Research Article

L-Fuzzy Connectedness in Stratified Order-Preserving Operator Spaces

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The concepts of stratified order-preserving operators and stratified continuity are introduced in *L*-fuzzy topological spaces. Their basic properties are discussed, and their characteristic properties are observed. The relationship between induced stratified order-preserving topological spaces and general order-preserving operator topological spaces is studied. Finally, stratified connectedness is introduced, and its properties are studied systematically.

1. Introduction

Since the concept of *L*-fuzzy order-preserving operator was introduced by Professor Chen in 2002 (see [1]), a lot of papers have been devoted to it (see [2–5]). In general topology, similar researches were established (see [5]). However, it has not yet been found in stratified space. As we know, it is quite different between *L*-fuzzy stratified topological spaces and *L*-fuzzy topological spaces. Thus, it is important and meaningful to introduce the concept.

2. Preliminaries

In this paper, a lattice *L* is called a completely distributive lattice with an order-reserving involution', if the following conditions hold.

- (1) *L* is a completely lattice. That is, the largest element $1 \in L$, and smallest elements $0 \in L$. Besides, for all $a, b \in L$, $a \vee b \in L$, $a \wedge b \in L$.
- (2) Distributive law holds in *L*. That is, for all $i \in I \neq \emptyset$, for all $j \in J_i \neq \emptyset$, $a_{i,j} \in L$, then

$$\bigwedge_{i \in I} \left(\bigvee_{j \in J_i} a_{i,j} \right) = \bigvee_{f \in \prod_{i \in I} J_i} \left(\bigwedge_{i \in I} a_{i,f(i)} \right),$$

$$\bigvee_{i \in I} \left(\bigwedge_{j \in J_i} a_{i,j} \right) = \bigwedge_{f \in \prod_{i \in I} J_i} \left(\bigvee_{i \in I} a_{i,f(i)} \right). \quad (1)$$

- (3) There is a mapping $' : L \rightarrow L$ satisfies: for all $a \in L$, then $(a')' = a$, and for all $a, b \in L$, $a \leq b$ implies $b' \leq a'$ (see [6]).

In this paper, *X, Y* will always denote nonempty crisp sets, A mapping $A : X \rightarrow L$ is called an *L*-fuzzy set. L^X is the set of all *L*-fuzzy sets on *X*. An element $e \in L$ is called an irreducible element in *L*, if $p \vee q = e$ implies $p = e$ or $q = e$, where $p, q \in L$. The set of all nonzero irreducible elements in *L* will be denoted by $M(L)$ (see [6]). If $x \in X$, $\alpha \in M(L)$, then x_α is called a molecule in L^X . The set of all molecules in L^X is denoted by $M^*(L^X)$ (see [7]). If $A \in L^X$, $\alpha \in M(L)$, take $A_{[\alpha]} = \{x \in X \mid A(x) \geq \alpha\}$. If $E \subset X$, the complement of *E*, denoted by E' , and $E' = X - E = \{y \in X \mid y \notin E\}$ (see [8]).

The following are the concepts of *L*-fuzzy order-preserving operator in *L*-fuzzy topological space and order-preserving operator in general topological space.

Let *X* be a nonempty set. An operator $\omega : L^X \rightarrow L^X$ is called an *L*-fuzzy order preserving operator in L^X , if it

satisfies: (1) $\omega(1_X) = 1_X$, (2) for all $A, B \in L^X$ and $A \leq B$ implies $\omega(A) \leq \omega(B)$. A set $A \in L^X$ is called an ω -set, if $\omega(A) = A$. The set of all ω -sets in L^X is denoted by Ω . And (L^X, Ω) is called an order-preserving operator space (briefly, L -fopos). A molecule $x_\alpha \in M^*(L^X)$, $P \in \Omega$, P is called an ω -remote neighborhood of x_α , if $x_\alpha \notin P$. The set of all ω -remote neighborhood of x_α is denoted by $\omega\eta(x_\alpha)$. Let $x_\alpha \in M^*(L^X)$, $A \in L^X$, x_α is called an ω -adherent point of A , if for all $P \in \omega\eta(x_\alpha)$, $A \not\subset P$. The union of all ω -adherent points of A is called the ω -closure of A , denoted by A_ω^- . A set $A \in L^X$ is called ω -closed, if $A_\omega^- = A$. The set of all ω -closed sets in L^X is denoted by Ω_ω^- . Ω_ω^- is finite union and infinite intersection preserving (see [1]).

Let X be a nonempty set, $\mathcal{P}(X)$ be the family of all subsets of X . An operator $\sigma : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is called an order preserving operator in X , if it satisfies: (1) $\sigma(X) = X$, (2) for all $A \subset B \subset X$ implies $\sigma(A) \subset \sigma(B)$. A set $A \subset X$ is called an σ -set, if $\sigma(A) = A$. The set of all σ -sets in X is denoted by Δ . And (X, Δ) is called an order-preserving operator space on X (briefly, opo). Let $x \in X$, $P \in \Delta$, P is called an σ -remote neighborhood of x , if there is $Q \subset X$, such that $x \notin Q$, $P \subset Q$. The set of all σ -remote neighborhood of x is denoted by $\sigma\eta(x)$. Let $x \in X$, $A \subset X$, x is called an σ -adherent point of A , if for all $P \in \sigma\eta(x)$, $A \not\subset P$. The union of all σ -adherent points of A is called the σ -closure of A , denoted by A_σ^- . An set $A \subset X$ is called σ -closed, if $A_\sigma^- = A$. The set of all σ -closed sets in X is denoted by Δ_σ^- . Δ_σ^- is finite union and infinite intersection preserving (see [5]).

