set of all the lower continuous functions, denoted by $\omega_L(\Delta)$, consists an L -fuzzy cotopology in L^X . The space $(L^X, \omega_L(\Delta))$ is called the induced L -fopos by (X, Δ) .

A nonempty subfamily $\mathfrak{I}$ of L^X is called an α -ideal, if $\mathfrak{I}$ satisfies the following conditions:

- (1) $A \in \mathfrak{I}$, and $B_{[\alpha]} \subset A_{[\alpha]}$ implies $B \in \mathfrak{I}$,
- (2) $A, B \in \mathfrak{I}$ implies $A \vee B \in \mathfrak{I}$.

It is easy to check if $B \in L^X$, $A \in \mathfrak{I}$, such that $B_{[\alpha]} = A_{[\alpha]}$, then $B \in \mathfrak{I}$. Moreover, $\mathfrak{I}$ of L^X is an α -idea if and only if $\mathfrak{I}_{[\alpha]} = \{A_{[\alpha]} : A \in \mathfrak{I}\}$ is an ideal on X . If $B \in L^X$, $A \in \mathfrak{I}$, such that $B_{[\alpha]} = A_{[\alpha]}$, then $B \in \mathfrak{I}$. If $\mathfrak{I}, \mathfrak{J}$ are two α -ideals, then $\mathfrak{I} \vee \mathfrak{J} = \{I \vee J : I \in \mathfrak{I}, J \in \mathfrak{J}\}$ and $\mathfrak{I} \cap \mathfrak{J}$ are α -ideals too.

In this paper, if $\mathfrak{F} \subset L^X$, we denote $\mathfrak{F}_{[\alpha]} = \{A : A \in \mathfrak{F}\}$ and $\mathfrak{F}' = \{A' : A \in \mathfrak{F}\}$.

3. ω_α -Closed Set and ω_α -opos

In this section, we list out the main results in [10], which we will use in the following sections. Proofs of the theorems in the following can be found in [10] as well.

Definition 1. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. An operator $\omega_\alpha : L^X \rightarrow L^X$ is defined by, for all $A \in L^X$,

$$\omega_\alpha(A) = \bigwedge \{G \in \Omega_\omega^- : G_{[\alpha]} \supset A_{[\alpha]}\}. \quad (1)$$

Theorem 2. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. $A, B \in L^X$, then the following statements hold:

- (1) $A_{[\alpha]} \subset B_{[\alpha]}$ implies $\omega_\alpha(A) \leq \omega_\alpha(B)$;
- (2) $A_{[\alpha]} \subset \omega_\alpha(A)_{[\alpha]}$;
- (3) $\omega_\alpha(A \vee B) = \omega_\alpha(A) \vee \omega_\alpha(B)$;
- (4) $\omega_\alpha(\omega_\alpha(A)) = \omega_\alpha(A)$.

Definition 3. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. A set $A \in L^X$ is called ω_α -closed, if $\omega_\alpha(A)_{[\alpha]} = A_{[\alpha]}$. The set of all ω_α -closed sets in L^X is denoted by $\omega_\alpha(\Omega)$. $(L^X, \omega_\alpha(\Omega))$ is called an ω_α -order-preserving operator space (briefly, ω_α -opos).

Theorem 4. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. Then,

- (1) $1_X \in \omega_\alpha(\Omega)$;
- (2) if $\{A_i : i = 1, 2, \dots, n\} \subset \omega_\alpha(\Omega)$, then $\bigvee_{i=1}^n A_i \in \omega_\alpha(\Omega)$;
- (3) if $\{A_i : i \in I\} \subset \omega_\alpha(\Omega)$, then $\bigwedge_{i \in I} A_i \in \omega_\alpha(\Omega)$.

Remark 5. The theorem shows that ω_α is finite union and infinite intersection preserving.

Corollary 6. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. Then, $\omega_\alpha(\Omega)$ consists an L -fuzzy cotopology on L^X .

Lemma 7. Let (X, Δ) be an opos, $(L^X, \omega_L(\Delta))$ be an L -fopos induced by (X, Δ) and let $E \subset X$. Then, $E \in \Delta_\sigma^-$ if and only if $\chi_E \in (\omega_L(\Delta))_\omega^-$.

Theorem 8. Let (X, Δ) be an opos, $(L^X, \omega_L(\Delta))$ an L -fopos induced by (X, Δ) . $A \in L^X$, $\alpha \in M(L)$. Then, $A \in \omega_\alpha(\omega_L(\Delta))$ if and only if $A_{[\alpha]} \in \Delta_\sigma^-$.

Theorem 9. Let (X, Δ) be an opos, $(L^X, \omega_L(\Delta))$ is an L -fopos induced by (X, Δ) . Then, $A \in \omega_L(\Delta)$ if and only if for all $\alpha \in M(L)$, $A \in \omega_\alpha(\omega_L(\Delta))$.

Definition 10. Let $(L^X, \omega_\alpha^1(\Omega_1))$, $(L^Y, \omega_\alpha^2(\Omega_2))$ be ω_α^1 -opos, ω_α^2 -opos, respectively. An L -fuzzy homomorphism $f^- : L^X \rightarrow L^Y$ is called $(\omega_\alpha^1(\Omega_1), \omega_\alpha^2(\Omega_2))$ -continuous, if for all $B \in \omega_\alpha^2(\Omega_2)$, then $f^-(B) \in \omega_\alpha^1(\Omega_1)$.

Theorem 11. Let $(L^X, \omega_\alpha^1(\Omega_1))$, $(L^Y, \omega_\alpha^2(\Omega_2))$ be ω_α^1 -opos, ω_α^2 -opos, respectively. $f^- : L^X \rightarrow L^Y$ is an L -fuzzy mapping. Then, the following statements are equivalent:

- (1) f^- is $(\omega_\alpha^1(\Omega_1), \omega_\alpha^2(\Omega_2))$ -continuous;
- (2) for all $B \in L^Y$, $\omega_\alpha^1(f^-(B))_{[\alpha]} \subset f^-(\omega_\alpha^2(B))_{[\alpha]}$;
- (3) for all $A \in L^X$, $f^-(\omega_\alpha^1(A))_{[\alpha]} \subset \omega_\alpha^2(f^-(A))_{[\alpha]}$.

Definition 12. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, $x_\alpha \in M^*(L^X)$, $A \in \omega_\alpha(\Omega)$. A is called an ω_α -closed remote neighborhood of a , if $x \notin A_{[\alpha]}$. The set of all ω_α -closed remote neighborhood of x_α will be denoted by $\eta_{\omega_\alpha}(x_\alpha)$. A fuzzy point x_α is called ω_α -adherent point of $B \in L^X$, if for every $P \in \eta_{\omega_\alpha}(x_\alpha)$, $B_{([\alpha])} \not\subset P_{([\alpha])}$.

Remark 13. $P \in \eta_{\omega_\alpha}(x_\alpha) \Leftrightarrow (P_{[\alpha]})' \in \mathcal{N}(x)$, where $\mathcal{N}(x)$ is the neighborhood system of x in $(X, \omega_\alpha(\Omega)_{[\alpha]})$. Hence, $(\eta_{\omega_\alpha}(x_\alpha)_{[\alpha]})' = \mathcal{N}(x)$.

Theorem 14. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, $x_\alpha \in M^*(L^X)$, $A \in L^X$, then

- (1) x_α is an ω_α -adherent point of $A \in L^X$ if and only if $x \in \omega_\alpha(A)_{[\alpha]}$,
- (2) $\omega_\alpha(A)$ is the union of all ω_α -adherent points of A .

4. ω_α - $\mathfrak{I}$ -Closed Sets

Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. $\mathfrak{I} \subset L^X$ be an α -ideal. $x_\alpha \in M^*(L^X)$, $\eta_{\omega_\alpha}(x_\alpha)$ be the ω_α -remote family of x_α . For any $A \in L^X$, take

$$A^*(\mathfrak{I}, \omega_\alpha) = \bigvee \{x_\alpha : A_{[\alpha]} \not\subset P_{[\alpha]} \cup I_{[\alpha]}, \forall P \in \eta_{\omega_\alpha}(x_\alpha), I \in \mathfrak{I}\}. \quad (2)$$

Then, $A^*(\mathfrak{I}, \omega_\alpha)$ is called the local ω_α -remote function of A with respect to $\mathfrak{I}$ and $\omega_\alpha(\Omega)$, simply denoted by $A^*(\mathfrak{I})$ or A^* .

Remark 15. If $L = [0, 1]$, ω is the closure operator, then $(A_{[\alpha]}^*)'$ is the local function of $A_{[\alpha]}$ in $(X, \omega_\alpha(\Omega)_{[\alpha]})$, and $\mathcal{A}^* = \bigvee_{\alpha \in L} (A_{[\alpha]}^*)'$ is the fuzzy local function of A in [7]. Furthermore, we have $(A_{[\alpha]}^*)' = (A_{[\alpha]})'^*$.

Theorem 16. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. $\mathfrak{I}, \mathfrak{J} \subset L^X$ be two α -ideals. $A, B \in L^X$. Then, the following statements hold.