Let (X, Δ) be an opo, L be an fuzzy lattice, for all $\alpha \in L$. A fuzzy set $A : L \rightarrow X$ is called a lower continuous function, if $\{x \in X \mid A(x) \leq \alpha\} \in \Delta_\sigma^-$. Then, the set of all the lower continuous functions, denoted by $\omega_L(\Delta)$ consists an L -fuzzy topology in L^X . The space $(L^X, \omega_L(\Delta))$ is called the induced L -fopos by (X, Δ) (see [5]).

Considering the above definitions, if we say there is an L -fopos (L^X, Ω) or an opo (X, Δ) , we mean there is an order-preserving operator ω on L^X , or σ on X . The two spaces are generated by them, respectively.

Let (L^X, δ) and (L^Y, τ) be two L -fuzzy topological spaces, $f^- : L^X \rightarrow L^Y$ be induced by simple mapping $f : X \rightarrow Y$. If f^- and its reverse mapping f^+ satisfies:

$$(1) \text{ for all } A \in L^X, \quad y \in Y, \quad f^-(A)(y) = \vee \{A(x) \mid x \in X, f(x) = y\};$$

$$(2) \text{ for all } B \in L^Y, \quad x \in X, \quad f^-(B)(x) = B(f(x)).$$

Then f^- is called an L -fuzzy mapping (see [6]).

An L -fuzzy mapping $f^- : L^X \rightarrow L^Y$ is called homomorphism, if it satisfies:

$$(1) f^- \text{ is union preserving. That is, for } \{A_i \in L^X, (i \in I)\} \subset L^X, \quad f^-(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} f^-(A_i);$$

$$(2) f^- \text{ is reserving involution preserving. That is, for } B \in L^Y, \quad f^-(B') = (f^-(B))' \text{ (see [6]).}$$

3. ω_α -Closed Set

Definition 1. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. An operator $\omega_\alpha : L^X \rightarrow L^X$ is defined by: for all $A \in L^X$,

$$\omega_\alpha(A) = \wedge \{G \in \Omega_\omega^- \mid G_{[\alpha]} \supset A_{[\alpha]}\}. \quad (2)$$

Since the operator ω_α is related to $\alpha \in M(L)$, such kind of operator is generally called stratified order-preserving operator. Clearly, if ω is a closure operator, then ω_α is D_α (see [9]).

Theorem 2. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. $A, B \in L^X$, then the following statements hold:

$$(1) A_{[\alpha]} \subset B_{[\alpha]} \text{ implies } \omega_\alpha(A) \leq \omega_\alpha(B);$$

$$(2) A_{[\alpha]} \subset \omega_\alpha(A)_{[\alpha]};$$

$$(3) \omega_\alpha(A \vee B) = \omega_\alpha(A) \vee \omega_\alpha(B);$$

$$(4) \omega_\alpha(\omega_\alpha(A)) = \omega_\alpha(A).$$

Proof. (1), (2) are easy.

(3) by (1), we have $\omega_\alpha(A \vee B) \geq \omega_\alpha(A) \vee \omega_\alpha(B)$. On the other hand,

$$\begin{aligned} \omega_\alpha(A) \vee \omega_\alpha(B) &= (\wedge \{G \in \Omega_\omega^- \mid G_{[\alpha]} \supset A_{[\alpha]}\}) \\ &\quad \vee (\wedge \{H \in \Omega_\omega^- \mid H_{[\alpha]} \supset B_{[\alpha]}\}) \\ &= \wedge \{G \vee H \in \Omega_\omega^- \mid G_{[\alpha]} \supset A_{[\alpha]}, H_{[\alpha]} \supset B_{[\alpha]}\} \\ &\geq \wedge \{E \in \Omega_\omega^- \mid E_{[\alpha]} \supset A_{[\alpha]} \cup B_{[\alpha]}\} \\ &= \omega_\alpha(A \vee B). \end{aligned} \quad (3)$$

(4) For all $A \in L^X$, we have

$$\begin{aligned} \omega_\alpha(\omega_\alpha(A)) &= \wedge \{G \in \Omega_\omega^- \mid G_{[\alpha]} \supset \omega_\alpha(A)_{[\alpha]}\} \\ &\geq \wedge \{G \in \Omega_\omega^- \mid G_{[\alpha]} \supset A_{[\alpha]}\} \\ &= \omega_\alpha(A). \end{aligned} \quad (4)$$

On the other hand, as $\omega_\alpha(A) \in \Omega_\omega^-$, then by Definition 1, we have $\omega_\alpha(A) \leq \omega_\alpha(\omega_\alpha(A))$. Hence, (4) holds. $\square$

Definition 3. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. A set $A \in L^X$ is called ω_α -closed, if $\omega_\alpha(A)_{[\alpha]} = A_{[\alpha]}$. The set of all ω_α -closed sets in L^X is denoted by $\omega_\alpha(\Omega)$. $(L^X, \omega_\alpha(\Omega))$ is called ω_α -order-preserving operator space, (briefly, ω_α -opo).

By Definition 3, if ω is the closure operator, then an ω_α -closed set is a D_α -closed set (see [9]).

Theorem 4. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. Then

$$(1) 1_X \in \omega_\alpha(\Omega);$$

$$(2) \text{ If } \{A_i \mid i = 1, 2, \dots, n\} \subset \omega_\alpha(\Omega), \text{ then } \bigvee_{i=1}^n A_i \in \omega_\alpha(\Omega);$$

$$(3) \text{ If } \{A_i \mid i \in I\} \subset \omega_\alpha(\Omega), \text{ then } \bigwedge_{i \in I} A_i \in \omega_\alpha(\Omega).$$

Proof. (1) As $(1_X)_{[\alpha]} = X$, it is easy to check $\omega_\alpha(1_X)_{[\alpha]} = X = (1_X)_{[\alpha]}$. Thus, $1_X \in \omega_\alpha(\Omega)$.

(2) By Theorem 2, if $\{A_i \mid i = 1, 2, \dots, n\} \subset \omega_\alpha(\Omega)$. We have $\omega_\alpha(\bigvee_{i=1}^n A_i) = \bigvee_{i=1}^n \omega_\alpha(A_i)$. Then

$$\begin{aligned} \omega_\alpha\left(\bigvee_{i=1}^n A_i\right)_{[\alpha]} &= \left(\bigvee_{i=1}^n \omega_\alpha(A_i)\right)_{[\alpha]} = \bigcup_{i=1}^n \omega_\alpha(A_i)_{[\alpha]} \\ &= \bigcup_{i=1}^n (A_i)_{[\alpha]} = \left(\bigvee_{i=1}^n A_i\right)_{[\alpha]}. \end{aligned} \quad (5)$$

Hence, (2) holds.

(3) If $\{A_i \mid i \in I\} \subset \omega_\alpha(\Omega)$, so

$$\begin{aligned} \omega_\alpha\left(\bigwedge_{i \in I} A_i\right)_{[\alpha]} &\subset \left(\bigwedge_{i \in I} \omega_\alpha(A_i)\right)_{[\alpha]} \\ &= \bigcap_{i \in I} \omega_\alpha(A_i)_{[\alpha]} = \bigcap_{i \in I} (A_i)_{[\alpha]} = \left(\bigwedge_{i \in I} A_i\right)_{[\alpha]}. \end{aligned} \quad (6)$$

Remark 5. The theorem shows that ω_α is finite union and infinite intersection preserving.

Corollary 6. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. If there is $A \in \Omega_{\omega}^-$, such that $A_{[\alpha]} = \emptyset$. Then $\omega_\alpha(\Omega)$ consists an L -fuzzy co-topology on L^X .

Proof. By Theorem 4, we only need to prove $0_X \in \omega_\alpha(\Omega)$. In fact, as there is $A \in \Omega_{\omega}^-$, such that $A_{[\alpha]} = \emptyset$. Then

$$\omega_\alpha(0_X)_{[\alpha]} = \bigwedge \{G \in \Omega_{\omega}^- \mid G_{[\alpha]} \supset \emptyset\} = \emptyset = (0_X)_{[\alpha]}. \quad (7)$$

Hence, $0_X \in \omega_\alpha(\Omega)$. $\square$

Remark 7. In fact, the smallest fuzzy set 0_X is ω -closed. That is, $(0_X)_{\omega}^- = 0_X$. Otherwise, $\exists x_\lambda \leq (0_X)_{\omega}^-, x_\lambda \not\leq 0_X$. It means x_λ is an ω -adherent point of 0_X . However, for all $P \in \omega\eta(x_\lambda) \neq \emptyset$, $0_X \leq P$. This is a contradiction with the hypothesis that x_λ is an ω -adherent point of 0_X .

Taking the fact above, we can conclude that given an L -fopos (L^X, Ω) , $(L^X, \omega_\alpha(\Omega))$ is always an L -fuzzy co-topology.

Corollary 8. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. $\omega_\alpha(\Omega)$ is an L -fuzzy co-topology on L^X . Then $\omega_\alpha(\Omega)_{[\alpha]}$ is a co-topology on X . So $(X, \omega_\alpha(\Omega)_{[\alpha]})$ is an opos on X . Moreover, one has for all $A \in L^X$, $(A_{[\alpha]})_{\sigma_\omega}^- = \omega(A)_{[\alpha]}$.

Proof. We only prove $(A_{[\alpha]})_{\sigma_\omega}^- = \omega(A)_{[\alpha]}$. By Theorem 2 (1), $(A_{[\alpha]})_{\sigma_\omega}^- \subset (\omega(A)_{[\alpha]})_{\sigma_\omega}^-$. Again, by Theorem 2 (4), we have $\omega(A)_{[\alpha]} \in \omega_\alpha(\Omega)_{[\alpha]}$. So $(A_{[\alpha]})_{\sigma_\omega}^- \subset \omega(A)_{[\alpha]}$.