- (1) $\mathfrak{I}_{[\alpha]} = \{\emptyset\} \Rightarrow A^*(\mathfrak{I})_{[\alpha]} = \omega_\alpha(A)_{[\alpha]}$, $\mathfrak{I}_{[\alpha]} = \mathcal{P}(X) \Rightarrow A^*(\mathfrak{I})_{[\alpha]} = \emptyset$.
- (2) $A_{[\alpha]} \subset B_{[\alpha]} \Rightarrow A^*(\mathfrak{I})_{[\alpha]} \subset B^*(\mathfrak{I})_{[\alpha]}$.
- (3) $\mathfrak{I}_{[\alpha]} \subset \mathfrak{J}_{[\alpha]} \Rightarrow A^*(\mathfrak{I})_{[\alpha]} \subset A^*(\mathfrak{J})_{[\alpha]}$.
- (4) $A^*(\mathfrak{I})_{[\alpha]} = \omega_\alpha(A^*(\mathfrak{I}))_{[\alpha]} \subset \omega_\alpha(A)_{[\alpha]}$.
- (5) $(A^*(\mathfrak{I}))^*(\mathfrak{I})_{[\alpha]} \subset A^*(\mathfrak{I})_{[\alpha]}$.
- (6) $A \in \mathfrak{I} \Rightarrow A^*(\mathfrak{I})_{[\alpha]} = \emptyset$.
- (7) $(A \vee B)^*(\mathfrak{I})_{[\alpha]} = A^*(\mathfrak{I})_{[\alpha]} \cup B^*(\mathfrak{I})_{[\alpha]}$.
- (8) $(A^* - B^*)_{[\alpha]} = ((A - B)^* - B^*)_{[\alpha]} \subset (A - B)^*_{[\alpha]}$.
- (9) $B \in \mathfrak{I} \Rightarrow (A \vee B)^*_{[\alpha]} = A^*_{[\alpha]} = (A - B)^*_{[\alpha]}$.
- (10) $P \in \omega_\alpha(\Omega) \Rightarrow P_{[\alpha]}^* \subset P_{[\alpha]}$.

Proof. (1) Suppose $\mathfrak{I}_{[\alpha]} = \{\emptyset\}$, by Theorem 14, we have

$$\begin{aligned} x_\alpha &\leq A^*(\mathfrak{I}, \omega_\alpha) \\ \Leftrightarrow x_\alpha &\leq \bigvee \{x_\alpha : A_{[\alpha]} \not\subset P_{[\alpha]}, \forall P \in \eta_{\omega_\alpha}(x_\alpha)\} \quad (3) \\ \Leftrightarrow x_\alpha &\leq \omega_\alpha(A). \end{aligned}$$

If $X \in \mathfrak{I}_{[\alpha]}$, then there must be $B \in \mathfrak{I}$, such that $A_{[\alpha]} = B_{[\alpha]} = \{\emptyset\}$. Therefore, $A^*(\mathfrak{I})_{[\alpha]} = \emptyset$.

(2), (3), and (6) Easy.

(4) $x \notin \omega_\alpha(A)_{[\alpha]}$ implies $\omega_\alpha(A) \in \eta_\alpha(x_\alpha)$. By $A_{[\alpha]} \subset \omega_\alpha(A)_{[\alpha]}$, we have $x \notin A_{[\alpha]}^*$. So $A_{[\alpha]}^* \subset \omega_\alpha(A)_{[\alpha]}$. Thus, $\omega_\alpha(A)_{[\alpha]}^* \subset \omega_\alpha(\omega_\alpha(A))_{[\alpha]} = \omega_\alpha(A)_{[\alpha]}$. Besides, $x \in \omega_\alpha(A^*)_{[\alpha]}$ implies for all $P \in \eta_{\omega_\alpha}(x_\alpha)$, $A_{[\alpha]}^* \not\subset P_{[\alpha]}$. Take $y \in A_{[\alpha]}^* - P_{[\alpha]}$, so $P \in \eta_{\omega_\alpha}(y_\alpha)$ and for all $Q \in \eta_{\omega_\alpha}(y_\alpha)$, $I \in \mathfrak{I}$, $A_{[\alpha]} \not\subset Q_{[\alpha]} \cup I_{[\alpha]}$. Particularly, let $P = Q$. So $A_{[\alpha]} \not\subset P_{[\alpha]} \cup I_{[\alpha]}$. This means $x \in A_{[\alpha]}^*$. Hence, $\omega_\alpha(A^*)_{[\alpha]} \subset A_{[\alpha]}^*$. Therefore, $\omega_\alpha(A^*)_{[\alpha]} = A_{[\alpha]}^*$.

(5) By (4), $(A^*(\mathfrak{I}))^*(\mathfrak{I})_{[\alpha]} = (\omega_\alpha(A^*(\mathfrak{I})))^*(\mathfrak{I})_{[\alpha]} = \omega_\alpha(\omega_\alpha(A^*(\mathfrak{I})))_{[\alpha]} = (\omega_\alpha(A^*(\mathfrak{I})))_{[\alpha]} = A^*(\mathfrak{I})_{[\alpha]}$.

(7) By (2), $(A \vee B)^*(\mathfrak{I})_{[\alpha]} \supset A^*(\mathfrak{I})_{[\alpha]} \cup B^*(\mathfrak{I})_{[\alpha]}$. Conversely, if and only if $x \in A^*(\mathfrak{I})_{[\alpha]} \cup B^*(\mathfrak{I})_{[\alpha]}$, then for all $P \in \eta_{\omega_\alpha}(x_\alpha)$, $I \in \mathfrak{I}$, $A_{[\alpha]} \not\subset P_{[\alpha]} \cup I_{[\alpha]}$, and $B_{[\alpha]} \not\subset P_{[\alpha]} \cup I_{[\alpha]}$. Hence, $(A \vee B)_{[\alpha]} \not\subset P_{[\alpha]} \cup I_{[\alpha]}$. This implies $x \in (A \vee B)^*(\mathfrak{I})_{[\alpha]}$. So $(A \vee B)^*(\mathfrak{I})_{[\alpha]} \subset A^*(\mathfrak{I})_{[\alpha]} \cup B^*(\mathfrak{I})_{[\alpha]}$.

(8) For each $x \in (A^* - B^*)_{[\alpha]}$, we may remark $x \in (A - B)^*_{[\alpha]}$. If not, there are $P \in \eta_{\omega_\alpha}(x_\alpha)$, $I \in \mathfrak{I}$, such that $(A - B)_{[\alpha]} \subset P_{[\alpha]} \cup I_{[\alpha]}$. Thus, $A_{[\alpha]} \subset P_{[\alpha]} \cup I_{[\alpha]} \cup B_{[\alpha]}$. Since $x \notin B_{[\alpha]}^*$, there are $Q \in \eta_{\omega_\alpha}(x_\alpha)$, $J \in \mathfrak{I}$, $B_{[\alpha]} \subset Q_{[\alpha]} \cup J_{[\alpha]}$. This shows

$$\begin{aligned} A_{[\alpha]} &\subset P_{[\alpha]} \cup I_{[\alpha]} \cup B_{[\alpha]} \subset P_{[\alpha]} \cup I_{[\alpha]} \cup Q_{[\alpha]} \cup J_{[\alpha]} \\ &= (P \vee Q)_{[\alpha]} \cup (I \vee J)_{[\alpha]}. \end{aligned} \quad (4)$$

Consequently, $x \notin A_{[\alpha]}^*$. But this contradicts with $x \in (A^* - B^*)_{[\alpha]}$. Therefore, $(A^* - B^*)_{[\alpha]} \subset ((A - B)^* - B^*)_{[\alpha]}$. The reverse inclusion is obvious. So $(A^* - B^*)_{[\alpha]} = ((A - B)^* - B^*)_{[\alpha]}$.

(9) Obviously, $(A \vee B)^*_{[\alpha]} = A_{[\alpha]}^* \cup B_{[\alpha]}^* = A_{[\alpha]}^*$. And $(A - B)^*_{[\alpha]} \subset A_{[\alpha]}^*$. Conversely, $x \notin (A - B)^*_{[\alpha]}$, there are $P \in \eta_{\omega_\alpha}(x_\alpha)$, $I \in \mathfrak{I}$, $(A - B)_{[\alpha]} \subset P_{[\alpha]} \cup I_{[\alpha]}$. So $A_{[\alpha]} = (A - B)_{[\alpha]} \cup B_{[\alpha]} \subset P_{[\alpha]} \cup I_{[\alpha]} \cup B_{[\alpha]} = P_{[\alpha]} \cup (I \vee B)_{[\alpha]}$. Thus, $x \notin A_{[\alpha]}^*$. Then, (9) holds.