Conversely, if there is $x \in \omega(A)_{[\alpha]}$, but $x \notin (A_{[\alpha]})_{\sigma_\omega}^-$. Since $(A_{[\alpha]})_{\sigma_\omega}^- = \cap \{H_{[\alpha]} \mid H \in \omega_\alpha(\Omega), H_{[\alpha]} \supset A_{[\alpha]}\}$, there is $H \in \omega_\alpha(\Omega)$, such that $H_{[\alpha]} \supset A_{[\alpha]}$ and $x \notin H_{[\alpha]}$. So we have $H_{[\alpha]} = \omega_\alpha(H)_{[\alpha]} \supset \omega_\alpha(A)_{[\alpha]}$. This is a contradiction, because $x \in \omega(A)_{[\alpha]}$. Therefore, $(A_{[\alpha]})_{\sigma_\omega}^- \supset \omega(A)_{[\alpha]}$. $\square$

Lemma 9. Let (X, Δ) be an opos, $(L^X, \omega_L(\Delta))$ be an L -fopos induced by (X, Δ) . $E \subset X$. Then $E \in \Delta_{\sigma}^-$ if and only if $\chi_E \in (\omega_L(\Delta))_{\omega}^-$.

Proof. For any $\alpha \in M(L)$. If $\alpha \neq 1$, then $\{x \in X \mid \chi_E \leq \alpha\} = E'$. If $\alpha = 1$, then $\{x \in X \mid \chi_E \leq \alpha\} = X$. Thus, by the definition of lower continuous function, we have $E \in \Delta_{\sigma}^-$ if and only if $\chi_E \in (\omega_L(\Delta))_{\omega}^-$. $\square$

Theorem 10. Let (X, Δ) be an opos, $(L^X, \omega_L(\Delta))$ be an L -fopos induced by (X, Δ) . $A \in L^X$, $\alpha \in M(L)$. Then $A \in \omega_\alpha(\omega_L(\Delta))$ if and only if $A_{[\alpha]} \in \Delta_{\sigma}^-$.

Proof. For any $G \in L^X$, by Lemma 9, we have $G \in (\omega_L(\Delta))_{\omega}^-$ iff $G_{[\alpha]} \in \Delta_{\sigma}^-$. Let $A \in \omega_\alpha(\omega_L(\Delta))$. Then

$$\begin{aligned} A_{[\alpha]} &= \omega_\alpha(A)_{[\alpha]} = \left(\bigwedge \{G \in (\omega_L(\Delta))_{\omega}^- \mid G_{[\alpha]} \supset A_{[\alpha]}\}\right)_{[\alpha]} \\ &= \bigcap \{G_{[\alpha]} \mid G \in (\omega_L(\Delta))_{\omega}^-, G_{[\alpha]} \supset A_{[\alpha]}\} \\ &= \bigcap \{G_{[\alpha]} \in \Delta_{\omega}^- \mid G_{[\alpha]} \supset A_{[\alpha]}\} \\ &= (A_{[\alpha]})_{\sigma}^-. \end{aligned} \quad (8)$$

So $A_{[\alpha]} \in \Delta_{\sigma}^-$.

Conversely, if $A_{[\alpha]} \in \Delta_{\sigma}^-$, by Lemma 9, we have $\chi_{A_{[\alpha]}} \in (\omega_L(\Delta))_{\omega}^-$. And $(\chi_{A_{[\alpha]}})_{[\alpha]} = A_{[\alpha]}$, so

$$\begin{aligned} \omega_\alpha(A)_{[\alpha]} &= \bigcap \{G_{[\alpha]} \mid G \in (\omega_L(\Delta))_{\omega}^-, G_{[\alpha]} \supset A_{[\alpha]}\} \\ &\subset (\chi_{A_{[\alpha]}})_{[\alpha]} = A_{[\alpha]}. \end{aligned} \quad (9)$$

Hence, $\omega_\alpha(A)_{[\alpha]} = A_{[\alpha]}$. This implies $A \in \omega_\alpha(\omega_L(\Delta))$. $\square$

Theorem 11. Let (X, Δ) be an opos, $(L^X, \omega_L(\Delta))$ is an L -fopos induced by (X, Δ) . Then $A \in \omega_L(\Delta)$ iff for all $\alpha \in M(L)$, $A \in \omega_\alpha(\omega_L(\Delta))$.

Proof. By Theorem 10, we have

$$A \in (\omega_L(\Delta))' \iff A_{[\alpha]} \in \Delta_{\sigma}^- \iff A \in \omega_\alpha(\omega_L(\Delta)). \quad (10)$$

Definition 12. Let $(L^X, \omega_\alpha^1(\Omega_1))$, $(L^Y, \omega_\alpha^2(\Omega_2))$ be ω_α^1 -opos, ω_α^2 -opos, respectively. An L -fuzzy homomorphism $f^- : L^X \rightarrow L^Y$ is called $(\omega_\alpha^1(\Omega_1), \omega_\alpha^2(\Omega_2))$ -continuous, if for all $B \in \omega_\alpha^2(\Omega_2)$, then $f^-(B) \in \omega_\alpha^1(\Omega_1)$.

Theorem 13. Let $(L^X, \omega_\alpha^1(\Omega_1))$, $(L^Y, \omega_\alpha^2(\Omega_2))$ be ω_α^1 -opos, ω_α^2 -opos, respectively. $f^- : L^X \rightarrow L^Y$ is an L -fuzzy mapping. Then the following statements are equivalent.