(10) Suppose there is $x \in P_{[\alpha]}^* - P_{[\alpha]}$. So $P \in \eta_{\omega_\alpha}(x_\alpha)$. By $P_{[\alpha]} \subset (P \vee I)_{[\alpha]}$ for each $I \in \mathfrak{I}$, we have $x \notin P_{[\alpha]}$, a contradiction. $\square$

Theorem 17. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. Let $\mathfrak{I} \subset L^X$ be an ideal. $A \in L^X$. Take $\omega_\alpha^*(A) = A \vee A^*$, then

- (1) $\omega_\alpha^*(A)_{[\alpha]} = \emptyset$, if and only if $A_{[\alpha]} = \emptyset$,
- (2) $\omega_\alpha^*(A \vee B)_{[\alpha]} = \omega_\alpha^*(A)_{[\alpha]} \cup \omega_\alpha^*(B)_{[\alpha]}$,
- (3) $\omega_\alpha^*(\omega_\alpha^*(A))_{[\alpha]} = \omega_\alpha^*(A)_{[\alpha]}$,
- (4) $(\omega_\alpha^*(A^*))_{[\alpha]}^* = \omega_\alpha^*(A^*)_{[\alpha]} = (\omega_\alpha^*(A))_{[\alpha]}^* = (\omega_\alpha(A^*))_{[\alpha]}^* = A_{[\alpha]}^*$.

Proof. According to (5), (7) in Theorem 16, the proof is trivial.

By Theorem 17, we know if and only if we take $\omega_\alpha^*(\mathfrak{I}, \Omega) = \{A \in L^X : \omega_\alpha^*(A)_{[\alpha]} = A_{[\alpha]}\}$, then $\omega_\alpha^*(\mathfrak{I}, \Omega)$ (simply denoted by $\omega_\alpha^*(\mathfrak{I})$) consists an L -fuzzy cotopology on L^X . So it is called ω_α -cotopology formed by the ideal $\mathfrak{I}$ and $\omega_\alpha(\Omega)$. The pair $(L^X, \omega_\alpha^*(\mathfrak{I}, \Omega))$, simply denoted by $(L^X, \omega_\alpha^*(\mathfrak{I}))$, is called $\omega_\alpha^*(\mathfrak{I})$ -opos. By Theorem 16 (1), we have $\mathfrak{I}_{[\alpha]} = \{\emptyset\} \Rightarrow \omega_\alpha^*(A)_{[\alpha]} = \omega_\alpha(A)_{[\alpha]}$, thus $\omega_\alpha^*(\mathfrak{I}, \Omega) = \omega_\alpha(\Omega)$, and $\mathfrak{I}_{[\alpha]} = \mathcal{P}(X) \Rightarrow \omega_\alpha^*(A)_{[\alpha]} = A_{[\alpha]}$, thus $\omega_\alpha^*(\mathfrak{I}, \Omega) = L^X$. But in general, $\{\emptyset\} \subset \mathfrak{I} \subset \mathcal{P}(X)$, $A_{[\alpha]} \subset \omega_\alpha^*(A)_{[\alpha]} \subset \omega_\alpha(A)_{[\alpha]}$, consequently, $\omega_\alpha(\Omega) \subset \omega_\alpha^*(\mathfrak{I}, \Omega) \subset L^X$. Moreover, if and only if $\omega_\alpha(\Omega)_{[\alpha]} = \{\emptyset, X\}$, then $\omega_\alpha^*(\Omega) = \{A : A_{[\alpha]} \in \mathfrak{I}_{[\alpha]}\}$. $\square$

Example 18. Let $X = \{x, y\}$, $L = \{0, 1/2, 1\}$. An L -fuzzy set $G \in L^X$ satisfying $G(x) = a, G(y) = b$ will be denoted by (a, b) . Let $\Omega = \{(0, 0), (0, 1/2), (1, 1/2), (1, 1)\}$ be a cotopology on L^X . $\alpha = 1/2$. Put $\omega_\alpha(A) = \bigwedge \{E \in \Omega : E_{[\alpha]} \supset A_{[\alpha]}\}$. Then, $\omega_\alpha(\Omega) = \{A : \omega_\alpha(A)_{[\alpha]} = A_{[\alpha]}\}$ consists an ω_α -cotopology, and $\eta_{\omega_\alpha}(x_\alpha) = \{(0, 0), (0, 1/2), (0, 1)\}$, $\eta_{\omega_\alpha}(y_\alpha) = \{(0, 0)\}$. Totally, there are four kinds of α -ideals in L^X , namely, $\mathfrak{L} = \{(0, 0)\}$, $\mathfrak{M} = L^X$, $\mathfrak{I} = \{(0, 0), (0, 1/2), (0, 1)\}$ and $\mathfrak{J} = \{(0, 0), (1/2, 0), (1, 0)\}$. $\mathfrak{L}$ and $\mathfrak{M} = L^X$ are the trivial α -ideals. Let us study $\mathfrak{I}$ and $\mathfrak{J}$.

Denote $B = (0, 1/2), D = (1/2, 0), E = (1/2, 1/2)$. Then, $A^*(\mathfrak{I}) = B^*(\mathfrak{I}) = (0, 0), D^*(\mathfrak{I}) = E^*(\mathfrak{I}) = (1/2, 1/2)$. So $A, B, E \in \omega_\alpha^*(\mathfrak{I})$, But $D \notin \omega_\alpha^*(\mathfrak{I})$. In addition, $\omega_\alpha^*(\mathfrak{I}, \Omega) = \omega_\alpha(\Omega)$. However, $D^*(\mathfrak{J}) = (0, 0)$, which means $D \in \omega_\alpha^*(\mathfrak{J})$. It is easy to check $\omega_\alpha^*(\mathfrak{J}, \Omega) = L^X$.

In this example, we see $(\omega_\alpha(B))^*(\mathfrak{I})_{[\alpha]} = B^*(\mathfrak{I})_{[\alpha]} = \emptyset$. However, $\omega_\alpha^*(B)_{[\alpha]} = (B \vee B^*(\mathfrak{I}))_{[\alpha]} = \{y\}$. Thus, $(\omega_\alpha(B))^*(\mathfrak{I})_{[\alpha]} \neq \omega_\alpha^*(B)_{[\alpha]}$. Furthermore, $(\omega_\alpha(\Omega))^*(\mathfrak{I}) = \{A^*(\mathfrak{I}) : A \in \omega_\alpha(\Omega)\} = \{(0, 0), (1/2, 1/2)\}$ is not an ω_α -cotopology.

Theorem 19. Let $(L^X, \omega_\alpha(\Omega_i))$ ($i = 1, 2$) be an ω_α -opos. $\omega_\alpha(\Omega_1) \subset \omega_\alpha(\Omega_2)$. $\mathfrak{I}$ is an α -ideal. $A \in L^X$, then

- (1) $A^*(\mathfrak{I}, \omega_\alpha(\Omega_2))_{[\alpha]} \subset A^*(\mathfrak{I}, \omega_\alpha(\Omega_1))_{[\alpha]}$,
- (2) $A^*(\mathfrak{I}, \omega_\alpha^*(\mathfrak{I}, \Omega_1))_{[\alpha]} \subset A^*(\mathfrak{I}, \omega_\alpha(\Omega_1))_{[\alpha]}$,
- (3) $\omega_\alpha^*(\mathfrak{I}, \Omega_1) \subset \omega_\alpha^*(\mathfrak{I}, \Omega_2)$.

Theorem 20. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, an α -ideal $\mathfrak{I} \subset \omega_\alpha(\Omega)$. Then, $\omega_\alpha^*(\mathfrak{I}) = \omega_\alpha(\Omega)$.

Proof. Obviously, $\omega_\alpha^*(\mathfrak{I}) \supset \omega_\alpha(\Omega)$. If $A \notin \omega_\alpha(\Omega)$, that is, $A_{[\alpha]} \neq \omega_\alpha(A)_{[\alpha]}$. Take $y \in \omega_\alpha(A)_{[\alpha]} - A_{[\alpha]}$. Let us prove $y \in A^*(\mathfrak{I})_{[\alpha]}$. In fact, if $y \notin A^*(\mathfrak{I})_{[\alpha]}$, there are $Q \in \eta_{\omega_\alpha}(y_\alpha), I \in \mathfrak{I}$, such that $A_{[\alpha]} \subset (Q \vee I)_{[\alpha]}$. So

$$\begin{aligned} y \in \omega_\alpha(A)_{[\alpha]} &\subset \omega_\alpha(Q \vee I)_{[\alpha]} = (\omega_\alpha(Q) \vee \omega_\alpha(I))_{[\alpha]} \\ &= (Q \vee I)_{[\alpha]}. \end{aligned} \quad (5)$$

It means $y \in I_{[\alpha]}$. On the other hand, as $A_{[\alpha]} \subset (Q \vee I)_{[\alpha]}$, we have $(A \wedge I)_{[\alpha]} \subset Q_{[\alpha]}$. Then,

$$\begin{aligned} \omega_\alpha(A \wedge I)_{[\alpha]} &= (\omega_\alpha(A) \wedge \omega_\alpha(I))_{[\alpha]} \\ &= \omega_\alpha(A)_{[\alpha]} \cap I_{[\alpha]} \subset \omega_\alpha(Q)_{[\alpha]} \\ &= Q_{[\alpha]}. \end{aligned} \quad (6)$$

Notice that $y \in \omega_\alpha(A)_{[\alpha]} \cap I_{[\alpha]}$, we get $y \in Q_{[\alpha]}$, a contradiction. Hence, $y \in A^*(\mathfrak{I})_{[\alpha]}$. So $A^*(\mathfrak{I})_{[\alpha]} \supset \omega_\alpha(\Omega)$, which implies $A \notin \omega_\alpha^*(\mathfrak{I})$. Therefore, $\omega_\alpha^*(\mathfrak{I}) \subset \omega_\alpha(\Omega)$. The proof is completed. $\square$

Theorem 21. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, and let $\mathfrak{I}$ be an α -ideal. Take $\omega_\alpha(\Omega) \vee \mathfrak{I}$ as the supremum ω_α -cotopology generated by $\omega_\alpha(\Omega) \cup \mathfrak{I}$. Then, $\omega_\alpha^*(\mathfrak{I}) = \omega_\alpha(\Omega) \vee \mathfrak{I}$.