- (1) f^- is $(\omega_\alpha^1(\Omega_1), \omega_\alpha^2(\Omega_2))$ -continuous;
- (2) for all $B \in L^Y$, $\omega_\alpha^1(f^-(B))_{[\alpha]} \subset f^-(\omega_\alpha^2(B))_{[\alpha]}$;
- (3) for all $A \in L^X$, $f^-(\omega_\alpha^1(A))_{[\alpha]} \subset \omega_\alpha^2(f^-(A))_{[\alpha]}$.

Proof. (1) $\Rightarrow$ (2) for all $B \in L^Y$, so $\omega_\alpha^2(B)_{[\alpha]} \supset B_{[\alpha]}$, then $f^-(\omega_\alpha^2(B))_{[\alpha]} \supset f^-(B)_{[\alpha]}$. Therefore, $\omega_\alpha^1(f^-(\omega_\alpha^2(B)))_{[\alpha]} \supset \omega_\alpha^1(f^-(B))_{[\alpha]}$.

On the other hand, by Theorem 2, we get that $\omega_\alpha^2(B) \in \omega_\alpha^2(\Omega_2)$. And by hypothesis, f^- is $(\omega_\alpha^1(\Omega_1), \omega_\alpha^2(\Omega_2))$ -continuous, then, $f^-(\omega_\alpha^2(B)) \in \omega_\alpha^1(\Omega_1)$. Hence, $\omega_\alpha^1(f^-(\omega_\alpha^2(B)))_{[\alpha]} = f^-(\omega_\alpha^2(B))_{[\alpha]}$. Therefore, we have

$$\omega_\alpha^1(f^-(B))_{[\alpha]} \subset \omega_\alpha^1(f^-(\omega_\alpha^2(B)))_{[\alpha]} = f^-(\omega_\alpha^2(B))_{[\alpha]}. \quad (11)$$

(2) $\Rightarrow$ (3) for all $A \in L^X$, so $f^-(A) \in L^Y$. By (2), we get $\omega_\alpha^1(f^-(f^-(A)))_{[\alpha]} \subset f^-(\omega_\alpha^2(f^-(A)))_{[\alpha]}$. So,

$$\begin{aligned} & (f^- f^-(\omega_\alpha^2(f^-(A))))_{[\alpha]} \\ & \supset f^-(\omega_\alpha^1(f^-(f^-(A))))_{[\alpha]} \supset f^-(\omega_\alpha^1(A))_{[\alpha]}. \end{aligned} \quad (12)$$

And obviously, $(\omega_\alpha^2(f^-(A)))_{[\alpha]} = (f^- f^-(\omega_\alpha^2(f^-(A))))_{[\alpha]}$. Thus, (2) holds.

(3) $\Rightarrow$ (1). for all $B \in \omega_\alpha^2(\Omega_2)$. By (3), we have

$$\begin{aligned} & f^-(\omega_\alpha^1(f^-(B)))_{[\alpha]} \subset \omega_\alpha^2(f^-(f^-(B)))_{[\alpha]} \\ & \subset \omega_\alpha^2(B)_{[\alpha]} = B_{[\alpha]}. \end{aligned} \quad (13)$$

So, $\omega_\alpha^1(f^-(B))_{[\alpha]} \subset f^-(B)_{[\alpha]}$.

On the other hand, obviously, $f^-(B)_{[\alpha]} \subset \omega_\alpha^1(f^-(B))_{[\alpha]}$. Therefore, $f^-(B)_{[\alpha]} = \omega_\alpha^1(f^-(B))_{[\alpha]}$. This shows that $f^-(B) \in \omega_\alpha^1(\Omega_1)$. $\square$

4. ω_α -Connectedness

Definition 14. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. $A, B \in L^X$ are called ω_α -separated, if $\omega_\alpha(A)_{[\alpha]} \cap B_{[\alpha]} = A_{[\alpha]} \cap \omega_\alpha(B)_{[\alpha]} = \emptyset$. Otherwise, $A, B \in L^X$ are called ω_α -connected.

Clearly, if ω is the closure operator in L^X , then ω_α -connected is D_α -connected [8].

Definition 15. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. $A \in L^X$ is called an ω_α -connected set, if there not exist ω_α -separated sets $B, C \in L^X$, such that, $B_{[\alpha]} \neq \emptyset, C_{[\alpha]} \neq \emptyset$, and $A_{[\alpha]} = B_{[\alpha]} \cup C_{[\alpha]}$. Specially, if 1_X is ω_α -connected, then, (L^X, Ω) is called ω_α -connected.

Theorem 16. Let (L^X, Ω) be an L -fopos, $\alpha \in M(L)$. Then (L^X, Ω) is not ω_α -connected if and only if there are $A, B \in \omega_\alpha(\Omega)$, such that, $A_{[\alpha]} \neq \emptyset, B_{[\alpha]} \neq \emptyset$, and $A_{[\alpha]} \cap B_{[\alpha]} = \emptyset, A_{[\alpha]} \cup B_{[\alpha]} = X$.

Proof. Necessity. Let (L^X, Ω) be not ω_α -connected. That is, 1_X is not ω_α -connected. By Definition 15, there are two ω_α -separated sets $A, B \in L^X$, such that, $A_{[\alpha]} \neq \emptyset, B_{[\alpha]} \neq \emptyset$, and $A_{[\alpha]} \cup B_{[\alpha]} = (1_X)_{[\alpha]} = X$.