Proof. Put $\mathfrak{B} = \{P \vee I : P \in \omega_\alpha(\Omega), I \in \mathfrak{I}\}$. It is easy to prove $\mathfrak{B} \subset \omega_\alpha^*(\mathfrak{I})$ is a base of $\omega_\alpha^*(\mathfrak{I})$. $\square$

Lemma 22. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. Let $\mathfrak{I}, \mathfrak{J} \subset L^X$ be two ideals. $I \in \mathfrak{I}, A \in L^X$. Then, $(A - I)^*(\mathfrak{J}, \omega_\alpha(\Omega)) = A^*(\mathfrak{J}, \omega_\alpha^*(\mathfrak{I}))$.

Proof. By Theorem 21,

$$\begin{aligned} x \notin (A - I)^*(\mathfrak{J}, \omega_\alpha(\Omega)) &\iff \exists P \in \eta_{\omega_\alpha}(x_\alpha), I \in \mathfrak{I}, \\ &\quad (A - I)_{[\alpha]} \subset (P \vee I)_{[\alpha]} \\ &\iff \exists P \in \eta_{\omega_\alpha}(x_\alpha), I \in \mathfrak{I}, \\ &\quad A_{[\alpha]} \subset (P \vee I \vee I)_{[\alpha]} \quad (7) \\ &\iff \exists Q \in \eta_{\omega_\alpha^*(\mathfrak{I})}(x_\alpha), I \in \mathfrak{J}, \\ &\quad A_{[\alpha]} \subset (Q \vee I)_{[\alpha]} \\ &\iff x \notin A^*(\mathfrak{J}, \omega_\alpha^*(\mathfrak{I})). \end{aligned}$$

$\square$

Theorem 23. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. Let $\mathfrak{I}, \mathfrak{J} \subset L^X$ be two α -ideals. $A \in L^X$, then

- (1) $A^*(\mathfrak{I} \cap \mathfrak{J})_{[\alpha]} = A^*(\mathfrak{I})_{[\alpha]} \cup A^*(\mathfrak{J})_{[\alpha]}$,
- (2) $A^*(\mathfrak{I} \vee \mathfrak{J}, \omega_\alpha(\Omega))_{[\alpha]} = A^*(\mathfrak{I}, \omega_\alpha^*(\mathfrak{J}))_{[\alpha]} \cap A^*(\mathfrak{J}, \omega_\alpha^*(\mathfrak{I}))_{[\alpha]}$.

Proof. (1) Suppose $x \notin A^*(\mathfrak{I} \cap \mathfrak{J})_{[\alpha]}$, there are $P, Q \in \eta_{\omega_\alpha}(x_\alpha)$, and $I \in \mathfrak{I}, J \in \mathfrak{J}$, such that $A_{[\alpha]} \subset (P \vee I)_{[\alpha]}$, and $A_{[\alpha]} \subset (Q \vee J)_{[\alpha]}$. Thus, $A_{[\alpha]} \subset (P \vee I) \wedge (Q \vee J)_{[\alpha]}$. As $(P \wedge Q) \vee (P \wedge J) \vee (I \wedge Q) \in \eta_{\omega_\alpha}(x_\alpha)$, and $I \wedge J \in \mathfrak{I} \cap \mathfrak{J}$, it is clear that $x \notin A^*(\mathfrak{I} \cap \mathfrak{J})_{[\alpha]}$. So $A^*(\mathfrak{I} \cap \mathfrak{J})_{[\alpha]} \subset A^*(\mathfrak{I})_{[\alpha]} \cup A^*(\mathfrak{J})_{[\alpha]}$. The reverse inclusion is obvious according to (3) in Theorem 16. Therefore, (1) holds.

(2) Suppose $x \notin A^*(\mathfrak{I} \vee \mathfrak{J}, \omega_\alpha(\Omega))_{[\alpha]}$. There are $P \in \eta_{\omega_\alpha}(x_\alpha)$, and $I \in \mathfrak{I}, J \in \mathfrak{J}, K = I \vee J$, such that $A_{[\alpha]} \subset (P \vee K)_{[\alpha]}$. So $(A - J)_{[\alpha]} \subset (P \vee I)_{[\alpha]}$, $(A - I)_{[\alpha]} \subset (P \vee J)_{[\alpha]}$. Hence, by Lemma 22, we have $x \notin (A - J)^*(\mathfrak{I}, \omega_\alpha(\Omega))_{[\alpha]} = A^*(\mathfrak{I}, \omega_\alpha^*(\mathfrak{J}))_{[\alpha]}$, and $x \notin (A - I)^*(\mathfrak{J}, \omega_\alpha(\Omega))_{[\alpha]} = A^*(\mathfrak{J}, \omega_\alpha^*(\mathfrak{I}))_{[\alpha]}$. This shows that $x \notin A^*(\mathfrak{I}, \omega_\alpha^*(\mathfrak{J}))_{[\alpha]} \cap A^*(\mathfrak{J}, \omega_\alpha^*(\mathfrak{I}))_{[\alpha]}$. Therefore, $A^*(\mathfrak{I} \vee \mathfrak{J}, \omega_\alpha(\Omega))_{[\alpha]} \subset A^*(\mathfrak{I}, \omega_\alpha^*(\mathfrak{J}))_{[\alpha]} \cap A^*(\mathfrak{J}, \omega_\alpha^*(\mathfrak{I}))_{[\alpha]}$.

Conversely, suppose $x \notin A^*(\mathfrak{I}, \omega_\alpha^*(\mathfrak{J}))_{[\alpha]}$. Then, there exists $P \in \eta_{\omega_\alpha^*(\mathfrak{J})}(x_\alpha)$, and $I \in \mathfrak{I}$, such that $A_{[\alpha]} \subset (P \vee I)_{[\alpha]}$. Because of $P \in \omega_\alpha^*(\mathfrak{J}, \Omega) = \omega_\alpha(\Omega) \vee \mathfrak{J}$, there is $J \in \mathfrak{J}$, such that $\omega_\alpha(P - J) \in \eta_{\omega_\alpha}(x_\alpha)$. Hence, $A_{[\alpha]} \subset (P \vee I)_{[\alpha]} = (P - J)_{[\alpha]} \cup (I \vee J)_{[\alpha]}$. This shows $x \notin A^*(\mathfrak{I} \vee \mathfrak{J}, \omega_\alpha(\Omega))_{[\alpha]}$. Therefore, $A^*(\mathfrak{I} \vee \mathfrak{J}, \omega_\alpha(\Omega))_{[\alpha]} \subset A^*(\mathfrak{I}, \omega_\alpha^*(\mathfrak{J}))_{[\alpha]}$. Similarly,

we can prove $A^*(\mathfrak{I} \vee \mathfrak{J}, \omega_\alpha(\Omega))_{[\alpha]} \subset A^*(\mathfrak{J}, \omega_\alpha^*(\mathfrak{I}))_{[\alpha]}$. So $A^*(\mathfrak{I} \vee \mathfrak{J}, \omega_\alpha(\Omega))_{[\alpha]} \subset A^*(\mathfrak{I}, \omega_\alpha^*(\mathfrak{J}))_{[\alpha]} \cap A^*(\mathfrak{J}, \omega_\alpha^*(\mathfrak{I}))_{[\alpha]}$. The proof is completed. $\square$

By Theorem 20, we get two important results.

Corollary 24. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. Let $\mathfrak{I}$ be an α -ideal, then $A^*(\mathfrak{I}, \omega_\alpha(\Omega))_{[\alpha]} = A^*(\mathfrak{I}, \omega_\alpha^*(\mathfrak{I}))_{[\alpha]}$, and as a result $\omega_\alpha^*(\mathfrak{I}) = \omega_\alpha^*(\omega_\alpha^*(\mathfrak{I}))(\mathfrak{I})$.

Proof. By (2) in Theorem 23, take $\mathfrak{I} = \mathfrak{J}$, clearly. $\square$

Corollary 25. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. Let $\mathfrak{I}, \mathfrak{J} \subset L^X$ be two α -ideals. Then,

- (1) $\omega_\alpha^*(\mathfrak{I} \vee \mathfrak{J}) = \omega_\alpha^*(\omega_\alpha^*(\mathfrak{J}))(\mathfrak{I}) = \omega_\alpha^*(\omega_\alpha^*(\mathfrak{I}))(\mathfrak{J})$,
- (2) $\omega_\alpha^*(\mathfrak{I} \vee \mathfrak{J}) = \omega_\alpha^*(\mathfrak{I}) \vee \omega_\alpha^*(\mathfrak{J})$,
- (3) $\omega_\alpha^*(\mathfrak{I} \cap \mathfrak{J}) = \omega_\alpha^*(\mathfrak{I}) \cap \omega_\alpha^*(\mathfrak{J})$.

Proof. (1) By (2) in Theorem 23, easy.

(2) Since $\omega_\alpha(\Omega) \subset \omega_\alpha^*(\mathfrak{I})$ for every α -ideal $\mathfrak{I}$, by (1) and Theorem 20, we have $\omega_\alpha^*(\mathfrak{I} \vee \mathfrak{J}) = \omega_\alpha^*(\omega_\alpha^*(\mathfrak{J}))(\mathfrak{I}) = \omega_\alpha^*(\mathfrak{I}) \vee \mathfrak{I} = \omega_\alpha(\Omega) \vee \omega_\alpha^*(\mathfrak{I}) \vee \mathfrak{I} = \omega_\alpha^*(\mathfrak{I}) \vee \omega_\alpha^*(\mathfrak{J})$.

(3) Clearly, $\omega_\alpha^*(\mathfrak{I} \cap \mathfrak{J}) \subset \omega_\alpha^*(\mathfrak{I}) \cap \omega_\alpha^*(\mathfrak{J})$. Conversely, $P \in \omega_\alpha^*(\mathfrak{I}) \cap \omega_\alpha^*(\mathfrak{J})$. Then, $P^*(\mathfrak{I})_{[\alpha]} \subset P_{[\alpha]}$, and $P^*(\mathfrak{J})_{[\alpha]} \subset P_{[\alpha]}$. So by (1) in Theorem 23, $P^*(\mathfrak{I} \cap \mathfrak{J})_{[\alpha]} = P^*(\mathfrak{I})_{[\alpha]} \cup P^*(\mathfrak{J})_{[\alpha]} \subset P_{[\alpha]}$. Therefore, $P \in \omega_\alpha^*(\mathfrak{I} \cap \mathfrak{J})$. $\square$

Theorem 26. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, and let $\mathfrak{I}$ be an α -ideal. $A \in L^X$. Then, $\omega_\alpha^*(A)_{[\alpha]} = \omega_\alpha(A)_{[\alpha]}$ if and only if $\mathfrak{I} \subset \omega_\alpha(\Omega)$.

Proof. Sufficiency. according to (4) in Theorem 16, $\omega_\alpha^*(A)_{[\alpha]} \subset \omega_\alpha(A)_{[\alpha]}$. On the other hand, if $x \notin \omega_\alpha^*(A)_{[\alpha]}$, then $y \notin A_{[\alpha]}$ and $y \notin A_{[\alpha]}$. So there are $P \in \eta_{\omega_\alpha}(y_\alpha)$, $I \in \mathfrak{I}$, such that $A_{[\alpha]} \subset (P \vee I)_{[\alpha]}$. Here, we may assume $y \notin I_{[\alpha]}$, since $\mathfrak{I}$ is an α -ideal. Therefore, $\omega_\alpha(A)_{[\alpha]} \subset \omega_\alpha(P \vee I)_{[\alpha]} = (P \vee I)_{[\alpha]}$. This implies $y \notin \omega_\alpha(A)_{[\alpha]}$. $\omega_\alpha^*(A)_{[\alpha]} \supset \omega_\alpha(A)_{[\alpha]}$.

Necessary. if $A \in \mathfrak{I}$, then $A_{[\alpha]}^* = \emptyset$. Since $A_{[\alpha]} = (A \vee A^*)_{[\alpha]} = \omega_\alpha^*(A)_{[\alpha]} = \omega_\alpha(A)_{[\alpha]}$, $A \in \omega_\alpha(\Omega)$.

Since $(X, \omega_\alpha(\Omega)_{[\alpha]})$ is an opos on X . $\mathfrak{I}_{[\alpha]}$ is an ideal on X . We have the following results. $\square$

Theorem 27. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, and let $\mathfrak{I}$ be an α -ideal. Then, for every $A \in L^X$, $(A_{[\alpha]})^* = A_{[\alpha]}^*$, and therefore $\omega_\alpha^*(A)_{[\alpha]} = cl^*(A_{[\alpha]})$.

Proof. Since

$$\begin{aligned} x \notin A_{[\alpha]}^* &\iff \exists P \in \eta_{\omega_\alpha}(x_\alpha), I \in \mathfrak{I}, A_{[\alpha]} \subset (P \vee I)_{[\alpha]} \\ &\iff \exists P_{[\alpha]} \in \omega_\alpha(\Omega)_{[\alpha]}, x \in (P_{[\alpha]})', \\ &\quad I \in \mathfrak{I}, A_{[\alpha]} \cap (P_{[\alpha]})' = I_{[\alpha]} \\ &\iff x \notin (A_{[\alpha]})^*, \end{aligned} \quad (8)$$

then $(A_{[\alpha]})^* = A_{[\alpha]}^*$, $\omega_\alpha^*(A)_{[\alpha]} = (A \vee A^*)_{[\alpha]} = A_{[\alpha]} \cup (A_{[\alpha]})^* = cl^*(A_{[\alpha]})$. $\square$

Corollary 28. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, and let $\mathfrak{I}$ be an α -ideal. Then, for every $A \in L^X$, $\mathfrak{X}_{A_{[\alpha]}}^* = \mathfrak{X}_{A_{[\alpha]}} = \mathfrak{X}_{(A_{[\alpha]})^*}$.

Proof. $x \notin (\mathfrak{X}_{A_{[\alpha]}}^*)_{[\alpha]} \iff$ there exists $P \in \eta_{\omega_\alpha}(x_\alpha)$, $I \in \mathfrak{I}, A_{[\alpha]} \subset (P \vee I)_{[\alpha]} \iff x \notin A_{[\alpha]}^* \iff x \notin (\mathfrak{X}_{A_{[\alpha]}}^*)_{[\alpha]}$. So $\mathfrak{X}_{A_{[\alpha]}}^* = \mathfrak{X}_{A_{[\alpha]}}$. Besides, by Theorem 27, the later equation is easy. $\square$

Corollary 29. Let (X, Δ) be an opos on X , $\mathfrak{I} \subset X$ an ideal. The induced ω_α -opos and α -ideal are denoted, respectively, by $(L^X, \omega_\alpha(\omega_L(\Delta)))$ and $\mathfrak{I} = \{\mathfrak{X}_I : I \in \mathfrak{I}\}$. Then, $\omega_\alpha^*(\mathfrak{I}, \omega_\alpha(\omega_L(\Delta)))_{[\alpha]} = \Delta^*$.

Proof. By Theorem 27 and Corollary 28, $\omega_\alpha^*(\mathfrak{I}, \omega_\alpha(\omega_L(\Delta)))_{[\alpha]} \subset \Delta^*$. On other hand, again by Corollary 28, for every $U \subset X$, $\omega_\alpha^*(\mathfrak{X}_U)_{[\alpha]} = (\mathfrak{X}_U \vee \mathfrak{X}_U^*)_{[\alpha]} = U \cup (\mathfrak{X}_U^*)_{[\alpha]} = U \cup U^* = cl^*(U)$. $\square$

5. Compatibility of $\omega_\alpha(\Omega)$ with $\mathfrak{I}$

Definition 30. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. $\mathfrak{R} \subset \omega_\alpha(\Omega)$ is called an α - ω_α -remote neighborhood family of A (briefly, α - ω_α -RF of A , if for all $x \in A_{[\alpha]}$, there is $P \in \mathfrak{R}$, such that $P \in \eta_{\omega_\alpha}(x_\alpha)$).

Definition 31. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. Let $\mathfrak{I}$ be an α -ideal. $\omega_\alpha(\Omega)$ is said to be compatible with $\mathfrak{I}$, denoted by $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}$, if for any $A \in L^X$ and $x \in A_{[\alpha]}$, there are $P \in \eta_{\omega_\alpha}(x_\alpha)$ and $I \in \mathfrak{I}$, such that $A_{[\alpha]} \subset (P \vee I)_{[\alpha]}$, then $A \in \mathfrak{I}$.

Example 32. In Example 18, we know $\omega_\alpha = \{(0,0), (0,1/2), (0,1), (1/2,1/2), (1/2,1), (1,1/2), (1,1)\}$, $\eta_{\omega_\alpha}(x_\alpha) = \{(0,0), (0,1/2), (0,1)\}$, $\eta_{\omega_\alpha}(y_\alpha) = \{(0,0)\}$. Take $\mathfrak{I}_1 = L^X$, $\mathfrak{I}_2 = \{(0,0)\}$, $\mathfrak{I}_3 = \{(0,0), (1/2,0), (1,0)\}$, $\mathfrak{I}_4 = \{(0,0), (0,1/2), (0,1)\}$.