Since, $A, B \in L^X$ are ω_α -separated, we have

$$\begin{aligned} \omega_\alpha(A)_{[\alpha]} &= \omega_\alpha(A)_{[\alpha]} \cap X \\ &= \omega_\alpha(A)_{[\alpha]} \cap (A_{[\alpha]} \cup B_{[\alpha]}) \\ &= (\omega_\alpha(A)_{[\alpha]} \cap A_{[\alpha]}) \cup (\omega_\alpha(A)_{[\alpha]} \cap B_{[\alpha]}) \\ &= A_{[\alpha]} \cup \emptyset \\ &= A_{[\alpha]}. \end{aligned} \quad (14)$$

Therefore, $A_{[\alpha]} \cap B_{[\alpha]} \subset \omega_\alpha(A)_{[\alpha]} \cap B_{[\alpha]} = \emptyset$. Hence, the necessity holds.

Sufficiency. Take $A, B \in \omega_\alpha(\Omega)$ satisfies the conditions stated in the theorem. So $(1_X)_{[\alpha]} = A_{[\alpha]} \cup B_{[\alpha]} = X$. As $A_{[\alpha]} \cap B_{[\alpha]} = \emptyset$, and $\omega_\alpha(A)_{[\alpha]} = A_{[\alpha]}$, we have $\omega_\alpha(A)_{[\alpha]} \cap B_{[\alpha]} = A_{[\alpha]} \cap B_{[\alpha]} = \emptyset$. Similarly, $\omega_\alpha(B)_{[\alpha]} \cap A_{[\alpha]} = B_{[\alpha]} \cap A_{[\alpha]} = \emptyset$. Hence, A, B are ω_α -separated. By Definition 15, we have (L^X, Ω) is not ω_α -connected. $\square$

Theorem 17. Let (L^X, Ω) is an L -fopos, $\alpha \in M(L)$. A be ω_α -connected, then $A_{[\alpha]} \subset B_{[\alpha]} \subset \omega_\alpha(A)_{[\alpha]}$ implies B is ω_α -connected.

Proof. Since A is ω_α -connected. $A_{[\alpha]}$ is connected in $(X, \omega_\alpha(\Omega))$. By Theorem 20 in reference [10] (page 54) and Corollary 8, $\omega_\alpha(A)_{[\alpha]}$ is connected. So $B_{[\alpha]}$ is connected. Therefore, B is ω_α -connected. $\square$

Theorem 18. Let $(L^X, \omega_\alpha(\Omega))$ be an ω_α -opos. $\{A_t\}_{t \in T} \subset L^X$, A_t is ω_α -connected, and there is $s \in T$, such that for any $t \in T$, A_t and A_s are ω_α -connected. Then $A = \bigvee_{t \in T} A_t$ is ω_α -connected.

Proof. Since A_t is ω_α -connected, $A_{t[\alpha]}$ is connected in $(X, \omega_\alpha(\Omega))$. $A_{s[\alpha]}$ and $A_{t[\alpha]}$ are collected for each $s \in T$. Then by Theorem 21 in reference [10] (page 54) and Corollary 8, $A_{[\alpha]} = (\bigvee_{t \in T} A_t)_{[\alpha]} = \bigcup_{t \in T} A_{t[\alpha]}$ is connected. Therefore, $\bigvee_{t \in T} A_t$ is ω_α -connected. $\square$

Theorem 19. Let (X, Δ) be an opos, and $(L^X, \omega_L(\Delta))$ be the induced L -opos. Then $A \in \omega_\alpha(\omega_L(\Delta))$ is ω_α -connected if and only if $A_{[\alpha]} \subset X$ is connected.

Proof. Necessity. Let $A \in \omega_\alpha(\omega_L(\Delta))$ be ω_α -connected. Suppose that $A_{[\alpha]}$ is not connected, then there are nonempty $U, V \subset X$, such that $A_{[\alpha]} = U \cup V$, and $U_\sigma^- \cap V = V_\sigma^- \cap U = \emptyset$. By Theorem 10, $\omega_\alpha(1_U)_{[\alpha]} = U_\sigma^-$, $\omega_\alpha(1_V)_{[\alpha]} = V_\sigma^-$. So

$$\omega_\alpha(1_U)_{[\alpha]} \cap (1_V)_{[\alpha]} = \omega_\alpha(1_V)_{[\alpha]} \cap (1_U)_{[\alpha]} = \emptyset. \quad (15)$$

This means $1_U, 1_V$ are ω_α -separated sets, and $A_{[\alpha]} = (1_U)_{[\alpha]} \cup (1_V)_{[\alpha]}$. A contradiction.