It is easy to check $\mathfrak{I}_i$, ($i \leq 4$) are all α -ideals, $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}_1$ and $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}_2$. Take $G = (1/2, 1/2)$, then $x \in G_{[\alpha]}$ and $R = (0, 1/2) \in \eta_{\omega_\alpha}(x_\alpha)$, $J = (1/2, 0) \in \mathfrak{I}_3$, such that $G_{[\alpha]} \subset (R \vee J)_{[\alpha]}$. But $G \notin \mathfrak{I}_3$, therefore $\omega_\alpha(\Omega) \not\sim_\alpha \mathfrak{I}_3$. Furthermore, for all $A \in L^X$, if $A_{[\alpha]} = \emptyset$ or $A_{[\alpha]} = \{y\}$, then $A \in \mathfrak{I}_4$. If $A_{[\alpha]} = \{x\}$ or $A_{[\alpha]} = \{x, y\}$, there does not exist $E \in \eta_{\omega_\alpha}(x_\alpha)$, $I \in \mathfrak{I}_4$, such that $A_{[\alpha]} \subset (R \vee J)_{[\alpha]}$. Therefore, $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}_4$.

Theorem 33. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, $\mathfrak{I}$ an ideal, $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}$. Then, $\mathfrak{B} = \omega_\alpha^*(\mathfrak{I})$.

Proof. For every $B \in \omega_\alpha^*(\mathfrak{I})$, $B_{[\alpha]}^* \subset B_{[\alpha]}$. Put $A = B - B^*(\mathfrak{I})$. If $x \in A_{[\alpha]}$, then $x \notin B_{[\alpha]}^*$. So there are $P \in \eta_{\omega_\alpha}(x_\alpha)$ and $I \in \mathfrak{I}$, such that $B_{[\alpha]} \subset (P \vee I)_{[\alpha]}$. Thus, $A_{[\alpha]} \subset B_{[\alpha]} \subset (P \vee I)_{[\alpha]}$. Since $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}$, we get $A \in \mathfrak{I}$. Besides, by (4) in Theorem 16, $B^* \in \omega_\alpha(\Omega)$. Notice that $B_{[\alpha]} = (A \vee B^*)_{[\alpha]}$. Thus, $B \in \mathfrak{B}$, and $\omega_\alpha^*(\mathfrak{I}) \subset \mathfrak{B}$. The reverse inclusion is obvious. Therefore, $\mathfrak{B} = \omega_\alpha^*(\mathfrak{I})$. $\square$

Lemma 34. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, $\mathfrak{I}$ an α -ideal. Then, for every $A \in L^X$, $(A - A^*(\mathfrak{I}))_{[\alpha]} \cap (A - A^*(\mathfrak{I}))^*(\mathfrak{I})_{[\alpha]} = \emptyset$.

Proof. If $x \in (A - A^*(\mathfrak{I}))_{[\alpha]}$, so $x \notin A^*(\mathfrak{I})_{[\alpha]}$. That is, there are $P \in \eta_{\omega_\alpha}(x_\alpha)$, $I \in \mathfrak{I}$, such that $A_{[\alpha]} \subset (P \vee I)_{[\alpha]}$. As $(A - A^*(\mathfrak{I}))_{[\alpha]} \subset A_{[\alpha]}$, we have $x \notin (A - A^*(\mathfrak{I}))^*(\mathfrak{I})_{[\alpha]}$. Therefore, the conclusion holds. $\square$

Theorem 35. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, $\mathfrak{I}$ an ideal. Then, the following statements are equivalent.

- (1) $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}$.
- (2) If $A \in L^X$ has an α - ω_α -RF $\mathfrak{R}$, satisfying for all $P \in \mathfrak{R}$, there is $I \in \mathfrak{I}$, such that $A_{[\alpha]} \subset (P \vee I)_{[\alpha]}$, then $A \in \mathfrak{I}$.
- (3) For all $A \in L^X$, $(A \wedge A^*(\mathfrak{I}))_{[\alpha]} = \emptyset \Rightarrow A \in \mathfrak{I}$.
- (4) For all $A \in L^X$, $A - A^*(\mathfrak{I}) \in \mathfrak{I}$.
- (5) For all $A \in \omega_\alpha^*(\mathfrak{I})$, $A - A^*(\mathfrak{I}) \in \mathfrak{I}$.
- (6) For all $A \in L^X$, if $\nexists \emptyset \neq B_{[\alpha]} \subset B^*(\mathfrak{I})_{[\alpha]}$, $B_{[\alpha]} \subset A_{[\alpha]}$, then $A \in \mathfrak{I}$.

Proof. (1) $\Rightarrow$ (2) If $A \in L^X$ has a α - ω_α -RF $\mathfrak{R}$, satisfying the condition in (2), then for every $x \in A_{[\alpha]}$, there are $P \in \mathfrak{R}$ and $I \in \mathfrak{I}$ such that $P \in \eta_{\omega_\alpha}(x_\alpha)$ and $A_{[\alpha]} \subset (P \vee I)_{[\alpha]}$. As $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}$, we have $A \in \mathfrak{I}$.

(2) $\Rightarrow$ (1) By Definitions 30 and 31, clearly.

(1) $\Rightarrow$ (3) For all $A \in L^X$, $(A \wedge A^*(\mathfrak{I}))_{[\alpha]} = \emptyset$. If $A_{[\alpha]} = \emptyset$, then $A \in \mathfrak{I}$. If $x \in A_{[\alpha]}$, then $x \notin A^*(\mathfrak{I})_{[\alpha]}$, there are $P \in \eta_{\omega_\alpha}(x_\alpha)$ and $I \in \mathfrak{I}$, such that $A_{[\alpha]} \subset (P \vee I)_{[\alpha]}$. Because $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}$, we have $A \in \mathfrak{I}$.

(3) $\Rightarrow$ (1) if $A \in L^X$ satisfying, for every $x \in A_{[\alpha]}$, there are $P \in \eta_{\omega_\alpha}(x_\alpha)$ and $I \in \mathfrak{I}$, such that $A_{[\alpha]} \subset (P \vee I)_{[\alpha]}$. Then, $x \notin A^*(\mathfrak{I})_{[\alpha]}$. By (3), we have $A \in \mathfrak{I}$. Hence, $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}$.

(3) $\Rightarrow$ (4) By Lemma 34, obviously.

(4) $\Rightarrow$ (5) Directly.

(5) $\Rightarrow$ (1) If for every $A \in L^X$ and $x \in A_{[\alpha]}$, there are $P \in \eta_{\omega_\alpha}(x_\alpha)$ and $I \in \mathfrak{I}$, such that $A_{[\alpha]} \subset (P \vee I)_{[\alpha]}$, this means $(A \wedge A^*(\mathfrak{I}))_{[\alpha]} = \emptyset$. Put $B = A \vee A^*$, then $B_{[\alpha]}^* = (A \vee A^*)_{[\alpha]}^* = A_{[\alpha]}^* \cup (A^*)_{[\alpha]}^* = A_{[\alpha]}^*$. So $\omega_\alpha^*(B)_{[\alpha]} = (B \vee B^*(\mathfrak{I}))_{[\alpha]} = B_{[\alpha]}$. This means $B \in \omega_\alpha^*(\mathfrak{I})$. By (5), we have $B - B^* \in \mathfrak{I}$. Since

$$\begin{aligned} (B - B^*)_{[\alpha]} &= ((A \vee A^*) - (A \vee A^*)^*)_{[\alpha]} \\ &= (A \vee A^*)_{[\alpha]} - A_{[\alpha]}^* = A_{[\alpha]}, \end{aligned} \quad (9)$$

we draw a conclusion: for every $x \in A_{[\alpha]}$, $x \notin B_{[\alpha]}^* = A_{[\alpha]}^*$. Therefore, $A \in \mathfrak{I}$.

(4) $\Rightarrow$ (6) For any $A \in L^X$, $A = (A - A^*) \vee (A \wedge A^*)$. So $A_{[\alpha]}^* = ((A - A^*)_{[\alpha]}^* \cup (A \wedge A^*)_{[\alpha]}^*)_{[\alpha]}$. By (4), $A - A^* \in \mathfrak{I}$, so $(A - A^*)_{[\alpha]}^* = \emptyset$. Thus $A_{[\alpha]}^* = (A \wedge A^*)_{[\alpha]}^*$. Since $(A \wedge A^*)_{[\alpha]} \subset A_{[\alpha]}^*$, we have $(A \wedge A^*)_{[\alpha]} \subset (A \wedge A^*)_{[\alpha]}^*$. Here by the assumption in (6), $(A \wedge A^*)_{[\alpha]} = \emptyset$. This implies $A_{[\alpha]} = (A - A^*)_{[\alpha]}^*$. Again by (4), $A \in \mathfrak{I}$.

(6) $\Rightarrow$ (4) By Lemma 34, for any $A \in L^X$, $(A - A^*)_{[\alpha]} \cap (A - A^*)_{[\alpha]}^* = \emptyset$. Let's prove $A - A^*$ satisfies the condition in (6). In fact, if not, there is $\emptyset \neq B_{[\alpha]} \subset B_{[\alpha]}^*$, $B_{[\alpha]} \subset (A - A^*)_{[\alpha]}$, then $B_{[\alpha]} \subset B_{[\alpha]}^* \subset (A - A^*)_{[\alpha]}^*$. We have $\emptyset \neq B_{[\alpha]} \subset (A - A^*)_{[\alpha]} \cap (A - A^*)_{[\alpha]}^*$. A contradiction. Therefore, $A - A^* \in \mathfrak{I}$. $\square$

Theorem 36. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, $\mathfrak{I}$ be an α -ideal. Then the following statements are equivalent, and implied by $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}$.

- (1) For every $A \in L^X$, $(A \wedge A^*)_{[\alpha]} = \emptyset \Rightarrow A_{[\alpha]}^* = \emptyset$.
- (2) For every $A \in L^X$, $(A - A^*)_{[\alpha]}^* = \emptyset$.
- (3) For every $A \in L^X$, $(A \wedge A^*)_{[\alpha]}^* = A_{[\alpha]}^*$.

Proof. $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I} \Rightarrow$ (1). By (3) in Theorem 35, If $(A \wedge A^*)_{[\alpha]} = \emptyset \Rightarrow A \in \mathfrak{I}$. Thus $A^*(\mathfrak{I})_{[\alpha]} = \emptyset$.

(1) $\Rightarrow$ (2) By Lemma 34, $(A - A^*(\mathfrak{I}))_{[\alpha]} \cap (A - A^*(\mathfrak{I}))^*(\mathfrak{I})_{[\alpha]} = \emptyset$. Then by (1), we have $(A - A^*)_{[\alpha]}^* = \emptyset$.

(2) $\Rightarrow$ (3) $(A - A^*)_{[\alpha]} = (A - (A \wedge A^*))_{[\alpha]}$. By (8) in Theorem 16, $(A - A^*)_{[\alpha]}^* = (A - (A \wedge A^*))_{[\alpha]}^* \supset A_{[\alpha]}^* - (A \wedge A^*)_{[\alpha]}^*$. According to (2), $(A - A^*)_{[\alpha]}^* = \emptyset$. Thus, $A_{[\alpha]}^* = (A \wedge A^*)_{[\alpha]}^*$.

(3) $\Rightarrow$ (1) Straightforward. $\square$

Theorem 37. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos, $\mathfrak{I}$ be an α -ideal. $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}$. Then for every $A \in \omega_\alpha^*(\mathfrak{I})$ if and only if there are $B \in \omega_\alpha(\Omega)$ and $I \in \mathfrak{I}$, such that $A_{[\alpha]} = (B \vee I)_{[\alpha]}$.

Proof. Necessity. suppose $A \in \omega_\alpha^*(\mathfrak{I})$. So $A_{[\alpha]}^* \subset A_{[\alpha]}$ and $A_{[\alpha]} \subset ((A - A^*) \vee A^*)_{[\alpha]}$. By (4) in Theorem 2, and (5) in Theorem 17, we have $A^* \in \omega_\alpha(\Omega)$ and $A - A^* \in \mathfrak{I}$.

Sufficiency. suppose $A_{[\alpha]} = (B \vee I)_{[\alpha]}$, $\omega_\alpha(B)_{[\alpha]} = B_{[\alpha]}$ and $I \in \mathfrak{I}$. Then $A_{[\alpha]}^* = B_{[\alpha]}^* \cup I_{[\alpha]}^* = B_{[\alpha]}^* \subset \omega_\alpha(B)_{[\alpha]} = B_{[\alpha]}$ $\subset A_{[\alpha]}$. Therefore, $A \in \omega_\alpha^*(\mathfrak{I})$. $\square$

6. ω_α -Connectivity and ω_α^* -Connectivity

In this section, we shall introduce four kinds of connectedness and discuss their relations.

Concepts of $\omega_\alpha^*(\mathfrak{I})$ -separated sets and ω_α^* -connected are defined as:

Definition 38. Let (L^X, ω_α) and $(L^X, \omega_\alpha^*(\mathfrak{I}))$ be ω_α -opos and $\omega_\alpha^*(\mathfrak{I})$ -opos, respectively.

- (1) $A, B \in L^X$ are called ω_α -separated sets (resp. $\omega_\alpha^*(\mathfrak{I})$ -separated set), if $\omega_\alpha(A)_{[\alpha]} \cap B_{[\alpha]} = \omega_\alpha(B)_{[\alpha]} \cap A_{[\alpha]} = \emptyset$, (resp. $\omega_\alpha^*(A)_{[\alpha]} \cap B_{[\alpha]} = \omega_\alpha^*(B)_{[\alpha]} \cap A_{[\alpha]} = \emptyset$).
- (2) $G \in L^X$ is called an ω_α -connected set (resp. ω_α^* -connected set), if there not exist ω_α -separated sets $B, C \in L^X$, such that, $B_{[\alpha]} \neq \emptyset, C_{[\alpha]} \neq \emptyset$, and $G_{[\alpha]} = B_{[\alpha]} \cup C_{[\alpha]}$, (resp. there not exist $\omega_\alpha^*(\mathfrak{I})$ -separated sets $B, C \in L^X$, such that, $B_{[\alpha]} \neq \emptyset, C_{[\alpha]} \neq \emptyset$, and $G_{[\alpha]} = B_{[\alpha]} \cup C_{[\alpha]}$).

Theorem 39. Let $(L^X, \omega_\alpha^*(\Omega))$ be an ω_α^* -opos. A is ω_α^* -connected and $A_{[\alpha]}^* \subset B_{[\alpha]}^* \subset \omega_\alpha^*(A)_{[\alpha]}$. Then, B is ω_α^* -connected.

Proof. Let $B_{[\alpha]} = E_{[\alpha]} \cup F_{[\alpha]}$, and E, F are ω_α^* -separated. Take

$$E_{[\alpha]}^\Delta = A_{[\alpha]} \cap E_{[\alpha]}, \quad F_{[\alpha]}^\Delta = A_{[\alpha]} \cap F_{[\alpha]}. \quad (10)$$

Then, $A_{[\alpha]} = E_{[\alpha]}^\Delta \cup F_{[\alpha]}^\Delta$, and

$$\begin{aligned} \omega_\alpha(E_{[\alpha]}^\Delta)_{[\alpha]} \cap F_{[\alpha]}^\Delta &= ((E_{[\alpha]}^\Delta)^* \vee E_{[\alpha]}^\Delta)_{[\alpha]} \\ &\cap F_{[\alpha]}^\Delta \subset \omega_\alpha^*(E)_{[\alpha]} \cap F_{[\alpha]} \\ &= \emptyset. \end{aligned} \quad (11)$$

Similarly, we have $\omega_\alpha(F^\Delta)_{[\alpha]} \cap E_{[\alpha]}^\Delta = \emptyset$. Thus, E^Δ, F^Δ are ω_α^* -separated.

As A is $\omega_{[\alpha]}^*$ -connected, so $E_{[\alpha]}^\Delta = \emptyset$, or $F_{[\alpha]}^\Delta = \emptyset$. Take $E_{[\alpha]}^\Delta = \emptyset$, for example. Then, $A_{[\alpha]} = F_{[\alpha]}^\Delta$. This means $A_{[\alpha]} \subset F_{[\alpha]}$. As $\omega_\alpha^* A_{[\alpha]} \subset \omega_\alpha^* F_{[\alpha]}$, and

$$E_{[\alpha]} \subset E_{[\alpha]} \cap \omega_\alpha^*(A)_{[\alpha]} \subset E_{[\alpha]} \cap \omega_\alpha^*(F)_{[\alpha]} = \emptyset. \quad (12)$$

Then, B is ω_α^* -connected. $\square$

Theorem 40. Let (X, Δ) be an opos, $\mathcal{I} \subset X$ an ideal. Take $\mathfrak{I} = \{\mathcal{X}_I : I \in \mathcal{I}\}$. The induced ω_α -opos and $\omega_\alpha^*(\mathfrak{I})$ -opos are denoted, respectively, by $(L^X, \omega_\alpha(\omega_L(\Delta)))$ and $(L^X, \omega_\alpha^*(\mathfrak{I}, \omega_\alpha(\omega_L(\Delta))))$. Then, for every $A \in L^X$,

- (1) A is ω_α -connected in $(L^X, \omega_\alpha(\omega_L(\Delta)))$ if and only if $A_{[\alpha]} \subset X$ is connected in (X, Δ) ;
- (2) A is ω_α^* -connected in $(L^X, \omega_\alpha^*(\mathfrak{I}, \omega_\alpha(\omega_L(\Delta))))$ if and only if $A_{[\alpha]} \subset X$ is connected in (X, Δ^*) ;
- (3) $A_{[\alpha]} \subset X$ is connected in $(X, \Delta) \Rightarrow A_{[\alpha]} \subset X$ is connected in (X, Δ^*) .