Sufficiency. Suppose $A \in \omega_\alpha(\omega_L(\Delta))$ is not ω_α -connected. Then There are ω_α -separated sets $B, C \in L^X$, such that $B_{[\alpha]}, C_{[\alpha]} \neq \emptyset$, and $A_{[\alpha]} = B_{[\alpha]} \cup C_{[\alpha]}$. Since $\omega_\alpha(B) \in \omega_\alpha(\omega_L(\Delta))$, by Theorem 10, we have $\omega_\alpha(B)_{[\alpha]} \in \Delta_\sigma^-$. Thus, $(B_{[\alpha]})_\sigma^- \subset (\omega_\alpha(B)_{[\alpha]})_\sigma^- = \omega_\alpha(B)_{[\alpha]}$. Therefore,

$$(B_{[\alpha]})_\sigma^- \cap C_{[\alpha]} \subset \omega_\alpha(B)_{[\alpha]} \cap C_{[\alpha]} = \emptyset. \quad (16)$$

Similarly, we get that $(B_{[\alpha]})_\sigma^- \cap C_{[\alpha]} = \emptyset$. This means $A_{[\alpha]} \subset X$ is not connected. A contradiction. $\square$

Theorem 20. Let (L^X, Ω_i) ($i = 1, 2$) be two L -opos, $f^- : L^X \rightarrow L^X$ be (ω^1, ω^2) -continuous, $A \in L^X$ be ω_α^1 -connected. Then $f^-(A)$ is ω_α^2 -connected.

Proof. Let $f^-(A)_{[\alpha]} = B_{[\alpha]} \cup C_{[\alpha]}$, $\omega_\alpha^2(B)_{[\alpha]} \cap C_{[\alpha]} = \omega_\alpha^2(C)_{[\alpha]} \cap B_{[\alpha]} = \emptyset$. Since $f^{-1}(B_{[\alpha]}) = f^-(B)_{[\alpha]}$. Take

$$E_{[\alpha]} = f^{-1}(B_{[\alpha]}), \quad F_{[\alpha]} = f^{-1}(C_{[\alpha]}), \quad (17)$$

we have

$$A_{[\alpha]} \subset (f^- f^-(A))_{[\alpha]} = f^-(B)_{[\alpha]} \cup f^-(C)_{[\alpha]} = E_{[\alpha]} \cup F_{[\alpha]}. \quad (18)$$

Besides, by

$$\begin{aligned} \omega_\alpha^1(E)_{[\alpha]} &= \omega_\alpha^1(f^-(B))_{[\alpha]} \subset f^-(\omega_\alpha^2(B))_{[\alpha]}, \\ \omega_\alpha^1(F)_{[\alpha]} &= \omega_\alpha^1(f^-(C))_{[\alpha]} \subset f^-(\omega_\alpha^2(C))_{[\alpha]} \end{aligned} \quad (19)$$

we get

$$\begin{aligned} \omega_\alpha^1(E)_{[\alpha]} \cap F_{[\alpha]} &\subset f^-(\omega_\alpha^2(B))_{[\alpha]} \cap f^-(\omega_\alpha^2(C))_{[\alpha]} \\ &= f^{-1}(\omega_\alpha^2(B)_{[\alpha]} \cap C_{[\alpha]}) = f^{-1}(\emptyset) = \emptyset, \\ \omega_\alpha^1(F)_{[\alpha]} \cap E_{[\alpha]} &\subset f^-(\omega_\alpha^2(C))_{[\alpha]} \cap f^-(\omega_\alpha^2(B))_{[\alpha]} \\ &= f^{-1}(\omega_\alpha^2(C)_{[\alpha]} \cap B_{[\alpha]}) = f^{-1}(\emptyset) = \emptyset. \end{aligned} \quad (20)$$

Denote $G_{[\alpha]} = A_{[\alpha]} \cup E_{[\alpha]}$, $H_{[\alpha]} = A_{[\alpha]} \cup F_{[\alpha]}$. Then $A_{[\alpha]} = G_{[\alpha]} \cup H_{[\alpha]}$, and

$$\omega_\alpha^1(G)_{[\alpha]} \cap H_{[\alpha]} = \omega_\alpha^1(H)_{[\alpha]} \cap G_{[\alpha]} = \emptyset. \quad (21)$$

Since A is ω_α^1 -connected, then $G_{[\alpha]} = \emptyset$, or $H_{[\alpha]} = \emptyset$. Take the former, for example, $A_{[\alpha]} = H_{[\alpha]} \subset F_{[\alpha]}$. Thus,

$$f^-(A)_{[\alpha]} \subset f^-(F)_{[\alpha]} = (f^- f^-(C))_{[\alpha]} \subset C_{[\alpha]}. \quad (22)$$

Thus,

$$\begin{aligned} B_{[\alpha]} &= B_{[\alpha]} \cap f^-(A)_{[\alpha]} \subset B_{[\alpha]} \cap C_{[\alpha]} \\ &\subset B_{[\alpha]} \cap \omega_\alpha^2(C)_{[\alpha]} = \emptyset. \end{aligned} \quad (23)$$

□

Theorem 21. Let (L^X, Ω) be an L -opos, $A \in L^X$, for all $\alpha \in M(L)$, $M^*(A) = \{a \in M^*(L^X) \mid a_{[\alpha]} \in A_{[\alpha]}\}$. Then A is ω_α -connected if and only if for all $a, b \in M^*(A)$, for all $P : M^*(A) \rightarrow \cup \{\eta_{\omega_\alpha}(x) \mid x \in M^*(A)\}$, there are finite molecular $x_1 = a, x_2, \dots, x_n = b \in M^*(A)$, such that $A_{[\alpha]} \not\subset P(x_j)_{[\alpha]} \cup P(x_j)_{[\alpha]}$, ($j = 1, 2, \dots, n-1$).