Proof. (1) *Necessity.* suppose $A_{[\alpha]}$ is not connected, then there are nonempty $U, V \subset X$, such that $A_{[\alpha]} = U \cup V$, and $U_\sigma^- \cap V = V_\sigma^- \cap U = \emptyset$. By Theorem 8, $\omega_\alpha(1_U)_{[\alpha]} = U_\sigma^-$, $\omega_\alpha(1_V)_{[\alpha]} = V_\sigma^-$. So

$$\omega_\alpha(1_U)_{[\alpha]} \cap (1_V)_{[\alpha]} = \omega_\alpha(1_V)_{[\alpha]} \cap (1_U)_{[\alpha]} = \emptyset. \quad (13)$$

This means $1_U, 1_V$ are ω_α -separated sets, and $A_{[\alpha]} = (1_U)_{[\alpha]} \cup (1_V)_{[\alpha]}$. This is a contradiction with A is ω_α -connected.

Sufficiency. suppose A is not ω_α -connected. Then, there are ω_α -separated sets $B, C \in L^X$, such that $B_{[\alpha]}, C_{[\alpha]} \neq \emptyset$, and $A_{[\alpha]} = B_{[\alpha]} \cup C_{[\alpha]}$. Since $\omega_\alpha(B) \in \omega_\alpha(\omega_L(\Delta))$, by Theorem 8, we have $\omega_\alpha(B)_{[\alpha]} \in \Delta_\sigma^-$. Thus, $(B_{[\alpha]})_\sigma^- \subset (\omega_\alpha(B)_{[\alpha]})_\sigma^- = \omega_\alpha(B)_{[\alpha]}$. Therefore,

$$(B_{[\alpha]})_\sigma^- \cap C_{[\alpha]} \subset \omega_\alpha(B)_{[\alpha]} \cap C_{[\alpha]} = \emptyset. \quad (14)$$

Similarly, we get $(C_{[\alpha]})_\sigma^- \cap B_{[\alpha]} = \emptyset$. This means $A_{[\alpha]} \subset X$ is not connected. It is another contradiction.

(2) By Theorem 27, similarly.

(3) Notice that $(A_{[\alpha]})_\sigma^- \supset cl^*(A_{[\alpha]})$; easily. $\square$

Theorem 41. Let $(L^X, \omega_\alpha^*(\mathfrak{I}))$ be an $\omega_\alpha^*(\mathfrak{I})$ -opos. $A, B \in L^X$. Then,

- (1) ω_α -separated $\Rightarrow \omega_\alpha^*$ -separated,
- (2) A and B are ω_α^* -separated $\Rightarrow A^*$ and B^* are ω_α -separated $\Leftrightarrow A^*$ and B^* are ω_α^* -separated,
- (3) $A, B \in \omega_\alpha(\Omega) \cap \mathfrak{I}$, then A and B are ω_α -separated $\Leftrightarrow A$ and B are ω_α^* -separated $\Leftrightarrow A_{[\alpha]} \cap B_{[\alpha]} = \emptyset$,
- (4) $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I} \Rightarrow A - A^*$ and A^* are ω_α^* -separated.

Proof. (1) Since $\omega_\alpha^*(A)_{[\alpha]} \subset \omega_\alpha(A)_{[\alpha]}$ for every $A \in L^X$; easy.

(2) By Theorem 16 (4) and (5), we have $\omega_\alpha(A^*)_{[\alpha]} = A_{[\alpha]}^* \subset \omega_\alpha^*(A)_{[\alpha]}$ and $\omega_\alpha^*(A^*)_{[\alpha]} = (A^*)_{[\alpha]}^* \cup A_{[\alpha]}^* = A_{[\alpha]}^* = \omega_\alpha^*(A)_{[\alpha]}$; obvious.

(3) Since $A \in \omega_\alpha(\Omega) \cap \mathfrak{I}$, $A_{[\alpha]}^* = \emptyset$ and $\omega_\alpha(A)_{[\alpha]} = A_{[\alpha]} = \omega_\alpha^*(A)_{[\alpha]}$. The result follows.

(4) Since $\omega_\alpha(\Omega) \sim_\alpha \mathfrak{I}$, $A - A^* \in \mathfrak{I}$. So $\omega_\alpha^*(A - A^*)_{[\alpha]} \cap A_{[\alpha]}^* = (A - A^*)_{[\alpha]} \cap A_{[\alpha]}^* = \omega_\alpha^*(A^*)_{[\alpha]} \cap (A - A^*)_{[\alpha]} = \emptyset$. $\square$

Example 42. In Example 18, take $B = (0, 1/2)$, $D = (1/2, 0)$, $E = (1/2, 1/2)$. Then, $\omega_\alpha^*(B)(\mathfrak{I})_{[\alpha]} = \omega_\alpha(B)_{[\alpha]} = B_{[\alpha]}$, $\omega_\alpha^*(D)(\mathfrak{I})_{[\alpha]} = \{x\}$. $\omega_\alpha(D)_{[\alpha]} = \{x, y\}$. Thus, B and D are $\omega_\alpha^*(\mathfrak{I})$ -separated, but not ω_α -separated. Therefore, $E = (1/2, 1/2)$ is ω_α -connected, but not $\omega_\alpha^*(\mathfrak{I})$ -connected. Moreover, $B, E \in \mathfrak{I}$ implies $B^*(\mathfrak{I})_{[\alpha]} = \emptyset$, $E^*(\mathfrak{I})_{[\alpha]} = \emptyset$. So A^* and B^* are ω_α -separated. However, since $\omega_\alpha^*(\mathfrak{I})(B)_{[\alpha]} = B_{[\alpha]}$, $\omega_\alpha^*(\mathfrak{I})(E)_{[\alpha]} = E_{[\alpha]}$, A and B are not ω_α^* -separated.

Theorem 43. Let $(L^X, \omega_\alpha^*(\Omega))$ be an ω_α -opos, $\mathfrak{I}, \mathfrak{J}$ two α -ideals. $A, B \in L^X$. Then,

- (1) if $\mathfrak{I} \subset \mathfrak{J}$, $\omega_\alpha^*(\mathfrak{I})$ -separated $\Rightarrow \omega_\alpha^*(\mathfrak{J})$ -separated,
- (2) $\omega_\alpha^*(\mathfrak{I} \cap \mathfrak{J})$ -separated if and only if both $\omega_\alpha^*(\mathfrak{I})$ -separated and $\omega_\alpha^*(\mathfrak{J})$ -separated,
- (3) $\omega_\alpha^*(\mathfrak{I} \vee \mathfrak{J})$ -separated if and only if $\omega_\alpha^*(\omega_\alpha^*(\mathfrak{I}))(\mathfrak{J})$ -separated if and only if $\omega_\alpha^*(\omega_\alpha^*(\mathfrak{J}))(\mathfrak{I})$ -separated,
- (4) $\omega_\alpha^*(\omega_\alpha^*(\mathfrak{I}))(\mathfrak{I})$ -separated $\Leftrightarrow \omega_\alpha^*(\mathfrak{I})$ -separated.

Proof. By Theorems 19(1), 4.4 and Corollary 24, easy. $\square$

Corollary 44. Let $(L^X, \omega_\alpha^*(\Omega))$ be an ω_α -opos, $\mathfrak{I}$ an α -ideals. Then,

- (1) ω_α^* -connected $\Rightarrow \omega_\alpha$ -connected,
- (2) A^* and B^* are ω_α -connected $\Leftrightarrow A^*$ and B^* are ω_α^* -connected $\Rightarrow A$ and B are ω_α^* -connected,
- (3) $A \in \omega_\alpha(\Omega) \cap \mathfrak{I}$, then A is ω_α -connected $\Leftrightarrow A$ is ω_α^* -connected.

Corollary 45. Let $(L^X, \omega_\alpha^*(\Omega))$ be an ω_α -opos, $\mathfrak{I}, \mathfrak{J}$ two α -ideals. $A, B \in L^X$. Then,

- (1) if $\mathfrak{I} \subset \mathfrak{J}$, $\omega_\alpha^*(\mathfrak{I})$ -connected $\Rightarrow \omega_\alpha^*(\mathfrak{J})$ -connected,
- (2) $\omega_\alpha^*(\mathfrak{I} \cap \mathfrak{J})$ -connected if and only if both $\omega_\alpha^*(\mathfrak{I})$ -connected and $\omega_\alpha^*(\mathfrak{J})$ -connected,
- (3) $\omega_\alpha^*(\mathfrak{I} \vee \mathfrak{J})$ -connected if and only if $\omega_\alpha^*(\omega_\alpha^*(\mathfrak{I}))(\mathfrak{J})$ -connected if and only if $\omega_\alpha^*(\omega_\alpha^*(\mathfrak{J}))(\mathfrak{I})$ -connected.

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