Proof. Necessity. Suppose the result does not holds. That is, there is $a, b \in M^*(A)$, $a \neq b$. And there is a mapping

$$P : M^*(A) \longrightarrow \cup \{\eta_{\omega_\alpha}(x) \mid x \in M^*(A)\}, \quad (24)$$

such that for any finite $x_1 = a, x_2, \dots, x_n = b \in M^*(A)$, $A_{[\alpha]} \not\subset P(x_j)_{[\alpha]} \cup P(x_j)_{[\alpha]}$ does not holds.

For the convenient sake, $a, b \in M^*(A)$, a and b are called joined, if there are finite $x_1 = a, x_2, \dots, x_n = b \in M^*(A)$, such that $A_{[\alpha]} \not\subset P(x_j)_{[\alpha]} \cup P(x_j)_{[\alpha]}$, denoted by $a \sim b$. Otherwise, a and b are called disjointed, denoted by $a \not\sim b$. Let

$$\mu = \{x \in M^*(A) \mid a \sim x\}, \quad \nu = \{x \in M^*(A) \mid a \not\sim x\}, \quad (25)$$

and $B = \vee \mu$, $C = \vee \nu$. Obviously, $a \sim a$, so $a \in \mu$, $a \leq B$. and $a \not\sim b$, so $b \in \nu$, $b \leq C$. Hence, $B_{[\alpha]} \neq \emptyset$, $C_{[\alpha]} \neq \emptyset$.

Since for all $a \in M^*(A)$, $a \in \mu$, or $a \in \nu$. So $A_{[\alpha]} = B_{[\alpha]} \cup C_{[\alpha]}$. Next, let us prove B, C are not ω_α -connected.

Suppose that $\omega_\alpha(B)_{[\alpha]} \cap C_{[\alpha]} \neq \emptyset$, then there is $x_{[\alpha]} \in \omega_\alpha(B)_{[\alpha]} \cap C_{[\alpha]}$. By $x_{[\alpha]} \in \omega_\alpha(B)_{[\alpha]}$, we have $B_{[\alpha]} \not\subset P(x)_{[\alpha]}$. So there is $y \in \mu$, such that $y_{[\alpha]} \notin P(y)_{[\alpha]}$, and $y_{[\alpha]} \in B_{[\alpha]} \subset A_{[\alpha]}$, hence $A_{[\alpha]} \not\subset P(x)_{[\alpha]} \cup P(y)_{[\alpha]}$. This means that $y \sim a$, and $x \sim a$.

On the other hand, by $x_{[\alpha]} \in C_{[\alpha]}$, then $C_{[\alpha]} \not\subset P(x)_{[\alpha]}$, so there is $z \in \nu$, such that $z_{[\alpha]} \notin P(x)_{[\alpha]}$. Therefore, $z_{[\alpha]} \notin P(x)_{[\alpha]} \cup P(x)_{[\alpha]}$. And by $z_{[\alpha]} \in C_{[\alpha]} \subset A_{[\alpha]}$, we have $A_{[\alpha]} \not\subset P(x)_{[\alpha]} \cup P(z)_{[\alpha]}$. This means $x \sim a$ and $a \sim z$. This is a contradiction with $z \in \nu$. Therefore $\omega_\alpha(B)_{[\alpha]} \cap C_{[\alpha]} = \emptyset$. Similarly, we get that $\omega_\alpha(C)_{[\alpha]} \cap B_{[\alpha]} = \emptyset$.

Sufficiency. Suppose A is not ω_α -connected. Then there are $B, C \in L^X$, such that B, C are ω_α -separated, and $B_{[\alpha]} \neq \emptyset$, $C_{[\alpha]} \neq \emptyset$, $A_{[\alpha]} = B_{[\alpha]} \cup C_{[\alpha]}$. Define a mapping

$$\begin{aligned} P : M^*(A) &\longrightarrow \cup \{\eta_{\omega_\alpha}(x) \mid x \in M^*(A)\}, \\ P(x) &= \begin{cases} \omega_\alpha(C), & x_{[\alpha]} \in B_{[\alpha]}, \\ \omega_\alpha(B), & x_{[\alpha]} \in C_{[\alpha]}. \end{cases} \end{aligned} \quad (26)$$

Then for all $x \in M^*(A)$, we have $x \not\subset P(x)$, $P(x) \in \omega_\alpha(\Omega)$.

Take $a_{[\alpha]} \in B_{[\alpha]}$, $b_{[\alpha]} \in C_{[\alpha]}$. Then for any finite $x_1 = a, x_2, \dots, x_n = b \in M^*(A)$, $x_{i[\alpha]} \in B_{[\alpha]}$, or $x_{i[\alpha]} \in C_{[\alpha]}$. Thus, $P(x_i)_{[\alpha]} \in \omega_\alpha(C)$, or $P(x_i)_{[\alpha]} \in \omega_\alpha(B)$, ($i = 1, 2, \dots, n$). However, since $P(x_1)_{[\alpha]} \in \omega_\alpha(C)$, $P(x_n)_{[\alpha]} \in \omega_\alpha(B)$, there is $1 \leq j \leq n-1$, such that $P(x_j) = \omega_\alpha(C)$, $P(x_{j+1}) = \omega_\alpha(B)$. Therefore, $A_{[\alpha]} = B_{[\alpha]} \cup C_{[\alpha]} \subset P(x_j)_{[\alpha]} \cup P(x_{j+1})_{[\alpha]}$. A contradiction. □

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